B4.4 Fourier Analysis HT21

Lecture 4: The Fourier inversion formula in $\mathscr{S}$ and L^1

1. The Fourier inversion formula in $\mathscr{S}(\mathbb{R}^n)$
2. The Fourier inversion formula in $\mathrm{L}^1(\mathbb{R}^n)$
3. The other convolution rule

The material corresponds to pp. 16–20 in the lecture notes and should be covered in Week 2.

The Fourier transform on $\mathscr{S}$

In Lecture 3 we saw that the Fourier transform defined for $f \in \mathrm{L}^1(\mathbb{R}^n)$ by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-\mathrm{i}\xi \cdot x} \, \mathrm{d}x$$

is an $\mathscr{S}$ continuous linear map $\mathcal{F}\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ and that the $\mathscr{S}$ continuity is quantified through the Fourier bounds: for all k, $l \in \mathbb{N}_0$ there exists a constant $c = c(n,k,l)$ so

$$\overline{S}_{k,l}(\widehat{\phi}) \leq c \overline{S}_{l+n+1,k}(\phi)$$

holds for all $\phi \in \mathscr{S}(\mathbb{R}^n)$.

Regularity versus decay at infinity

In Lecture 3 we also observed and formulated some instances of the important principle stating that *regularity of f implies decay at infinity of $\widehat{f}$ and that decay at infinity of f implies regularity of $\widehat{f}$*:

(a) Let $m \in \mathbb{N}_0$. If $f \in \mathrm{W}^{m,1}(\mathbb{R}^n)$, then

$$\frac{\widehat{f}(\xi)}{|\xi|^m} \to 0 \text{ as } |\xi| \to \infty,$$

(b) Let $m \in \mathbb{N}$ and $m \geq n+1$. If $\left(1+|x|^2\right)^{\frac{m}{2}} f(x) \in \mathrm{L}^\infty(\mathbb{R}^n)$, then $\widehat{f} \in \mathrm{C}^{m-n-1}(\mathbb{R}^n)$,

(b1) Let $m \in \mathbb{N}_0$. If $\left(1+|x|^2\right)^{\frac{m}{2}} f(x) \in \mathrm{L}^1(\mathbb{R}^n)$, then $\widehat{f} \in \mathrm{C}^m(\mathbb{R}^n)$.

The Fourier inversion formula in L^1, and its generalizations considered in later lectures, will among other things allow us to swap the roles of f and $\widehat{f}$ in (a), (b), (b1) above.

The Fourier inversion formula in $\mathscr{S}$

Theorem The Fourier transform $\mathcal{F}\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is bijective with inverse given by

$$\mathcal{F}^{-1}(\phi)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \phi(\xi) \mathrm{e}^{\mathrm{i}x\cdot\xi} \, \mathrm{d}\xi.$$

Consequently we have in compact symbolic form

$$\mathcal{F}^{-1} = (2\pi)^{-n} \widetilde{\mathcal{F}},$$

where we recall that the operations $\widetilde{(\cdot)}$ and $\mathcal{F}$ commute.

The proof is a beautiful calculation that starts with the product rule:

$$\int_{\mathbb{R}^n} \widehat{\phi} \psi \, \mathrm{d}x = \int_{\mathbb{R}^n} \phi \widehat{\psi} \, \mathrm{d}x$$

holds for all ϕ, $\psi \in \mathsf{L}^1(\mathbb{R}^n)$. The idea is to make a good choice for ψ that allows us to relate ϕ and $\widehat{\phi}$.

Proof of the Fourier inversion formula in $\mathscr{S}$

Lemma 1 If $G(x) = \mathrm{e}^{-\frac{|x|^2}{2}}$, $x \in \mathbb{R}^n$, then $\widehat{G} = (2\pi)^{\frac{n}{2}} G$.

Proof of Lemma 1. We start by reducing to the one-dimensional case. First note that

$$G(x) = \prod_{j=1}^{n} \mathrm{e}^{-\frac{x_j^2}{2}},$$

and so by use of Fubini's theorem

$$\widehat{G}(\xi) = \prod_{j=1}^{n} \mathcal{F}_{x_j \to \xi_j}\left(\mathrm{e}^{-\frac{x_j^2}{2}}\right)(\xi_j).$$

If therefore we can prove the lemma when $n = 1$, then the general case will follow.

Proof of the Fourier inversion formula in $\mathscr{S}$

Assume now that $n = 1$, so that

$$G(x) = \mathrm{e}^{-\frac{x^2}{2}},\ x \in \mathbb{R}.$$

Clearly $G(0) = 1$ and $G'(x) = -xG(x)$ for all $x \in \mathbb{R}$, that is, G is a solution to the initial value problem

$$\begin{cases} y' + xy = 0, & x \in \mathbb{R} \\ y(0) = 1. \end{cases}$$

It is easy to check, using the Leibniz rule and the constancy theorem, that this ODE admits a unique solution defined on $\mathbb{R}$, that then must be G. Now Fourier transform the identity $G' + xG = 0$ by use of the differentiation rules to get

$$\widehat{G}' + \xi \widehat{G} = 0 \text{ on } \mathbb{R}.$$

Next check that $\widehat{G}(0) = \int_{-\infty}^{\infty} G(x)\,\mathrm{d}x = \sqrt{2\pi}$ (a standard integral). Consequently $\widehat{G}/\sqrt{2\pi}$ solves the initial value problem, and so by uniqueness of solutions, $\widehat{G}/\sqrt{2\pi} = G$. This concludes the proof. □

Proof of the Fourier inversion formula in $\mathscr{S}$

The next result is an approximation that generalizes aspects of our result for the standard mollifier on $\mathbb{R}^n$.

Lemma 2 Let $K \in \mathrm{L}^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} K(x)\,\mathrm{d}x = 1$. Let K_t be the L^1 dilation of K by $t > 0$, so

$$K_t(x) = \frac{1}{t^n} K\left(\frac{x}{t}\right),\ x \in \mathbb{R}^n \ \text{and} \ \ t > 0.$$

Then we have

(i) when $\phi \in \mathscr{S}(\mathbb{R}^n)$, $K_t * \phi \to \phi$ in $\mathrm{L}^1(\mathbb{R}^n)$ and uniformly on $\mathbb{R}^n$ as $t \searrow 0$,

(ii) when $f \in \mathrm{L}^1(\mathbb{R}^n)$, $K_t * f \to f$ in $\mathrm{L}^1(\mathbb{R}^n)$ as $t \searrow 0$.

Remark The family $\left(K_t\right)_{t>0}$ is called an *approximate unit* because if $\phi \in \mathscr{D}(\mathbb{R}^n)$, then

$$\langle K_t, \phi \rangle = \int_{\mathbb{R}^n} K(x)\phi(tx)\,\mathrm{d}x \to \phi(0) \ \text{ as } \ t \searrow 0,$$

that is, $K_t \to \delta_0$ in $\mathscr{D}'(\mathbb{R}^n)$ as $t \searrow 0$.

Proof of Lemma 2. [The proof is not examinable]
We start with (i) and fix $\phi \in \mathscr{S}(\mathbb{R}^n)$. Let $\varepsilon > 0$. Now clearly

$$\big|(K_t * \phi)(x) - \phi(x)\big| \leq \int_{\mathbb{R}^n} |K(y)||\phi(x+ty) - \phi(x)| \, \mathrm{d}y.$$

We split the integral into two parts corresponding to $|y| \leq m$ and $|y| > m$, respectively, where we choose $m > 0$ so

$$\int_{|y|>m} |K(y)| \, \mathrm{d}y < \frac{\varepsilon}{2(1+2\|\phi\|_\infty)}.$$

Accordingly we estimate

$$\begin{aligned}\big|(K_t * \phi)(x) - \phi(x)\big| &\leq \int_{|y|\leq m} |K(y)||\phi(x+ty) - \phi(x)| \, \mathrm{d}y \\ &\quad + 2\|\phi\|_\infty \int_{|y|>m} |K(y)| \, \mathrm{d}y \\ &< \int_{|y|\leq m} |K(y)||\phi(x+ty) - \phi(x)| \, \mathrm{d}y + \frac{\varepsilon}{2}.\end{aligned}$$

Proof of Lemma 2 continued...

In order to estimate the integral over $|y| \leq m$ we use the fundamental theorem of calculus:

$$|\phi(x+ty) - \phi(x)| \leq \int_0^1 |\nabla\phi(x+sty)|t|y|\,\mathrm{d}s \leq \|\nabla\phi\|_\infty mt.$$

Consequently

$$\begin{aligned}\big|\big(K_t * \phi\big)(x) - \phi(x)\big| &< \int_{|y|\leq m} |K(y)|\|\nabla\phi\|_\infty mt\,\mathrm{d}y + \frac{\varepsilon}{2} \\ &\leq \|K\|_1\|\nabla\phi\|_\infty mt + \frac{\varepsilon}{2} < \varepsilon\end{aligned}$$

provided we take

$$t < \frac{\varepsilon}{2(1+\|K\|_1\|\nabla\phi\|_\infty m)}.$$

This establishes the uniform convergence.

Proof of Lemma 2 continued...

In order to see that the convergence also takes place in the L^1 sense we proceed similarly, but this time we take m so

$$\int_{|y|>m} |K(y)| \, \mathrm{d}y < \frac{\varepsilon}{2(1+2\|\phi\|_1)}.$$

Then we get, using Tonelli's theorem to swap the integration order:

$$\begin{aligned} \|K_t * \phi - \phi\|_1 &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |K(y)||\phi(x+ty) - \phi(x)| \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_{\mathbb{R}^n} |K(y)| \int_{\mathbb{R}^n} |\phi(x+ty) - \phi(x)| \, \mathrm{d}x \, \mathrm{d}y. \end{aligned}$$

Splitting the y-integral and estimating with the fundamental theorem of calculus as before results in

$$\|K_t * \phi - \phi\|_1 \leq \frac{\varepsilon}{2} + \|K\|_1 \|\nabla \phi\|_1 mt < \varepsilon$$

provided we take

$$t < \frac{\varepsilon}{2(1+\|K\|_1 \|\nabla \phi\|_1 m)}.$$

Proof of Lemma 2 continued...

Finally, for (ii) we pick $\phi \in \mathscr{S}(\mathbb{R}^n)$ with $\|f - \phi\|_1 < \frac{\varepsilon}{2}$. Then we estimate using the triangle inequality

$$\begin{aligned} \|K_t * f - f\|_1 &\leq \|K_t * (f - \phi)\|_1 + \|K_t * \phi - \phi\|_1 + \|\phi - f\|_1 \\ &\leq 2\|f - \phi\|_1 + \|K_t * \phi - \phi\|_1 \\ &< \varepsilon + \|K_t * \phi - \phi\|_1 \end{aligned}$$

and the conclusion follows from (i) □

We can now return to the main line of proof.

Proof of the Fourier inversion formula in $\mathscr{S}$

By Lemma 1 we have

$$\int_{\mathbb{R}^n} \widehat{G}\,\mathrm{d}\xi = \int_{\mathbb{R}^n} (2\pi)^{\frac{n}{2}} G\,\mathrm{d}\xi = (2\pi)^{\frac{n}{2}} \widehat{G}(0) = (2\pi)^n$$

and so with $K = (2\pi)^{-n}\widehat{G}$ we have $\int_{\mathbb{R}^n} K \mathrm{d}x = 1$, hence accoding to Lemma 2, $K_t * \phi \to \phi$ uniformly on $\mathbb{R}^n$ as $t \searrow 0$. We now calculate:

$$\begin{aligned}
\big(K_t * \phi\big)(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \phi(x-y)\big(\widehat{G}\big)_t(y)\,\mathrm{d}y \\
&\overset{\text{dilation rule}}{=} (2\pi)^{-n} \int_{\mathbb{R}^n} \phi(x-y)\mathcal{F}_{\xi\to y}\big(G(t\xi)\big)\,\mathrm{d}y \\
&\overset{\text{product rule}}{=} (2\pi)^{-n} \int_{\mathbb{R}^n} \mathcal{F}_{\xi\to y}\big(\phi(x-\xi)\big)\,G(ty)\,\mathrm{d}y \\
&\overset{\text{translation rule}}{=} (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{\phi}(-y)\mathrm{e}^{-\mathrm{i}y\cdot x} G(ty)\,\mathrm{d}y
\end{aligned}$$

Proof of the Fourier inversion formula in $\mathscr{S}$

Here we can use Lebesgue's dominated convergence theorem to find the limit of the right-hand side as $t \searrow 0$:

$$\begin{aligned}(2\pi)^{-n}\int_{\mathbb{R}^n}\widehat{\phi}(-y)\mathrm{e}^{-\mathrm{i}y\cdot x}G(ty)\,\mathrm{d}y \quad &\to \quad (2\pi)^{-n}\int_{\mathbb{R}^n}\widehat{\phi}(-y)\mathrm{e}^{-\mathrm{i}y\cdot x}\,\mathrm{d}y\\ &= \quad (2\pi)^{-n}\int_{\mathbb{R}^n}\widehat{\phi}(y)\mathrm{e}^{\mathrm{i}y\cdot x}\,\mathrm{d}y\end{aligned}$$

and the proof is finished. □

The Fourier inversion formula in L^1

Theorem Let $f \in \mathsf{L}^1(\mathbb{R}^n)$. Then

$$f(x) = \lim_{t \searrow 0} (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi) \mathrm{e}^{\mathrm{i}\xi \cdot x - \frac{t^2|\xi|^2}{2}} \, \mathrm{d}\xi \text{ in } \mathsf{L}^1(\mathbb{R}^n).$$

Consequently, when also $\widehat{f} \in \mathsf{L}^1(\mathbb{R}^n)$, then

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi) \mathrm{e}^{\mathrm{i}\xi \cdot x} \, \mathrm{d}\xi \tag{1}$$

holds almost everywhere.

Note that when $\widehat{f} \in \mathsf{L}^1(\mathbb{R}^n)$ the Riemann-Lebesgue lemma says that right-hand side of (1) belongs to $\mathsf{C}_0(\mathbb{R}^n)$. Therefore any $f \in \mathsf{L}^1(\mathbb{R}^n)$ whose Fourier transform $\widehat{f}$ is also in $\mathsf{L}^1(\mathbb{R}^n)$ has a representative in $\mathsf{C}_0(\mathbb{R}^n)$! It was therefore no accident that $\widehat{\mathbf{1}_{(-1,1)}} = 2\mathrm{sinc} \notin \mathsf{L}^1(\mathbb{R})$.

Proof of the Fourier inversion formula in L^1

Following the proof of the inversion formula in $\mathscr{S}$ we get by use of the product, translation and dilation rules that

$$\left(\left((2\pi)^{-n}\widehat{G}\right) * f\right)(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{f}(\xi)\mathrm{e}^{\mathrm{i}\xi\cdot x - \frac{t^2|\xi|^2}{2}}\,\mathrm{d}\xi$$

for each $x \in \mathbb{R}^n$ and $t > 0$. By Lemma 2(ii) the left-hand side converges to f in $\mathrm{L}^1(\mathbb{R}^n)$ as $t \searrow 0$ concluding the proof of the general case.
If additionally $\widehat{f} \in \mathrm{L}^1(\mathbb{R}^n)$, then we can pass to the limit under the integral sign on the right-hand side by use of Lebesgue's dominated convergence theorem and the desired identity follows. $\square$

The other convolution rule

Proposition If ϕ, $\psi \in \mathscr{S}(\mathbb{R}^n)$, then

$$\widehat{(\phi\psi)} = (2\pi)^{-n}\widehat{\phi} * \widehat{\psi}.$$

Proof. Because $\widehat{\phi}$, $\widehat{\psi} \in \mathscr{S}(\mathbb{R}^n) \subset \mathrm{L}^1(\mathbb{R}^n)$ we have by the usual convolution rule,

$$\mathcal{F}\big(\widehat{\phi} * \widehat{\psi}\big) = \widehat{\widehat{\phi}}\widehat{\widehat{\psi}}. \tag{2}$$

Here we have by the Fourier inversion formula in $\mathscr{S}$, $\mathcal{F}^{-1} = (2\pi)^{-n}\widetilde{\mathcal{F}}$, so $\mathcal{F}^2 = (2\pi)^n\widetilde{(\cdot)}$, and therefore

$$\widehat{\widehat{\phi}}\widehat{\widehat{\psi}} = (2\pi)^{2n}\widetilde{\phi}\widetilde{\psi} = (2\pi)^{2n}\widetilde{\phi\psi}.$$

Fourier transforming this identity we get

$$(2\pi)^{2n}\widetilde{\widehat{\phi\psi}} = (2\pi)^{2n}\widehat{\widetilde{\phi\psi}} = \mathcal{F}^2\big(\widehat{\phi} * \widehat{\psi}\big).$$

The other convolution rule

By virtue of (2) and the Fourier inversion formula in $\mathscr{S}$ we have $\widehat{\phi} * \widehat{\psi} \in \mathscr{S}(\mathbb{R}^n)$, hence by another use of the Fourier inversion formula in $\mathscr{S}$ we conclude. □

Corollary If ϕ, $\psi \in \mathscr{S}(\mathbb{R}^n)$, then also $\phi * \psi \in \mathscr{S}(\mathbb{R}^n)$.

This can also be proved directly–see the lecture notes.