Fourier Analysis HT22

Problem Sheet 1

Problem 1. Consider the following four functions on $\mathbb{R}$:

$$f_1(x) = \mathrm{e}^{-x^2+2x}, \quad f_2(x) = \mathrm{e}^{-x}H(x), \quad f_3(x) = \mathrm{e}^{-|x|}, \quad f_4(x) = \tfrac{1}{x^2+1},$$

where H is Heaviside's function.
(i) Verify that these functions all belong to $\mathrm{L}^1(\mathbb{R})$. Which of them belong to $\mathscr{S}(\mathbb{R})$ and which to $\mathrm{L}^2(\mathbb{R})$?
(ii) Calculate the Fourier transforms of these functions. Deduce *Laplace's integral*

$$\int_0^\infty \frac{\cos(x\xi)}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2}\mathrm{e}^{-|\xi|} \quad (\xi \in \mathbb{R}).$$

(iii) For each of the Fourier transforms $\hat{f}_j$, determine whether it is a function in $\mathscr{S}(\mathbb{R})$, in $\mathrm{L}^1(\mathbb{R})$, or in $\mathrm{L}^2(\mathbb{R})$.

Problem 2. Let $f \in \mathrm{L}^1(\mathbb{R}^n)$ and denote by $(\mathbf{e}_j)_{j=1}^n$ the standard basis for $\mathbb{R}^n$. For $\xi \in \mathbb{R}^n$ we write $\xi = \xi_1\mathbf{e}_1 + \cdots + \xi_n\mathbf{e}_n$. Show that if $\xi_j \neq 0$, then

$$\hat{f}(\xi) = -\int_{\mathbb{R}^n} f\big(x + \frac{\pi}{\xi_j}\mathbf{e}_j\big)\mathrm{e}^{-ix\cdot\xi}\,\mathrm{d}x,$$

and conclude that

$$\big|\hat{f}(\xi)\big| \leq \frac{1}{2}\int_{\mathbb{R}^n}\big|f(x) - f\big(x + \frac{\pi}{\xi_j}\mathbf{e}_j\big)\big|\,\mathrm{d}x.$$

Using that $\mathscr{D}(\mathbb{R}^n)$ is dense in $\mathrm{L}^1(\mathbb{R}^n)$ deduce the *Riemann-Lebesgue Lemma*: $\hat{f}$ is continuous and $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

Problem 3. Let $t > 0$ and put $G_t(x) = \mathrm{e}^{-t|x|^2}$ for $x \in \mathbb{R}^n$. Use the Fourier transform to find a formula for the convolution $G_s * G_t$ for all $s, t > 0$.

Problem 4. Let $a > 0$ and $b, c \in \mathbb{R}$. Put $g(x) = \mathrm{e}^{-ax^2+bx+c}$, $x \in \mathbb{R}$. Calculate $\hat{g}$.

Problem 5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying $|f(x)| \leq \mathrm{e}^{-|x|}$ for almost all $x \in \mathbb{R}$. Prove that the Fourier transform $\hat{f}$ cannot have compact support unless $f(x) = 0$ for almost all $x \in \mathbb{R}$. (*Hint: Use a Differentiation Rule to see that $\hat{f}$ is C^∞ and consider a suitable Taylor expansion.*)

Problem 6. (Optional) Let $\phi \in \mathscr{S}(\mathbb{R}^n)$. Prove one of the following assertions and then derive the other:

(i) If $\phi(0) = 0$, then we may write $\phi = \sum_{j=1}^n x_j \phi_j$ with $\phi_j \in \mathscr{S}(\mathbb{R}^n)$.

(ii) If $\int_{\mathbb{R}^n} \phi \, \mathrm{d}x = 0$, then we may write $\phi = \sum_{j=1}^n \partial_j \phi_j$ with $\phi_j \in \mathscr{S}(\mathbb{R}^n)$.

Problem 7. (Optional) Let $f(x) = \mathrm{e}^{-|x|}$, $x \in \mathbb{R}^n$.
(a) Compute the Fourier transform $\hat{f}(\xi)$ when $n = 1$. Deduce for $\lambda \geq 0$ the identity

$$\mathrm{e}^{-\lambda} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1 + |\xi|^2} \mathrm{e}^{i\lambda\xi} \, \mathrm{d}\xi.$$

(b) Using $\frac{1}{1+|\xi|^2} = \int_0^\infty \mathrm{e}^{-(1+|\xi|^2)t} \, \mathrm{d}t$ and (a) show that for each $\lambda \geq 0$ the identity

$$\mathrm{e}^{-\lambda} = \int_0^\infty \frac{1}{\sqrt{\pi t}} \mathrm{e}^{-t-\frac{\lambda^2}{4t}} \, \mathrm{d}t$$

holds.
(c) Compute the Fourier transform $\hat{f}(\xi)$ in the general n-dimensional case, for instance by use of the formula from (b) with $\lambda = |x|$.