Algebraic Number Theory: Problem Sheet 0. 2022/23.

This sheet is for your own use (it is not intended to be handed in).

1. Let $q \in \mathbb{Q}$, let r be a non-zero square-free integer (that is: there is no prime p for which $p^2|r$), and let $q^2r \in \mathbb{Z}$. Show that $q \in \mathbb{Z}$.

2. Find the minimal polynomial of $\frac{1+i}{\sqrt{2}}$. What are the other roots of this polynomial?

3. Show that $\mathbb{Z}[i]$ is a Euclidean Domain. What are the units in this ring?

4. Factorise $6 + 12i$ into irreducibles in $\mathbb{Z}[i]$, and prove that your factors are indeed irreducible.

5. Let a be a non-zero element of $R := \mathbb{Z}[i]$, and define $A = \{ar : r \in R\}$. Show that R/A is finite. If a is prime show that R/A is an integral domain. Quote an appropriate theorem on finite integral domains, and deduce that A is a maximal ideal of R.

6. Let $S = \{m + n\sqrt{-6} : m, n \in \mathbb{Z}\}$, and let I be the ideal of S generated by 2 and $\sqrt{-6}$. Show that S/I has exactly two elements, and deduce that I is a maximal ideal of S.

Reading and Further Practice: Chapter 1 of Stewart and Tall, including the exercises.