Here we look at a solution for Example 9 on page 19 of the lecture notes where we do not presume the asymptotic scalings given there ($T = T_0 + O(\epsilon^2)$ and $u = \epsilon u + O(\epsilon^3)$ etc). Instead, let us suppose we do not anticipate those, and we expand both $u$ and $T$ as full regular series in integer powers of (small) $\epsilon$.

**Example** Consider consider the following nonlinear equation for $u(x)$:

$$u'' + Tu + u^3 = 0 \quad 0 < x < \pi, \quad u(0) = u(1) = 0.$$

Introduce a small parameter $\epsilon$ so that $T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \dots$, and consider small solutions as an asymptotic expansion

$$u = \epsilon \phi_1(x) + \epsilon^2 \phi_2(x) + \epsilon^3 \phi_3(x) + \dots$$

Substituting in and equating powers of $\epsilon$ we find to order $\epsilon$,

$$\phi_1'' + T_0 \phi_1 = 0 \quad \phi_1(0) = \phi_1(1) = 0.$$

So we must have $T_0 = n^2\pi^2$ and $\phi_1 = A \sin nx$ for some real constant $A \neq 0$, and thus non trivial for $A$ non zero.
This equation can be written $L\phi_1 = 0$, where $L$ is self-adjoint.
To order $\epsilon^2$ we have

$$\phi_2'' + T_0 \phi_2 = -T_1 \phi_0 \quad \phi_2(0) = \phi_2(1) = 0.$$

The Fredholm Alternative says that this has a solution if and only if the RHS is orthogonal to $\sin nx \propto \phi_0$, which spans the null space of $L$ (which is self adjoint). Thus we must take $T_1 = 0$. Then wlog we can take $\phi_2 = 0$.
To order $\epsilon^3$ we have

$$\phi_3'' + n^2 \phi_3 = -T_2 A \sin nx - A^3 \sin^3 nx \quad \phi_3(0) = \phi_3(1) = 0.$$

The Fredholm Alternative says that this has a solution if and only if RHS is orthogonal to $\sin nx \propto \phi_0$ (which spans the null space of the adjoint operator): hence we have

$$-T_2 \int_0^\pi \sin^2 nx dx = A^2 \int_0^\pi \sin^4 nx dx.$$

So $A = \pm\sqrt{-T_2 \int_0^\pi \sin^2 nx dx / \int_0^\pi \sin^4 nx dx} = \pm\sqrt{-4T_2/3}$, as required. $A$ is real iff $T_2 \leq 0$.
Of course what is $T_2$ here is written as $T_1$ in the notes; and $u_3$ here is $y_1$ in the notes.