# B4.4 Fourier Analysis HT22

Lecture 13: The uncertainty principle

1. Examples
2. The local Sobolev space $\mathrm{H}^s_{\mathrm{loc}}(\Omega)$
3. Heisenberg's uncertainty principle
4. Comparison with a Sobolev inequality

The material corresponds to pp. 46–48 in the lecture notes and should be covered in Week 7.

**Example 1** Let $f \in \mathrm{L}^2(\mathbb{R}^n)$. Show that the PDE

$$-\Delta u + u = f$$

has precisely *one* solution in $\mathscr{S}'(\mathbb{R}^n)$ and that it satisfies $u \in \mathrm{H}^2(\mathbb{R}^n)$ and $\|u\|_{\mathrm{H}^2} = \|f\|_2$.

From lecture 12 we know that the differential operator $-\Delta + 1$ is elliptic and has the Bessel kernel of order 2, $g_2$, as fundamental solution. Furthermore, $\widehat{g_2} = \left(1 + |\xi|^2\right)^{-1}$ is a moderate $\mathrm{C}^\infty$ function and hence it follows, by the extended convolution rule, that $g_2 * f \in \mathscr{S}'(\mathbb{R}^n)$ is a solution.

By the characterization of $\mathrm{H}^2$ as the space of Bessel potentials from lecture 9 we have $g_2 * f \in \mathrm{H}^2(\mathbb{R}^n)$ and $\|g_s * f\|_{\mathrm{H}^2} = \|f\|_2$.

Could there be other solutions?

**Example 1 continued...**

Assume $u \in \mathscr{S}'(\mathbb{R}^n)$ is a solution. Then we get by Fourier transformation of the PDE and use of the differentiation rule

$$\big(|\xi|^2 + 1\big)\widehat{u} = \widehat{f} \text{ in } \mathscr{S}'(\mathbb{R}^n),$$

hence by the Fourier inversion formula and the extended convolution rule,

$$u = \mathcal{F}_{\xi \to x}\left(\frac{1}{1+|\xi|^2}\widehat{f}\right) = g_2 * f.$$

The one and only!

**Example 2** Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $g \in \mathrm{L}^2_{\mathrm{loc}}(\Omega)$. Show that all solutions to the PDE

$$-\Delta v + v = g \ \text{ in } \ \mathscr{D}'(\Omega)$$

are regular distributions.

Fix $\omega \Subset \Omega$ and put $\chi = \rho_\varepsilon * \mathbf{1}_{B_\varepsilon(\omega)}$ with $\varepsilon > 0$ so small that $\chi \in \mathscr{D}(\Omega)$. Note that $\mathbf{1}_\omega \leq \chi \leq 1$ and that $\chi g \in \mathrm{L}^2(\mathbb{R}^n)$ if we define $\chi g \equiv 0$ off $\Omega$. Define $u = g_2 * (\chi g)$. Then $u \in \mathrm{H}^2(\mathbb{R}^n)$ and

$$u - \Delta u = \chi g \ \text{ in } \ \mathscr{S}'(\mathbb{R}^n).$$

It follows that (writing $u = u|_\omega$ also for the restriction)

$$u - \Delta u = g \ \text{ in } \ \mathscr{D}'(\omega).$$

Consequently $w = v - u \in \mathscr{D}'(\omega)$ satisfies

$$w - \Delta w = 0 \ \text{ in } \ \mathscr{D}'(\omega).$$

## Example 2 continued...

But in lecture 12 we proved that $\text{sing.supp}(g_2) = \{0\}$ and so that the differential operator $-\Delta + 1$ is hypoelliptic. It follows that $w \in C^\infty(\omega)$, and hence that

$$v = v|_\omega = w + u \in L^2_{\text{loc}}(\omega).$$

Because $\omega \Subset \Omega$ was arbitrary we conclude that $v \in L^2_{\text{loc}}(\Omega)$, and so in particular that $v$ is a regular distribution on $\Omega$.

**Remark** Above we actually have more since $u \in H^2(\mathbb{R}^n)$ we can say that $v \in H^2_{\text{loc}}(\Omega)$, where this means that $\chi v \in H^2(\mathbb{R}^n)$ for each $\chi \in \mathscr{D}(\Omega)$ provided we define $\chi v \equiv 0$ off $\Omega$. It is not difficult to check that this is the same as saying that $\partial^\alpha v \in L^2_{\text{loc}}(\Omega)$ for each multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq 2$.

## The local Sobolev space $\mathrm{H}^s_{\mathrm{loc}}(\Omega)$

**Definition** Let $\Omega$ be an open subset of $\mathbb{R}^n$. For $s \in \mathbb{R}$ we define

$$\mathrm{H}^s_{\mathrm{loc}}(\Omega) = \Big\{ v \in \mathscr{D}'(\Omega) : \, \chi v \in \mathrm{H}^s(\mathbb{R}^n) \quad \forall \chi \in \mathscr{D}(\Omega) \Big\}$$

**Proposition** Let $\Omega$ be an open subset of $\mathbb{R}^n$. If $k \in \mathbb{N}_0$, then

$$\mathrm{H}^k_{\mathrm{loc}}(\Omega) = \big\{ f \in \mathscr{D}'(\Omega) : \, \partial^\alpha f \in \mathsf{L}^2_{\mathrm{loc}}(\Omega) \text{ for } |\alpha| \leq k \big\}.$$

We defined the latter in B4.3 where we called it $\mathsf{W}^{k,2}_{\mathrm{loc}}(\Omega)$.

*Proof.* It is, by use of the Leibniz rule, clear that $\mathsf{W}^{k,2}_{\mathrm{loc}}(\Omega) \subseteq \mathrm{H}^k_{\mathrm{loc}}(\Omega)$. For the converse we take for $\omega \Subset \Omega$ the test function $\chi = \rho_\varepsilon * \mathbf{1}_{B_\varepsilon(\omega)}$ with $\varepsilon > 0$ so small that $\chi \in \mathscr{D}(\Omega)$. We have seen in an earlier lecture that $\mathrm{H}^k(\mathbb{R}^n) = \mathsf{W}^{k,2}(\mathbb{R}^n)$ and so $\chi f \in \mathsf{W}^{k,2}(\mathbb{R}^n)$. Because $\chi f = f$ on $\omega$ we conclude that $f|_\omega \in \mathsf{W}^{k,2}(\omega)$, and hence that $f \in \mathsf{W}^{k,2}_{\mathrm{loc}}(\Omega)$ concluding the proof. □

## Uncertainty principles

Under this header belongs any result that says something about the limits to the simultaneous localization of a tempered distribution and its Fourier transform. We have already seen *qualitative* forms of this stating that it is not possible for a tempered distribution and its Fourier transform to both have compact support unless it is the zero distribution.

**Example** We have that $\widehat{\delta_0} = 1$. Note that $\delta_0$ is localized at $\{0\}$, whereas its Fourier transform 1 is not localized at all. A less extreme example is $f = \mathbf{1}_{(-1,1)}$ whose Fourier transform is $\widehat{f} = 2\mathrm{sinc}(\xi)$. Clearly $f$ is localized in $(-1,1)$ whereas it is less clear where we should consider $\widehat{f}$ to be localized–somewhere in a symmetric interval around 0. It approaches 0 when $|\xi|$ increases to infinity, but not fast enough for it to be integrable.

It is possible to quantify this and the most famous such result is Heisenberg's uncertainty principle that he formulated in the context of quantum mechanics.

## Heisenberg's uncertainty principle

**Theorem** Let $x_0,\ \xi_0 \in \mathbb{R}^n$ and $\phi \in \mathscr{S}(\mathbb{R}^n)$. Then *Heisenberg's inequality*

$$\frac{n}{2}(2\pi)^{\frac{n}{2}}\|\phi\|_2^2 \leq \big\|(x-x_0)\phi\big\|_2\big\|(\xi-\xi_0)\widehat{\phi}\big\|_2 \qquad (1)$$

holds. It is sharp and *equality* holds if and only if $\phi$ is a modulated Gaussian:

$$\phi(x) = c\mathrm{e}^{\mathrm{i}\xi_0\cdot x-\varepsilon(x-x_0)^2},$$

where $c \in \mathbb{C}$ and $\varepsilon > 0$.

*Note*: when we write $\|f\|_p$ for a vector valued function, say $f\colon \mathbb{R}^n \to \mathbb{C}^d$, then it is understood that we use the usual hermitian norm on $\mathbb{C}^d$, so that

$$\big|f(x)\big| := \sqrt{f(x)\cdot \overline{f(x)}} = \sqrt{|f_1(x)|^2 + \ldots + |f_d(x)|^2} \text{ and } \big\|f\big\|_p := \big\||f|\big\|_p$$

**Remark** In somewhat vague terms, if $\phi$ is concentrated on the ball $B_r(x_0)$, then $\widehat{\phi}$ cannot be concentrated on a ball much smaller than $B_{\frac{1}{r}}(\xi_0)$.

## Heisenberg's uncertainty principle–proof

First note that it suffices to prove the inequality when $x_0 = \xi_0 = 0$: If you assume the inequality holds true for $x_0 = \xi_0 = 0$, then apply it to the function

$$x \mapsto \mathrm{e}^{-\mathrm{i}\xi_0 \cdot x} \phi(x + x_0)$$

and use the translation rules for the Fourier transform, you get the inequality in the general case.

By the differentiation rule $\mathrm{i}\xi\widehat{\phi}(\xi) = \widehat{\nabla\phi}(\xi)$, so from Plancherel's theorem

$$\|x\phi\|_2 \|\xi\widehat{\phi}\|_2 = (2\pi)^{\frac{n}{2}} \|x\phi\|_2 \|\nabla\phi\|_2$$

We continue by use of the Cauchy-Schwarz inequality, whereby

$$(2\pi)^{\frac{n}{2}} \|x\phi\|_2 \|\nabla\phi\|_2 \geq (2\pi)^{\frac{n}{2}} \left| \int_{\mathbb{R}^n} x\phi(x) \cdot \overline{\nabla f(x)} \, \mathrm{d}x \right|$$

## Heisenberg's uncertainty principle–proof

Next we use the elementary inequality $\left|a+\mathrm{i}b\right| \geq \left|a\right|$:

$$
\begin{aligned}
(2\pi)^{\frac{n}{2}} \left| \int_{\mathbb{R}^n} x\phi(x) \cdot \overline{\nabla\phi(x)} \, \mathrm{d}x \right| &\geq (2\pi)^{\frac{n}{2}} \left| \int_{\mathbb{R}^n} x \cdot \mathrm{Re}\big(\phi(x)\overline{\nabla\phi(x)}\big) \, \mathrm{d}x \right| \\
&= (2\pi)^{\frac{n}{2}} \left| \int_{\mathbb{R}^n} x \cdot \nabla\big(\frac{1}{2}\left|\phi(x)\right|^2\big) \, \mathrm{d}x \right|
\end{aligned}
$$

Finally integrate by parts to get

$$
(2\pi)^{\frac{n}{2}} \left| \int_{\mathbb{R}^n} x \cdot \nabla\big(\frac{1}{2}\left|\phi(x)\right|^2\big) \, \mathrm{d}x \right| = \frac{n}{2}(2\pi)^{\frac{n}{2}} \|\phi\|_2^2.
$$

The proof of the inequality is completed. To see that it is sharp and that equality holds if and only if $\phi$ is a modulated Gaussian we inspect the cases of equality in the two inequalities we employed above: see lecture notes for details. □

## Comparison with a Sobolev inequality

We have established Heisenberg's inequality:

$$\frac{n}{2}\left(2\pi\right)^{\frac{n}{2}}\|\phi\|_2^2 \leq \|x\phi\|_2\|\xi\widehat{\phi}\|_2$$

valid for all $\phi \in \mathscr{S}(\mathbb{R}^n)$. We can rewrite it using the differentiation rules:

$$\widehat{-\mathrm{i}x\phi} = \nabla\widehat{\phi} \text{ and } \mathrm{i}\xi\widehat{\phi} = \widehat{\nabla\phi},$$

whereby we get by use of Plancherel's theorem

$$\begin{aligned}\frac{n}{2}\left(2\pi\right)^{\frac{n}{2}}\|\phi\|_2^2 &\leq \|x\phi\|_2\|\xi\widehat{\phi}\|_2 \\ &= \left(2\pi\right)^{-\frac{n}{2}}\|\nabla\widehat{\phi}\|_2\left(2\pi\right)^{\frac{n}{2}}\|\nabla\phi\|_2 \\ &= \|\nabla\widehat{\phi}\|_2\|\nabla\phi\|_2.\end{aligned}$$

## Comparison with a Sobolev inequality

**A Sobolev inequality:** If $n \geq 3$, then

$$S_n \|\phi\|_{\frac{2n}{n-2}} \leq \|\nabla \phi\|_2$$

holds for all $\phi \in \mathscr{S}(\mathbb{R}^n)$, where the constant

$$S_n = \frac{n(n-2)}{4} \omega_{n-1}^{\frac{2}{n}}$$

is sharp.

We omit the proof.