# B4.4 Fourier Analysis HT22

Lecture 10: The Paley-Wiener theorem for compactly supported test functions

1. The Fourier transform of a compactly supported test function
2. The Fourier-Laplace transform
3. The Paley-Wiener theorem for test functions
4. An example

The material corresponds to pp. 38–41 in the lecture notes and should be covered in Week 5.

## What is the Fourier transform of $\phi \in \mathscr{D}(\mathbb{R}^n)$?

When $\phi \in \mathscr{D}(\mathbb{R}^n)$ its Fourier transform

$$\widehat{\phi}(\xi) = \int_{\mathbb{R}^n} \phi(x) \mathrm{e}^{-\mathrm{i}\xi \cdot x} \, \mathrm{d}x$$

is a Schwartz test function, but does it have other additional properties that reflect it has compact support? The Paley-Wiener theorem we discuss and prove in this lecture characterizes the Fourier transforms of functions from $\mathscr{D}(\mathbb{R}^n)$.

The starting point is the observation that the function

$$x \mapsto \phi(x) \mathrm{e}^{-\mathrm{i}\zeta \cdot x}$$

remains integrable over $x \in \mathbb{R}^n$ when $\zeta \in \mathbb{C}^n$. Note that this is clear exactly because $\phi$ has compact support.

## The Fourier-Laplace transform of $\phi \in \mathscr{D}(\mathbb{R}^n)$

**Definition** The *Fourier-Laplace transform* of $\phi \in \mathscr{D}(\mathbb{R}^n)$ is

$$\widehat{\phi}(\zeta) = \int_{\mathbb{R}^n} \phi(x) \mathrm{e}^{-\mathrm{i}\zeta \cdot x} \, \mathrm{d}x, \; \zeta \in \mathbb{C}^n.$$

Note that the Fourier-Laplace transform is denoted by the same symbol as the Fourier transform and that it will be clear from context in which capacity we consider $\widehat{\phi}$.

Write $\zeta \in \mathbb{C}^n$ as $\zeta = \xi + \mathrm{i}\eta$ with $\xi$, $\eta \in \mathbb{R}^n$ and consider the function

$$\mathbb{R}^{2n} \ni (\xi, \eta) \mapsto \widehat{\phi}(\xi + \mathrm{i}\eta)$$

A standard application of Lebesgue's dominated convergence theorem shows that $\widehat{\phi}$ is $\mathrm{C}^1$ and its partial derivatives can be computed by differentiation behind the integral sign.

## The Fourier-Laplace transform of $\phi \in \mathscr{D}(\mathbb{R}^n)$

Denote $\zeta_j = \xi_j + \mathrm{i}\eta_j \in \mathbb{C}$ corresponding to $j \in \{1, \ldots, n\}$. Then we can check the Cauchy-Riemann equation in the variables $\zeta_j$:

$$\frac{\partial}{\partial \overline{\zeta_j}} \widehat{\phi}(\zeta) = \int_{\mathbb{R}^n} \phi(x) \frac{\partial}{\partial \overline{\zeta_j}} \mathrm{e}^{-\mathrm{i}\zeta \cdot x} \, \mathrm{d}x = 0.$$

It follows that $\mathbb{C} \ni \zeta_j \mapsto \widehat{\phi}(\zeta)$ is holomorphic, where the remaining variables $\zeta_k$ for $k \neq j$ are kept fixed. The function $\widehat{\phi}(\zeta)$ is therefore separately holomorphic in the variables $\zeta = (\zeta_1, \ldots, \zeta_n)$ and we refer to such functions as simply holomorphic (or entire) functions on $\mathbb{C}^n$. We can quantify the fact that the support of $\phi$ is compact as follows. Take $R > 0$ so $\phi$ is supported in $\overline{B_R(0)}$. Then

$$\left| \widehat{\phi}(\zeta) \right| \leq \int_{B_R(0)} |\phi(x)| \mathrm{e}^{\eta \cdot x} \, \mathrm{d}x \leq \|\phi\|_1 \mathrm{e}^{R|\eta|}$$

holds for all $\zeta = \xi + \mathrm{i}\eta \in \mathbb{C}^n$. So the size of the ball centered at 0 containing the support is giving a bound on the growth of the Fourier-Laplace transform.

We can improve on this by a calculation similar to the proof for the differentiation rule: let $\alpha \in \mathbb{N}_0^n$ and calculate using integration by parts to get

$$\widehat{\partial^\alpha \phi}(\zeta) = \bigl(\mathrm{i}\zeta\bigr)^\alpha \int_{\mathbb{R}^n} \phi(x) \mathrm{e}^{-\mathrm{i}\zeta \cdot x} \, \mathrm{d}x = \bigl(\mathrm{i}\zeta\bigr)^\alpha \widehat{\phi}(\zeta)$$

and so

$$|\zeta^\alpha| \bigl| \widehat{\phi}(\zeta) \bigr| = \bigl| \widehat{\partial^\alpha \phi}(\zeta) \bigr| \leq \| \partial^\alpha \phi \|_1 \mathrm{e}^{R|\eta|}$$

holds for all $\zeta = \xi + \mathrm{i}\eta \in \mathbb{C}^n$. We combine this estimate with the following bound (a consequence of the bound (1) derived in lecture 9):

$$\bigl(1 + |\zeta|^2\bigr)^m \leq (2n)^{m-1} \sum_{|\alpha| \leq m} |\zeta^\alpha|^2$$

where $\zeta \in \mathbb{C}^n$, $m \in \mathbb{N}$. Here we write $|\zeta| = \sqrt{\zeta \cdot \overline{\zeta}} = \sqrt{|\xi|^2 + |\eta|^2}$ and

$$|\zeta^\alpha|^2 = \bigl| \prod_{j=1}^n \zeta_j^{\alpha_j} \bigr|^2 = \prod_{j=1}^n |\zeta_j|^{2\alpha_j}.$$

## The Fourier-Laplace transform of $\phi \in \mathscr{D}(\mathbb{R}^n)$

Combination of the bounds yields:

$$\begin{aligned}\left(1+|\zeta|^2\right)^m\left|\widehat{\phi}(\zeta)\right|^2 &\leq (2n)^{m-1}\sum_{|\alpha|\leq m}\left|\zeta^\alpha\widehat{\phi}(\zeta)\right|^2\\ &\leq (2n)^{m-1}\sum_{|\alpha|\leq m}\|\partial^\alpha\phi\|_1^2\mathrm{e}^{2R|\eta|}\end{aligned}$$

and so, taking square roots, we arrive at

$$\left(1+|\zeta|^2\right)^{\frac{m}{2}}\left|\widehat{\phi}(\zeta)\right| \leq c\mathrm{e}^{R|\eta|}$$

for all $\zeta = \xi + \mathrm{i}\eta \in \mathbb{C}^n$, where $c = c(n, m, \phi) \geq 0$ is a constant. By inspection it follows that we can take

$$c = (2n)^{\frac{m-1}{2}}\|\phi\|_{\mathrm{W}^{m,1}}.$$

## The Paley-Wiener theorem for test functions

**Theorem** (1) If $\phi \in \mathscr{D}(\mathbb{R}^n)$ has support in the closed ball $\overline{B_R(0)}$, then the Fourier transform $\widehat{\phi}$ admits an extension to $\mathbb{C}^n$ as an entire function (denoted $\widehat{\phi}(\zeta)$ and called the Fourier-Laplace transform of $\phi$) satisfying the *boundedness condition*: for each $m \in \mathbb{N}$ there exists a constant $c_m = c_m(n, \phi) \geq 0$ such that

$$\left|\widehat{\phi}(\zeta)\right| \leq c_m \left(1 + |\zeta|^2\right)^{-\frac{m}{2}} \mathrm{e}^{R|\eta|} \tag{1}$$

holds for all $\zeta = \xi + \mathrm{i}\eta \in \mathbb{C}^n$.
(2) If $\Phi \colon \mathbb{C}^n \to \mathbb{C}$ is an entire function satisfying the boundedness condition (1) for some $R \geq 0$, then there exists (a unique) $\phi \in \mathscr{D}(\mathbb{R}^n)$ supported in $\overline{B_R(0)}$ such that $\Phi = \widehat{\phi}$.

We have established the first part (1) and we turn to (2).

## The Paley-Wiener theorem for test functions–proof of (2)

We focus on the case $n = 1$. [The proof of (2) for $n > 1$ is not examinable]

Assume $\Phi\colon \mathbb{C} \to \mathbb{C}$ is an entire function satisfying the boundedness condition (1): there exists an $R \geq 0$ with the property that for each $m \in \mathbb{N}$ there exists a constant $c_m \geq 0$ such that

$$\bigl|\Phi(\zeta)\bigr| \leq c_m\bigl(1 + |\zeta|^2\bigr)^{-\frac{m}{2}}\mathrm{e}^{R|\eta|}$$

holds for all $\zeta = \xi + \mathrm{i}\eta \in \mathbb{C}$. Put $\varphi := \Phi|_{\mathbb{R}}$. Then $\varphi \in \mathrm{C}^{\infty}(\mathbb{R})$.
Our first aim is to prove that $\varphi \in \mathscr{S}(\mathbb{R})$ because then we can use the Fourier inversion formula in $\mathscr{S}$ to say that $\varphi$ is the Fourier transform of a Schwartz test function. Let $k$, $m \in \mathbb{N}_0$. We must show that

$$S_{k,m}(\varphi) = \sup_{\xi \in \mathbb{R}}\bigl|\xi^k \varphi^{(m)}(\xi)\bigr|$$

is finite.

## The Paley-Wiener theorem for test functions–proof of (2)

Since $\Phi$ is holomorphic we have that $\varphi^{(m)}(\xi) = \Phi^{(m)}(\xi)$ for $\xi \in \mathbb{R}$ and $m \in \mathbb{N}$, where the derivative on the right-hand side is the $m$-th complex derivative. We have a growth condition on $\Phi$ and use the Cauchy integral formula to get bounds on its derivatives:

$$\Phi^{(m)}(\zeta) = \frac{m!}{2\pi \mathrm{i}} \int_{|z-\zeta|=1} \frac{\Phi(z)}{(z-\zeta)^{m+1}} \, \mathrm{d}z.$$

Indeed in combination with the estimation lemma we find

$$\left|\Phi^{(m)}(\zeta)\right| \leq m! \max_{z \in \partial B_1(\zeta)} \left|\Phi(z)\right|.$$

These inequalities are sometimes called *Cauchy inequalities*.

## The Paley-Wiener theorem for test functions–proof of (2)

We now invoke the growth condition satisfied by $\Phi$ and corresponding to $k \in \mathbb{N}$ we find $c_k \geq 0$ such that

$$|\Phi(z)| \leq c_k \bigl(1+|z|^2\bigr)^{-\frac{k}{2}} \mathrm{e}^{R|y|}$$

holds for all $z = x + \mathrm{i}y \in \mathbb{C}$.

If $\zeta = \xi \in \mathbb{R}$ and $|z-\xi| = 1$, then $|y| \leq 1$ and $|z| \geq \bigl||\xi|-1\bigr|$, hence

$$\begin{aligned} \bigl|\varphi^{(m)}(\xi)\bigr| = \bigl|\Phi^{(m)}(\xi)\bigr| &\leq m! \max_{z \in \partial B_1(\xi)} \bigl|\Phi(z)\bigr| \\ &\leq m! \max_{z \in \partial B_1(\xi)} \left( c_k \bigl(1+|z|^2\bigr)^{-\frac{k}{2}} \mathrm{e}^{R|y|} \right) \\ &\leq c_k m! \bigl(1+(|\xi|-1)^2\bigr)^{-\frac{k}{2}} \mathrm{e}^{R} \end{aligned}$$

## The Paley-Wiener theorem for test functions–proof of (2)

Consequently,

$$\begin{aligned} \left|\xi^k \varphi^{(m)}(\xi)\right| &\leq c_k m! \left( \frac{\xi^2}{1 + (|\xi| - 1)^2} \right)^{\frac{k}{2}} \mathrm{e}^R \\ &\leq 2^k c_k m! \mathrm{e}^R \end{aligned}$$

holds for all $\xi \in \mathbb{R}$, and thus $S_{k,m}(\varphi) < \infty$. Because $k$, $m \in \mathbb{N}_0$ were arbitrary it follows that $\varphi \in \mathscr{S}(\mathbb{R})$.

We can now use the Fourier inversion formula in $\mathscr{S}$ and find $\phi \in \mathscr{S}(\mathbb{R})$ such that $\varphi = \widehat{\phi}$. Indeed, the function

$$\phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\xi) \mathrm{e}^{\mathrm{i}x\xi} \, \mathrm{d}\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\xi) \mathrm{e}^{\mathrm{i}x\xi} \, \mathrm{d}\xi \, , \; x \in \mathbb{R}$$

will do the job!

## The Paley-Wiener theorem for test functions–proof of (2)

A key trick that we will use now is that in the formula for $\phi(x)$ we can deform the integration contour using Cauchy's theorem.

We start by noting that for each fixed $x \in \mathbb{R}$ the function $\zeta \mapsto \Phi(\zeta)\mathrm{e}^{\mathrm{i}x\zeta}$ is entire so for $r > 0$ and $\eta \in \mathbb{R} \setminus \{0\}$ we have by Cauchy's theorem

$$\int_{\Gamma_r} \Phi(\zeta)\mathrm{e}^{\mathrm{i}x\zeta}\,\mathrm{d}\zeta = 0$$

where $\Gamma_r$ is the rectangular contour traversed anti-clockwise and with vertices $\pm r$, $\pm r + \mathrm{i}\eta$.

We seek to pass to the limit $r \to \infty$ and in order to estimate the integrals over the two vertical sides we invoke the boundedness property with $k = 2$. Hereby we find a constant $c = c_2 \geq 0$ such that

$$\left|\Phi(\zeta)\right| \leq \frac{c}{1+|\zeta|^2}\mathrm{e}^{R|\eta|} \tag{2}$$

holds for all $\zeta = \xi + \mathrm{i}\eta \in \mathbb{C}$.

## The Paley-Wiener theorem for test functions–proof of (2)

Using the bound (2) and the estimation lemma it is easy to show that the integrals over the two vertical sides vanish in the limit $r \to \infty$:

$$\left| \int_0^1 \Phi\big(\pm r + \mathrm{i}\eta t\big) \mathrm{e}^{\mathrm{i}x(\pm r + \mathrm{i}\eta t)} \mathrm{i}\eta \, \mathrm{d}t \right| \leq \int_0^1 \frac{c}{1 + |\pm r + \mathrm{i}\eta t|^2} \mathrm{e}^{R|\eta| - x\eta t} |\eta| \, \mathrm{d}t$$

$$\leq \frac{c|\eta| \mathrm{e}^{(R+|x|)|\eta|}}{1 + r^2} \to 0.$$

Consequently we get

$$\phi(x) = \frac{1}{2\pi} \int_{\mathbb{R}+\mathrm{i}\eta} \Phi(\zeta) \mathrm{e}^{\mathrm{i}x\zeta} \, \mathrm{d}\zeta \, , \quad x \in \mathbb{R}$$

for each $\eta \in \mathbb{R}$. We shall use this formula with the freedom in the choice of $\eta$ to show that $\phi$ is supported in $[-R, R]$.

## The Paley-Wiener theorem for test functions–proof of (2)

We estimate for $x \in \mathbb{R}$ and $\eta \in \mathbb{R}$:

$$\begin{aligned} \left|\phi(x)\right| &\leq \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\Phi\big(\xi+\mathrm{i}\eta\big)\mathrm{e}^{\mathrm{i}x(\xi+\mathrm{i}\eta)}\right|\,\mathrm{d}\xi \\ &\stackrel{(2)}{\leq} \frac{c}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi}{1+\xi^2+\eta^2}\mathrm{e}^{(R-|x|)|\eta|} \\ &\leq \frac{c}{2}\mathrm{e}^{(R-|x|)|\eta|} \end{aligned}$$

If we take $|x| > R$, then we get as $\eta \to \infty$ that $\phi(x) = 0$, that is, $\phi$ is supported in $[-R, R]$. □

## Example: The Fourier transform of a distribution supported in $\{0\}$

Assume $u \in \mathscr{E}'(\mathbb{R}^n)$ is supported in $\{0\}$. By a result from B4.3 we have that

$$u \in \mathrm{span}\{\partial^\alpha \delta_0 : \alpha \in \mathbb{N}_0^n\},$$

that is, for some $d \in \mathbb{N}_0$ and $c_\alpha \in \mathbb{C}$ we have

$$u = \sum_{|\alpha| \leq d} c_\alpha \partial^\alpha \delta_0.$$

Now $\widehat{\delta_0} = 1$ and so by the differentiation rule

$$\widehat{u} = \sum_{|\alpha| \leq d} c_\alpha \bigl(\mathrm{i}\xi\bigr)^\alpha =: p(\xi),$$

a polynomial. By the Fourier inversion formula we see that *any* polynomial is the Fourier transform of a distribution supported in $\{0\}$.

## Example: The Fourier transform of a distribution supported in $\{0\}$

When $u$ has Fourier transform $\widehat{u} = p$, then it clearly admits an extension as an entire function on $\mathbb{C}^n$. Furthermore, with $c = \max_{|\alpha| \leq d} |c_\alpha|$, we have

$$\left|\widehat{u}(\zeta)\right| \leq c\left(1 + |\zeta|^2\right)^{\frac{d}{2}} \tag{3}$$

for all $\zeta \in \mathbb{C}^n$.

In fact, the converse is also true: Assume $\Phi \colon \mathbb{C}^n \to \mathbb{C}$ is an entire function satisfying (3) (so is of polynomial growth). Then by Liouville's theorem $\Phi$ is a polynomial of degree at most $d$ and using the Fourier inversion formula in $\mathscr{S}'$ there exists $u \in \mathscr{E}'(\mathbb{R}^n)$ supported in $\{0\}$ and such that $\widehat{u} = \Phi$.

The Paley-Wiener theorem we discuss in the next lecture will address the situation when the distribution is supported in the ball $\overline{B_R(0)}$.