Applied Math Methods in Function Spaces: Preliminaries

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This is a short, sharp, background narrative that sets the analysis of differential and integral equations within a function space setting. This is not examinable but provides a context and big picture to the methods introduced in the course.

You may have already met function spaces within finite element methods, quantum mechanics, and elsewhere.

Inner Product Spaces

A Hilbert space is a complete normed space, X, that is equipped with an **inner product**, that is a real valued, bi-linear, function

$$\langle u, v \rangle : X \times X \to \mathbb{R},$$

such that the norm on X is given by

$$||u|| = \langle u, u \rangle^{1/2}.$$

"Completeness" merely requires that all Cauchy sequences have a convergent subsequence. So the rationals are incomplete (we can easily have a sequence of rationals converging to an irrational number which by definition is not in the space;) whereas as $\mathbb{R}$ is complete (indeed we can define the reals as *the completion* of the rationals).

We all know the n-dimensional Euclidean space, $\mathbb{R}^n$: then the inner product is the familiar vector "dot" product, or scalar product, $\langle u, v \rangle = u^T.v$.

Non-zero elements u and v in X are **orthogonal** iff $\langle u, v \rangle = 0$.

Note that if we work in the complex extension, $\mathbb{C}^n$, we must define

$$\langle u, v \rangle = u^T.\bar{v},$$

and so on, where $\bar{v}$ denotes the complex conjugate of v.

Let us think primarily of real spaces (since we often want to consider real valued functions as solutions to applied problems) yet introduce ideas about the spectra (eigenvalues and so forth) within their complexification (we know real matrices have complex valued spectra and eigenvectors, and so on) .

The space of pth power ($p > 1$), complex valued ,integrable function on $\Omega = [a, b]$ are those functions f for which the following integral exists:

$$\int_\Omega |f(t)|^p dt.$$

This space is called $L_p[\Omega]$ and is equipped with the norm

$$||f||_p \equiv \left(\int_\Omega |f(t)|^p dt \right)^{1/p}.$$

For $p = 2$ $L_p[\Omega]$ is a Hilbert space with the inner product

$$\langle u, v\rangle = \int_\Omega u(t)\bar{v}(t)dt.$$

For $p \neq 2$ but positive it is a complete normed pace (these are usually called Banach spaces), but it has no inner product, since if we try to define

$$\langle u, v\rangle = \int_\Omega u(t)^{p/2}\bar{v}(t)^{p/2}dt$$

this isn't linear in u and $\bar{v}$.

$L_2[\Omega]$ generalises to cases where Ω is a non trivial subset on $\mathbb{R}^m$ (where $m = 1, 2, 3, \dots$), usually with a piecewise smooth boundary $\delta\Omega$.

We only really need to know about $\mathbb{C}^n$ and $L_2[\Omega]$ for now.

We meet will $L_2[\Omega]$ when we consider integral operators and differential operators, made up of a differential form (in one or more dimensions) and imposing appropriate boundary conditions. We also meet it in finite element methods, within numerical analysis, and in the definition of weak solutions to PDEs.

The general theory of Hilbert spaces is available in chapter 3 of E. Kreyzig, "Introduction for Functional Analysis", which is very far from introductory! It is freely available in pdf form `https://physics.bme.hu/sites/physics.bme.hu/files/users/BMETE15AF53_kov/Kreyszig%20-%20Introductory%20Functional%20Analysis%20with%20Applications%20(1).pdf`. Enjoy.

Lots of this is peppered throughout the book by P. Grindrod, "Patterns and Waves", including the Fredholm alternative and applications and more general spectral theory of differential operations in some specific applied math settings: freely available, see `https://www.researchgate.net/publication/351122823_Patterns_And_Waves_The_Theory_and_Application_of_Reaction-Diffusion_Equations`

Linear Operators and Equations

A linear operator L acting on a Hilbert space, X, has both a domain, $D(L)$ and range in X, so that

$$L : D(L) \subset X \to X.$$

We define the adjoint operator L^* on X to be the linear operator such that

$$\langle Lu, v\rangle = \langle u, L^*v\rangle$$

for all u and v in X for which these expressions are finite and well-defined. If you are given X and L then you should always work directly from this definition to find L^* (see the examples below).

We are guaranteed that an adjoint operator L^* exists, since for all v fixed $\langle Lu, v\rangle$ is a linear functional in u and there is a theorem (the the Riesz representation theorem, see Kryezig) that states it can be written as $\langle u, w\rangle$ for some $w \in X$, so we can set $L^*v = w$, defining L^* point-wise. Moreover $\langle Lu, v\rangle$ is clearly linear in v. It is obvious that $(L^*)^* = L$. In any example, such as the differential or integral operators discussed below, these issues are very straightforward. Their general utility is often very useful, though, for modellers (not to get sucked into "the weeds").

Some operators are self adjoint ($L = L^*$) in which case all of the spectrum lives on the real axis.

In the case of equations, we might be interested in solutions of an inhomogeneous equation:

$$Lu = g,$$

with g given; where we also have the related homogeneous equation

$$Lu = 0.$$

Similarly we can consider an adjoint equation

$$L^* u = h,$$

with h given; and we also have the related homogeneous adjoint equation

$$L^* u = 0.$$

These equations (and their possible solutions) are the catalyst for Fredholm Alternative theory.

In fact, "$Lu = g$ has a solution iff g is orthogonal to the null space of L^*".

Say this to yourself every day. If L and thus L^* are invertible (so there is no null space in either case - that is, no non-zero solutions to the homogeneous equations above) then a unique solution of $Lu = g$ exists. *Alternatively* if the null spaces are non-trivial then this provides a *solvability condition*.

In the case of $X = \mathbb{R}^n$ then L is merely multiplication by an $n \times n$ real matrix A, and and L^* corresponds to multiplication by the real matrix A^T.

Suppose $X = L_2([a, b])$ the space of square integrable functions over the interval $[a.b]$. Consider the Fredholm operator

$$Lu(x) = u(x) - \lambda \int_a^b K(x, t)u(t)dt,$$

where, say, u is real valued.

To find the adjoint operator in this example we consider any real valued function v and the inner product

$$\langle Lu, v\rangle = \int_a^b v(x) \left(u(x) - \lambda \int_a^b K(x, t)u(t)dt \right) dx = \int_a^b uv - \lambda \int_a^b \int_a^b v(x)K(x, t)u(t)dtdx$$

swapping x and t in the last integraland reversing the order of integration we see

$$\langle Lu, v\rangle = \int_a^b uv - \lambda. \int_a^b \int_a^b v(t)K(t, x)u(x)dtdx = \int_a^b y(x) \left(v(x) - \lambda \int_a^b K(t, x)v(t)dt \right) dx.$$

Hence we take

$$L^* v(x) = v(x) - \lambda \int_a^b K(t, x)v(t)dt,$$

so that $\langle Lu, v\rangle = \langle u, L^* v\rangle$, as required in the above definition of the adjoint.

A linear differential operator defined over some spatial domain, Ω say, is a linear differential form defined over Ω (with either ordinary or partial derivatives depending upon the dimension of Ω), together with some suitable homogeneous boundary conditions to be imposed on the (assumed) piecewise smooth boundary, $\delta\Omega$.

Suppose, for example, $X = L_2([a, b])$ the space of square integrable functions over the interval $[a, b]$, with its intergral inner product.

Consider the linear operator

$$Lu(x) = u''(x) + A(x)u'(x) + B(x)u(x) \;\; a < x < b, \;\; u(a) = 0, \;\; u(b) = 0,$$

for given real valued functions A and B.

We have (using integration by parts, twice for the first term and once for the middle term),

$$\langle Lu, v\rangle = \int_a^b v(x) \left(u''(x) + A(x)u'(x) + B(x)u(x) \right) dx$$

$$= \left[v(x)u'(x)\right]_a^b - \left[v'(x)u(x)\right]_a^b + [v(x)A(x)u(x)]_a^b + \int_a^b u(x)(v''(x) - (A(x)v(x))' + B(x)v(x))dx.$$

Applying the boundary conditions on u, then $0 = [v'(x)u(x)]_a^b = [v(x)A(x)u(x)]_a^b$, and we have

$$\langle Lu, v\rangle = \left[v(x)u'(x)\right]_a^b + \int_a^b u(x)\left(v''(x) - (A(x)v(x))' + B(x)v(x)\right) dx.$$

But this must be true for a wide choice of u in X, and so we must impose

$$L^*v = v''(x) - (A(x)v(x))' + B(x)v(x) \;\; a < x < b, \;\; v(a) = 0, \; v(b) = 0.$$

Then $\langle Lu, v\rangle = \langle u, L^*v\rangle$ as required.

We might wish to solve an inhomogeneous equation (these often arise in asymptotic approaches to bifurcation analysis):

$$Lu(x) = u''(x) + A(x)u'(x) + B(x)u(x) = g(x) \;\; a < x < b, \;\; u(a) = 0, \; u(b) = 0.$$

Or we might have a PDE where $x \in \Omega$ a suitable subset of $\mathbb{R}^m$ together with suitable boundary conditions. Then all of the above considerations will be in play.

If Ω is in $\mathbb{R}^m$ then the divergence theorem (sometimes called Green's theorem) does the job of the integration by parts and the operator L and its adjoint L^* contain partial derivatives. Thus the corresponding equations will be PDEs.

For example, consider the operator

$$Lu = \Delta u(\mathbf{x}) + a(\mathbf{x})u \quad \mathbf{x} \in \Omega \subset \mathbb{R}^m, \;\; \mathbf{n}.\nabla u = 0 \;\; \mathbf{x} \in \delta\Omega.$$

Then, using the divergence theorem, and the no-flux (Neumann) boundary condition on u, we have

$$\int_\Omega \nabla.((\nabla u)v - (\nabla v)u)\, d\mathbf{x} = \int_{\delta\Omega} \mathbf{n}.((\nabla u)v - (\nabla v)u)\, dA - \int_{\delta\Omega} \mathbf{n}.(\nabla v)u\, dA.$$

So

$$\int_\Omega (\nabla.\nabla u + \alpha u)v\, d\mathbf{x} - \int_\Omega (\nabla.\nabla v + \alpha v)u\, d\mathbf{x} = -\int_{\delta\Omega} \mathbf{n}.(\nabla v)u\, dA.$$

Thus, if

$$L^*v = \Delta v(\mathbf{x}) + a(\mathbf{x})v \quad \mathbf{x} \in \Omega \subset \mathbb{R}^m, \quad \mathbf{n}.\nabla v = 0\, \mathbf{x} \in \delta\Omega,$$

that is $L^* = L$, we have

$$\langle Lu, v\rangle - \langle u, L^*v\rangle = 0.$$

Hence L is self-adjoint.

If you ever have a problem (an equation) to solve for a function $u(x)$ with inhomogeneous boundary conditions, then it is often rather useful to choose a function, $u_0(x)$ say, that satisfies the boundary given conditions, so that $u(x) = u_0(x) + \tilde{u}(x)$, and then $\tilde{u}(x)$ satisfies homogenous boundary conditions. The substitution may well change the inhomogeneous part of the full equation. But it makes the whole process much easier, since the resulting differential form has homogeneous boundary conditions: hence it is a linear operator. And so all of the above applies.

For differential operators there are some important subtleties we have glossed over, since the domain of 2nd order operators L, like the one above, is dense in $L_2([a, b])$: it contains those $u \in L_2([a, b])$ for which $u'' \in L_2([a, b])$; and which satisfy the homogeneous boundary conditions. Nevertheless that domain, $D(L)$, is itself a vector space and inherits the integral inner product and thus all notions of orthogonality.

For Fredholm integral operators we just need a well-behaved kernel, $K \in L_2$ with respect to both arguments, in order for L to be well defined.