Vector fields

Notation: I like to write all vectors as column vectors: so use $(u,v)^T$ to mean $\begin{pmatrix} u \\ v \end{pmatrix}$. This is a good practice to keep especially when you start to consider vector calculus and the nabla operations.
Let Ω be a bounded convex domain in two dimensions, with a piecewise smooth boundary denoted by $\delta\Omega$. Consider the functional, E:

$$J(u,v) = \int_\Omega F(x,y,u(x,y),u_x(x,y),u_y(x,y),v(x,y),v_x(x,y),v_y(x,y))\, dxdy,$$

defined for the C^2 vector field $(u(x,y),v(x,y))^T$ on Ω, satisfying boundary conditions where $(u,v)^T$ is given and continuous on the bpoundary, $\delta\Omega$.
Find a pair of partial differential equations that the extremal vector field $(u,v)^T$ must satisfy.

Solution
Suppose that $(u,v)^T$ is an extremal.
Consider $J(u+\eta, v+\psi)$ where both η and ψ vanish on the boundary, $\delta\Omega$, then

$$J(u+\eta,v+\psi) = \int_\Omega F\, dxdy + \int_\Omega F_u\eta + F_{u_x}\eta_x + F_{u_y}\eta_y + F_v\psi + F_{v_x}\psi_x + F_{v_y}\psi_y\, dxdy,$$

Here F and its partial derivates are all evaluated at

$$(x,y,u(x,y),u_x(x,y),u_y(x,y),v(x,y),v_x(x,y),v_y(x,u)).$$

We will write $(F_{u_x}, F_{u_y})^T$ to denote the column vector field over Ω.
The second integral must vanish at an extremal. It is equal to

$$\int_\Omega F_u\eta + F_v\psi + \begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix}.\nabla\eta + \begin{pmatrix} F_{v_x} \\ F_{v_y} \end{pmatrix}.\nabla\psi\, dxdy,$$

$$= \int_\Omega \left(F_u - \nabla.\begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix}\right)\eta + \left(F_v - \nabla.\begin{pmatrix} F_{v_x} \\ F_{v_y} \end{pmatrix}\right)\psi\, dxdy,$$

This is true by the Divergence theorem (Green's theorem), since we have the identities

$$\nabla.\left(\eta\begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix}\right) = \eta\nabla.\begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix} + \nabla\eta.\begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix}.$$

So

$$\int_{\delta\Omega} \nabla\left(\eta\begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix}\right).\mathbf{n}\, dS + \int_\Omega \left(F_u - \nabla.\begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix}\right)\eta\, dxdy$$

$$= \int_\Omega \eta F_u + \begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix}.\nabla.\eta\, dxdy$$

and the integrand within the boundary integral (from the Divergence theorem) vanishes due to the boundary condition on η. Similar for v and ψ.

Hence we have

$$F_u - \nabla . \begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix} = 0 \quad F_v - \nabla . \begin{pmatrix} F_{v_x} \\ F_{v_y} \end{pmatrix} = 0 \qquad (x, y)^T \in \Omega.$$