MMSC Further Mathematical Methods HT2022 — Sheet 1 Answers

1. Suppose A is a square matrix, $n \times n$. State (without proof) the Fredholm Alternative that gives necessary and sufficient conditions under which the system $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$.

Now consider the system $(A - \mu I)\mathbf{x} = \mathbf{b}$, where I is the $n \times n$ identity matrix, and μ is a constant. For what values of μ is there a unique solution? When μ is such that there is not a unique solution, what condition(s) must $\mathbf{b}$ satisfy in order for a solution to exist? When those conditions do hold, what is the most general solution $\mathbf{x}$?

Answer. Either

(i) $A^T\mathbf{y} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$, in which case $A\mathbf{x} = \mathbf{b}$ has a unique solution;

or

(ii) $A^T\mathbf{y} = \mathbf{0}$ has a maximal set of linearly independent nontrivial solutions $\mathbf{y}_1, \ldots, \mathbf{y}_r$ say. Then $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{y}_i \cdot \mathbf{b} = 0$ for each $i = 1, \ldots, r$. In this case the general solution is $\mathbf{x} = \mathbf{x}_p + \sum_{i=1}^{r} c_i\mathbf{x}_i$ where $\mathbf{x}_p$ is any particular solution, $c_1, \ldots, c_r$ are arbitrary constants, and the $\mathbf{x}_1, \ldots, \mathbf{x}_r$ are a maximal set of linearly independent solutions to $A\mathbf{x} = \mathbf{0}$.

For $(A - \mu I)\mathbf{x} = \mathbf{b}$ there is a unique solution if $A - \mu I$ is nonsingular, i.e. if μ is not an eigenvalue of A. If μ is an eigenvalue, then $\mathbf{b}$ must be orthogonal to all the corresponding left eigenvectors, i.e. $\mathbf{u} \cdot \mathbf{b} = 0$ for all $\mathbf{u}$ such that $(A - \mu I)^T\mathbf{u} = \mathbf{0}$ (equivalently $\mathbf{u}^T(A - \mu I) = \mathbf{0}^T$). If this holds then the general solution is

$$\mathbf{x} = \mathbf{x}_p + \sum_{i=1}^{r} c_i\mathbf{v}_i$$

where $\mathbf{x}_p$ is any particular solution, r is the dimension of the eigenspace, c_i are arbitrary constants, and $\mathbf{v}_i$ are linearly independent eigenvectors of A with eigenvalue μ.

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2. Suppose A is a square symmetric matrix and λ is a simple eigenvalue of A with corresponding normalised eigenvector $\boldsymbol{v}$. We wish to solve

$$(A - (\lambda + \epsilon)I)\,\boldsymbol{x} = \boldsymbol{b},$$

where I is the identity matrix and the vector $\boldsymbol{b}$ is such that $\boldsymbol{v} \cdot \boldsymbol{b} \neq 0$. Show that

$$\boldsymbol{x} \sim \frac{c}{\epsilon}\boldsymbol{v} + \boldsymbol{x}_1 + \cdots\,,$$

for some constant c which you should determine.

Answer. If we were to try $\boldsymbol{x} \sim \boldsymbol{x}_0 + \epsilon \boldsymbol{x}_1 + \cdots$ we find

At ϵ^0: $\quad A\boldsymbol{x}_0 - \lambda \boldsymbol{x}_0 = \boldsymbol{b}$,

which has no solution by the Fredholm alternative since $(A - \lambda I)^T \boldsymbol{v} = \mathbf{0}$ and $\boldsymbol{v} \cdot \boldsymbol{b} \neq 0$. Trying instead

$$\boldsymbol{x} \sim \frac{1}{\epsilon} \boldsymbol{x}_0 + \boldsymbol{x}_1 + \cdots,$$

we find

At ϵ^0: $\quad A\boldsymbol{x}_0 - \lambda \boldsymbol{x}_0 = \mathbf{0}$,

so that $\boldsymbol{x}_0 = c\mathbf{v}$ for some constant c.

At ϵ^1: $\quad A\boldsymbol{x}_1 - \lambda \boldsymbol{x}_1 = c\boldsymbol{v} + \boldsymbol{b}$.

By the Fredholm alternative there is a solution for $\boldsymbol{x}_1$ if and only if the right-hand side is orthogonal to $\boldsymbol{v}$, i.e. $c|\boldsymbol{v}|^2 + \boldsymbol{v} \cdot \boldsymbol{b} = 0$. Thus $c = -\boldsymbol{v} \cdot \boldsymbol{b}$ (since $\boldsymbol{v}$ is normalised).

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3. Find the eigenvalues and eigenfunctions of the integral equation

$$y(x) = \lambda \int_0^1 (g(x)h(t) + g(t)h(x))\, y(t)\, \mathrm{d}t, \qquad x \in [0, 1],$$

where g and h are continuous functions satisfying

$$\int_1^1 g(x)^2\, \mathrm{d}x = \int_0^1 h(x)^2\, \mathrm{d}x = 1, \qquad \int_0^1 g(x)h(x)\, \mathrm{d}x = 0.$$

Answer. The equation is

$$y(x) = \lambda X_1 g(x) + \lambda X_2 h(x)$$

where

$$X_1 = \int_0^1 h(t)y(t)\, \mathrm{d}t, \qquad X_2 = \int_0^1 g(t)y(t)\, \mathrm{d}t.$$

Multiplying by $g(x)$ and $h(x)$ in turn and integrating over x gives

$$X_2 = \lambda X_1$$
$$X_1 = \lambda X_2,$$

where we have used the integrals of g^2, h^2 and gh given in the question. Thus

$$\begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

A nonzero solution requires

$$\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1 = 0,$$

i.e. $\lambda = \pm 1$. The corresponding eigenvectors satisfy $X_1 = X_2$ when $\lambda = 1$ and $X_1 = -X_2$ when $\lambda = -1$. Thus the eigenvalues and eigenvectors are

$$\begin{aligned} &\lambda = 1, && y(x) = c(g(x) + h(x)), \\ &\lambda = -1, && y(x) = d(g(x) - h(x)), \end{aligned}$$

where c and d are arbitrary constants.

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4. Solve the equation

$$y(x) = 1 - x^2 + \lambda \int_0^1 (1 - 5x^2t^2)y(t)\,\mathrm{d}t.$$

Answer. The equation is

$$y(x) = 1 - x^2 + \lambda X_1 - 5\lambda X_2 x^2 = 1 + \lambda X_1 - (1 + 5\lambda X_2)x^2,$$

where

$$X_1 = \int_0^1 y(t)\,\mathrm{d}t, \qquad X_2 = \int_0^1 t^2 y(t)\,\mathrm{d}t.$$

Multiplying by 1 and x^2 respectively and integrating gives

$$\begin{aligned} X_1 &= 1 + \lambda X_1 - \frac{1}{3}(1 + 5\lambda X_2), \\ X_2 &= \frac{1}{3}(1 + \lambda X_1) - \frac{1}{5}(1 + 5\lambda X_2). \end{aligned}$$

Rearranging

$$\begin{pmatrix} 1 - \lambda & \frac{5\lambda}{3} \\ -\frac{\lambda}{3} & 1 + \lambda \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{15} \end{pmatrix}.$$

The matrix is invertible if

$$\begin{vmatrix} 1 - \lambda & \frac{5\lambda}{3} \\ -\frac{\lambda}{3} & 1 + \lambda \end{vmatrix} = 1 - \frac{4\lambda^2}{9} \neq 0,$$

i.e. if $\lambda \neq \pm 3/2$. In this case

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \frac{1}{1 - 4\lambda^2/9} \begin{pmatrix} 1 + \lambda & -\frac{5\lambda}{3} \\ \frac{\lambda}{3} & 1 - \lambda \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ \frac{2}{15} \end{pmatrix} = \frac{1}{9 - 4\lambda^2} \begin{pmatrix} 6 + 4\lambda \\ \frac{2}{5}(3 + 2\lambda) \end{pmatrix} = \begin{pmatrix} \frac{2}{3 - 2\lambda} \\ \frac{2}{5(3 - 2\lambda)} \end{pmatrix}.$$

Thus

$$y(x) = 1 + \frac{2\lambda}{3 - 2\lambda} - \left(1 + \frac{2\lambda}{3 - 2\lambda}\right) x^2 = \frac{3(1 - x^2)}{3 - 2\lambda}.$$

If $\lambda = 3/2$ the adjoint eigenvector satisfies

$$\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{5}{2} & \frac{5}{2} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so that $Y_1 = -Y_2$. Since

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ \frac{2}{15} \end{pmatrix} = \frac{8}{15} \neq 0$$

there is no solution when $\lambda = 3/2$. If $\lambda = -3/2$ the adjoint eigenvector satisfies

$$\begin{pmatrix} \frac{5}{2} & \frac{1}{2} \\ -\frac{5}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so that $Y_2 = -5Y_1$. Since

$$\begin{pmatrix} 1 & -5 \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ \frac{2}{15} \end{pmatrix} = 0,$$

there is a solution when $\lambda = -3/2$. Since the general solution of

$$\begin{pmatrix} \frac{5}{2} & -\frac{5}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{15} \end{pmatrix}$$

is $X_1 = \frac{4}{15} + a$, $X_2 = a$, for a constant, the general solution when $\lambda = -3/2$ is

$$y(x) = \frac{3}{5} - x^2 + c(1 - 5x^2).$$

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