Conformal Field Theory

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Conformal field theory is a vast subject. The aim of these notes is to give a pedagogical introduction to the subject in two and higher dimensions, where standard textbook material will be supplemented by more modern techniques. These notes are intended to be relatively self-contained, but some familiarity with the basics of quantum field theory is useful.

Chapters

Additional material

In preparing these notes I have used the following excellent books and reviews (specially the first two):

- Di Francesco, Mathieu and Senechal, "Conformal Field Theory", Springer.
- P. Ginsparg, "Applied Conformal Field Theory", "arXiv:hep-th/9108028".
- Jaume Gomis PIRSA 2011/2012 online lectures on CFT.
- John Cardy lecture notes on CFT.
- Slava Rychkov lecture notes on CFT, available at "arXiv:1601.05000".
- Xi Yin lecture notes on CFT.

1 Motivation

1.1 RG flows and scale invariance

Progress in science is possible thanks to the concept of *effective theories*. When describing a physical system we need to specify the relevant scale, and the description of such system will depend on this scale. Beyond such a scale our description will break down.

We will take the point of view that any quantum field theory is fundamentally defined with a ultra-violet (UV) cut-off scale Λ, corresponding to very small distances, or very large momenta. This scale is physically significant and the QFT description should not be trusted beyond it. The renormalization group flow gives us a systematic way to parametrise our ignorance of physics at energies above the scale Λ, in terms of coupling constants among the low energies degrees of freedom. In other words, it parametrises how quantities which are measurable at low energies depend on high energies degrees of freedom, with momenta of order Λ.

Let us see how this works in a simple example. Consider the following Lagrangian for a single scalar field in d Euclidean dimensions

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4, \tag{1.1}$$

and consider the functional integral

$$Z = \int [\mathcal{D}\phi]_\Lambda e^{-\int d^d x \mathcal{L}(\phi)} \tag{1.2}$$

where

$$[\mathcal{D}\phi]_\Lambda = \prod_{|k|<\Lambda} d\phi(k) \tag{1.3}$$

Namely, the independent degrees of freedom are given by the Fourier modes of the scalar field

$$\phi_{space}(x) = \int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x}\phi(k),$$

and according to (1.3) we integrate over the Fourier modes with momentum up to $|k| = \Lambda$ and set the rest to zero.

We want to study how the description in terms of low energy/momentum degrees of freedom depends on the cut-off Λ. In order to do this we integrate out the degrees of freedom with high energy. We introduce a real parameter $0 < b < 1$ and split the integration variables in two groups. The high momentum degrees of freedom are given by

$$\hat{\phi}(k) = \begin{cases} \phi(k) & \text{for } b\Lambda \leq |k| < \Lambda \\ \\ 0 & \text{for } |k| < b\Lambda \end{cases} \tag{1.4}$$

Then we define a new scalar field $\phi_{new}(k)$ identical to the old one in the region $|k| < b\Lambda$ and zero otherwise. Hence in the functional integral (1.2) we can replace

$$\phi(k) = \phi_{new}(k) + \hat{\phi}(k) \rightarrow \phi(k) + \hat{\phi}(k) \tag{1.5}$$

where we have renamed the new field simply by $\phi(k)$, which now includes the degrees of freedom with $|k|$ between zero and $b\Lambda$ only. Upon this replacement we obtain

$$\begin{aligned} Z &= \int [\mathcal{D}\phi]_{b\Lambda} \int \mathcal{D}\hat{\phi}\, e^{-\int d^dx \mathcal{L}(\phi+\hat{\phi})} \\ &= \int [\mathcal{D}\phi]_{b\Lambda} e^{-\int d^dx \mathcal{L}(\phi)} \int \mathcal{D}\hat{\phi} e^{-\int d^dx \left(\frac{1}{2}(\partial_\mu\hat{\phi})^2 + \frac{1}{2}m^2\hat{\phi}^2 + \lambda\frac{1}{6}\phi^3\hat{\phi} + \cdots\right)} \end{aligned} \tag{1.6}$$

Now we proceed as follows:

- Assume m, λ are small and expand the second exponential order by order in these parameters.
- Integrate over $\mathcal{D}\hat{\phi}$ term by term in this expansion.
- Recast everything as an exponential for the low energy degrees of freedom.

Although tedious, each step above is in principle a straightforward exercise in perturbation theory. The details can be found on chapter 12 of the QFT book by Peskin and Schroeder. One ends up with an expression of the form:

$$Z = \int [\mathcal{D}\phi]_{b\Lambda} e^{-\int d^dx \mathcal{L}_{eff}(\phi)} \tag{1.7}$$

where

$$\mathcal{L}_{eff} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{1}{2}\mu^2\phi^2 + \frac{\zeta}{4!}\phi^4 + \Delta C\,(\partial_\mu\phi)^4 + \Delta D\,\phi^6 + \cdots \tag{1.8}$$

In general, we can write

$$\mathcal{L}_{eff} = \frac{1}{2}(1+\Delta Z)(\partial_\mu\phi)^2 + \frac{1}{2}(m^2 + \Delta m^2)\phi^2 - \frac{1}{4!}(\lambda + \Delta\lambda)\phi^4 + \Delta C\,(\partial_\mu\phi)^4 + \cdots \tag{1.9}$$

where ΔZ, etc, depend on b in some complicated manner. The extra terms/corrections to the original Lagrangian take into account that now we are only considering degrees of freedom with $|k| < b\Lambda$: the interactions through higher energy degrees of freedom have been replaced by these extra terms. This has to be necesarilly the case. After all, the functional integrals (1.2) and (1.7) are actually equivalent!

Renormalization group flow

In order to compare these equivalent formulations purely at the level of the Lagrangian, consider the functional integral (1.7) and rescale distances and momenta:

$$x' = xb, \quad k' = k/b, \quad \to \quad |k'| < \Lambda \tag{1.10}$$

so that the integration regions of both functional integrals coincide. Furthermore

$$\int d^d x \mathcal{L}_{eff} = \int d^d x' b^{-d} \left(\frac{1}{2}(1+\Delta Z) b^2 (\partial'_\mu \phi)^2 + \frac{1}{2}(m^2 + \Delta m^2)\phi^2 + \cdots \right) \tag{1.11}$$

Next we rescale the scalar field as to have a canonical kinetic term (this will lead to a canonical propagator):

$$\phi' = \left(b^{2-d}(1+\Delta Z)\right)^{1/2} \phi, \tag{1.12}$$

to arrive to the following expression

$$\int d^d x \mathcal{L}_{eff} = \int d^d x \left(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m'^2 \phi^2 + \frac{\lambda'}{4!}\phi^4 + \cdots \right) \tag{1.13}$$

where we have suppressed the primes on the field. This has very much the form of the original Lagrangian, except the parameters have changed! More precisely

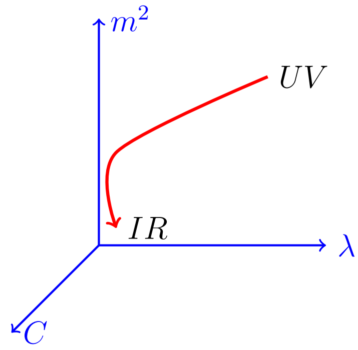
$$\begin{aligned} m'^2 &= (m^2 + \Delta m^2)(1+\Delta Z)^{-1} b^{-2} \\ \lambda' &= (\lambda + \Delta\lambda)(1+\Delta Z)^{-2} b^{d-4} \\ C' &= (C + \Delta C)(1+\Delta Z)^{-2} b^d \end{aligned} \tag{1.14}$$

and so on. In summary, we have managed to implement the process of integrating out high energy degrees of freedom as a transformation at the level of the Lagrangian! We can now assume $b = 1 - \delta$, very close to one, and consider redoing the above process many many times. This will generate a continuous transformation acting on the space of Lagrangians. This continuous transformation is called the *renormalization group flow.*

If one is interested in a process with typical energies $E^2 \ll \Lambda^2$ it is useful to follow the above procedure and flow

$$\mathcal{L}(\Lambda) \;\to\; \mathcal{L}_{eff}(E)$$

Note that the flow is always from the UV to the infra-red (IR), a description with smaller energy scale, see figure:



Scale invariant theories

Consider the free-field Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)^2 \tag{1.15}$$

This Lagrangian is actually a fixed point of the renormalization group transformations. Since the Lagrangian is quadratic, this is termed a Gaussian fixed point. Physically, a Lagrangian on a fixed point of the renormalization group flow describes a *scale invariant* theory.

Let us now turn into a more interesting example. Imagine a Lagrangian very close to $\mathcal{L}_0$, namely, m, λ, etc are very small parameters. In this case $\Delta m^2, \Delta\lambda, \Delta Z$, etc in (1.14) are of second order and can be discarded. Hence

$$\begin{aligned} m'^2 &= m^2 b^{-2} \\ \lambda' &= \lambda b^{d-4} \\ C' &= C b^d \end{aligned} \tag{1.16}$$

As we flow from the UV (high energies) to the IR (low energies) we apply the above transformation with $b = 1 - \delta$ and many many times. Three things can happen:

- A coupling gets multiplied by a negative power of b. In this case, it grows along the flow, taking us away from the fixed point (like an unstable perturbation). We say that the corresponding operator is *relevant* and that we have perturbed $\mathcal{L}_0$ by a relevant operator. An example is ϕ^2 above.
- A coupling gets multiplied by a positive power of b. In this case the perturbation decays along the flow. We call this an *irrelevant* perturbation. An example is $(\partial_\mu \phi)^4$ above.
- Some operators are *marginal*, such as ϕ^4 in $d = 4$. In this case their couplings do not scale with b [1].

[1] Usually this ceases to be true when higher orders in perturbation theory are taken into account. The existence of *exactly marginal* operators in a theory usually requires extra symmetries.

The power of b multiplying a given coupling, and whether an operator is relevant or irrelevant, can be understood by dimensional analysis. Consider an action of the form

$$S = \int d^d x \left(\frac{1}{2}(\partial_\mu \phi)^2 + \alpha_{m,n} \partial^m \phi^n \right)$$

where $\partial^m \phi^n$ is a generic insertion with m derivatives and n scalar fields. Recall that distances have mass dimension -1, while derivatives have mass dimension $+1$. In order for the action to be dimensionless the free scalar field in d dimensions should have mass dimension:

$$[\phi] = \frac{d-2}{2}$$

from this we can deduce

$$[\partial^m \phi^n] = n\frac{d-2}{2} + m \;\rightarrow\; [\alpha_{m,n}] = d - n\frac{d-2}{2} - m$$

Along the flow distances get rescaled according to (1.10) so that

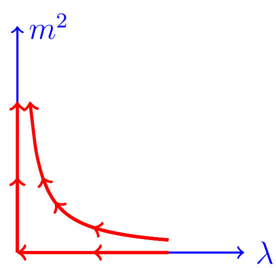
$$\alpha \rightarrow \alpha' = \alpha b^{-[\alpha]}$$

The three examples above correspond to $[\alpha_{0,2}] = 2$, $[\alpha_{0,4}] = 4 - d$ and $[\alpha_{4,4}] = -d$, consistent with the correct rescaling.

A very important consequence of this discussion is that, for $d > 2$, there is only a small number of relevant operators. Indeed, $\partial^m \phi^n$ will be relevant for

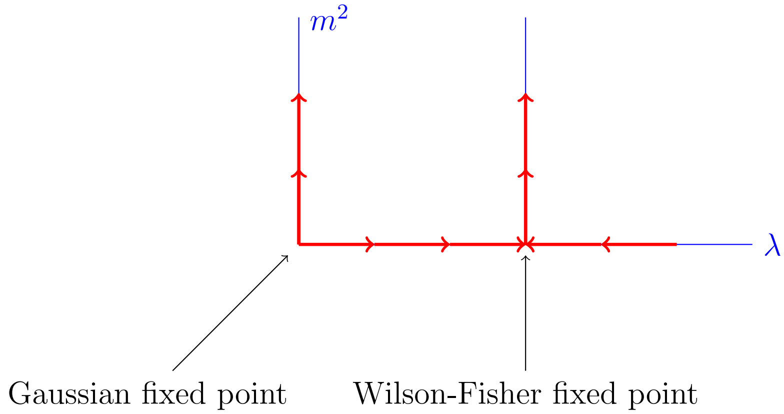
$$d - n\frac{d-2}{2} - m > 0$$

Going back to our example, let us consider the flow diagrams in the (m^2, λ) plane. Whether ϕ^4 is relevant or not depends on the number of dimensions d. For $d > 4$ ϕ^4 is irrelevant and we obtain the following flow diagram



As we move toward lower energies, from the UV to the IR, we see that the perturbation proportional to ϕ^4 dies off, while the relevant perturbation proportional to ϕ^2 take us away from the fixed point. From this example, we learn a very important and general lesson: no matter how complicated the Lagrangian is in the UV, just a few, relevant terms, will survive at low energies. This explains the success, and simplicity, of low energy effective theories.

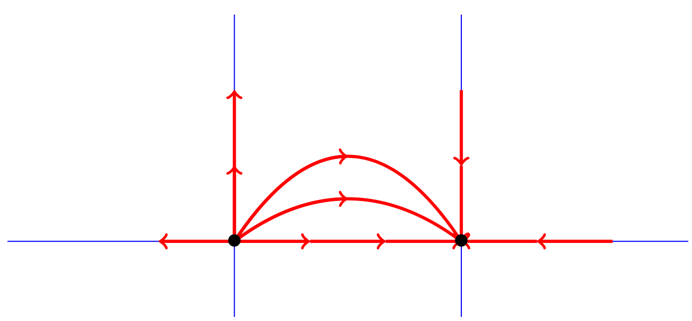
For $d = 4$ the flow diagram looks very much the same as for $d > 4$, although one would need to go to higher order in order to check this. For $d < 4$ things are more interesting. In this case the operator ϕ^4 is relevant, so that this perturbation also takes us away from the Gaussian fixed point. However, as λ gets larger, higher order corrections become more important. It turns out that the Lagrangian flows to another, non-gaussian, fixed point, with $\lambda \neq 0$, and the flow diagram looks like



This second, non-gaussian fixed point is called Wilson-Fisher fixed point. As already mentioned for the gaussian case, the theory will be scale invariant at that point. However, note that it will not be described by a weakly coupled Lagrangian (unless the number of dimensions is very close to four, in which case both fixed points are very close to each other).

General picture

In general, a Lagrangian may contain any combination of fields and their derivatives, consistent with the symmetries of the problem. Quantum field theories with fixed Lagrangian $\mathcal{L}(m, \lambda, \cdots)$ are points of a manifold. The renormalization group flow gives us flows (paths with orientation) in this manifold.



This flow has fixed points. We can think of the fixed points as fixing the ”topology” of the above manifold. It is furthermore believed (but it hasn’t been proven) that any QFT may be obtained by starting from a fixed point and adding an appropriate relevant deformation [2].

[2] Possible counterexamples to this would be a closed loop or a flow line that extends to infinity in both directions, not ending at any point.

Hence, the study of these fixed points is necessary to understand the structure of the space of QFTs.

A quantum field theory at an arbitrary point of the above manifold should be Poincare invariant. Furthermore, as we have discussed, at fixed points the theory is in addition scale invariant. We will assume that Poincare invariance together with scale invariance imply full conformal invariance (to be defined below). This is actually the case for every known relevant theory. This should be a very strong motivation to study conformal field theories!

1.2 Scale invariance in statistical mechanics

Before we proceed, let us discuss briefly how scale invariance arises in statistical models. The best known and simplest example is the Ising model, a quantum system consisting on spins $\sigma_i = \pm 1$ on a square lattice. The Hamiltonian of the system is given by

$$H = -\sum_{\langle i,j \rangle} \sigma_i \sigma_j \tag{1.17}$$

where the sum is only performed over nearest neighbours. For a total of N sites, there are 2^N configurations, which we denote as $\{\sigma\}$. The system has two ground states (states of minimal energy), namely all spins up or all spins down. The system can be considered at finite temperature T. The thermal partition function is given by

$$Z(\beta) = \sum_{\{\sigma\}} e^{-\beta H} \tag{1.18}$$

where $\beta = 1/T$. Thermal expectation values are then defined with respect to this partition function

$$\langle \mathcal{O} \rangle = \frac{\sum_{\{\sigma\}} \mathcal{O}(\{\sigma\}) e^{-\beta H}}{Z(\beta)} \tag{1.19}$$

A natural observable to study in any spin system is the spin correlator $\langle \sigma_i \sigma_j \rangle$, which for large lattices and due to translation invariance is expected to be a function of the distance $|i-j|$. The behaviour of the correlator depends on the temperature of the system. If the temperature is very small the system will choose one of its two grounds states and $\langle \sigma_i \sigma_j \rangle \to \langle \sigma \rangle^2$, which is a constant. If the temperature is very high (above certain critical value), we expect the spins to be randomly aligned and the correlator to fall quite fast as a function of the distance. Indeed:

$$\langle \sigma_i \sigma_j \rangle \sim e^{-|i-j|/\zeta(T)}, \quad T > T_c \tag{1.20}$$

where $\zeta(T)$ is a correlation length. As $T \to T_c$ from above, something very interesting happens. The correlation length diverges, with a specific behaviour

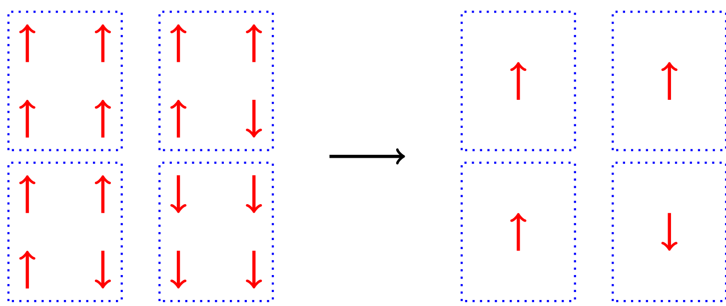
$$\zeta(T) \sim (T - T_c)^{-\nu}$$

This divergence leads to a change in the behaviour of the correlator and now:

$$\langle \sigma_i \sigma_j \rangle_{T_c} \sim \frac{1}{|i-j|^{d-2+\eta}} \tag{1.21}$$

ν and η are examples of what is called critical exponents. We will see that the behaviour (1.21) is characteristic of correlators in conformal field theories.

In order to understand how scale invariance arises near criticality, let us apply the ideas of the renormalization group flow to spin systems. Assume we are near criticality, so that the correlation length is very large. In this case nearby spins $|i-j| \ll \zeta$ should be well correlated and we should be able to replace a block of 2×2 spins by an effective spin (since inside each blocks the spins are approximately aligned):



this is called a block spin transformation. Note that in terms of the effective spin the correlation length changed as

$$\zeta \to \zeta' = \zeta/2 \tag{1.22}$$

Furthermore, we can describe the system in terms of a Hamiltonian for the effective spins, which can be in principle derived from the original Hamiltonian. Note the very close analogy between this discussion and the discussion above regarding QFT. At criticality the correlation length tends to infinity, and the system will present self-similarity under the block spin transformation (assuming the lattice has no boundaries) and the theory describing the effective degrees of freedom should be invariant under scale transformations.

2 Conformal transformations

A conformal theory is a theory which is invariant under conformal transformations, so we start by defining them. First we will discuss how they act on space-time coordinates and then their action on fields.

2.1 Conformal transformations

Denote by $g_{\mu\nu}$ the metric tensor in a $d-$dimensional space-time. A conformal transformation of coordinates is an invertible map $x \to x(x')$ which leaves the metric invariant up to (local) rescaling:

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x) \tag{2.1}$$

Two comments are in order:

- Isometries correspond to $\Lambda(x) = 1$, so that they are a subset of conformal transformations. When $g_{\mu\nu} = \eta_{\mu\nu}$, flat space, the group of isometries is simply the Poincare group, given by translations plus Lorentz rotations.
- Scale transformations/dilatations correspond to $\Lambda(x) = \textit{const}.$ so that a conformal transformation looks locally like a scale transformation plus a rotation plus a translation. In particular, they do not change the angle between intersecting curves, and from this derives their name: transformations which do not change the form.

Let us look at the condition (2.1) for an infinitesimal coordinate transformation:

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x) \tag{2.2}$$

where $\epsilon^\mu(x)$ is very small. Under a general coordinate transformation

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \tag{2.3}$$

so that

$$g_{\mu\nu} \to g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) + \mathcal{O}(\epsilon^2) \tag{2.4}$$

The requirement (2.1) with $\Lambda(x) \simeq 1 + f(x)$ is then equivalent to

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu} \tag{2.5}$$

Taking the trace of this relation fixes

$$f(x) = \frac{2}{d}\partial_\rho \epsilon^\rho \tag{2.6}$$

Although conformal transformations can be studied in general backgrounds, we will assume the original metric is $\eta_{\mu\nu} = diag(1, 1, \cdots, 1)$ (the treatment would be identical in Minkowski space). Taking partial derivative ∂_ρ of (2.5) plus permutations one can show

$$2\partial_\mu\partial_\nu\epsilon_\rho = \eta_{\mu\rho}\partial_\nu f + \partial_{\nu\rho}\partial_\mu f - \eta_{\mu\nu}\partial_\rho f \tag{2.7}$$

from where we deduce

$$2\partial^2\epsilon_\mu = (2-d)\partial_\mu f \tag{2.8}$$

But then, ∂_ν of this equation together with ∂^2 of (2.5) lead to

$$(2-d)\partial_\mu\partial_\nu f = \eta_{\mu\nu}\partial^2 f \;\rightarrow\; (d-1)\partial^2 f = 0 \tag{2.9}$$

It is clear from this equations that there is a crucial difference between $d = 2$ and $d > 2$. The case $d = 2$ will be studied later at length. For the moment, let us assume $d > 2$. In this case (2.9) imply $\partial_\mu\partial_\nu f = 0$, so that f is at most linear in the coordinates:

$$f = A + B_\mu x^\mu \tag{2.10}$$

where A, B_μ are some constants. At the level of ϵ this implies that ϵ is at most quadratic! So that

$$\epsilon^\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \tag{2.11}$$

with $c_{\mu\nu\rho} = c_{\mu\rho\nu}$. For now $a_\mu, b_{\mu\nu}$ and $c_{\mu\nu\rho}$ are arbitrary constants. Finally, we can plug this form of ϵ^μ into (2.5) and require the condition to be satisfied for all x. This leads to the following:

- a_μ is free of constraints, and corresponds to infinitesimal translations.
- $b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu}$, with $m_{\mu\nu} = -m_{\nu\mu}$. The trace part corresponds to an infinitesimal scale transformation. $m_{\mu\nu}$ corresponds to an infinitesimal Lorentz transformation/rotation.
- $c_{\mu\nu\rho} = \eta_{\mu\rho}b_\nu + \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu$, for a constant vector b_μ.

The last transformation acts on coordinates as

$$x^\mu \rightarrow x'^\mu = x^\mu + 2(x\cdot b)x^\mu - b^\mu x^2 \tag{2.12}$$

and is called a special conformal transformation. The finite versions of these transformations are given by

· Translations: $x'^{\mu} = x^{\mu} + a^{\mu}$

· Rigid rotations: $x'^{\mu} = M^{\mu}{}_{\nu} x^{\nu}$

· Dilatations: $x'^{\mu} = \alpha x^{\mu}$

· Special CT: $x'^{\mu} = \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b \cdot x + b^2 x^2}$

This is the complete set of conformal transformations. Note that the first two, translations and rigid rotations, form the Poincare transformations. Let us count the number of generators in d dimensions. There are d generators of translations, $\frac{d(d-1)}{2}$ generators of rotations, one dilatation generator and d generators of special conformal transformations. So that in total we have $\frac{(d+1)(d+2)}{2}$ generators.

Before proceeding, let us make one important remark. We can introduce the inversion I, a finite transformation with $I^2 = 1$, such that

$$I : x^{\mu} \to x'^{\mu} = \frac{x^{\mu}}{x^2} \tag{2.13}$$

It can be checked that performing an inversion, followed by a translation, followed by another inversion, is actually equivalent to a special conformal transformation.[3] We will return to this fact later.

2.2 The conformal group

Conformal transformations posses the structure of a group: the composition of conformal transformations is another conformal transformation and given a conformal transformation its inverse is also a conformal transformation. In the following we will construct a representation of the conformal generators as differential operators acting on functions/fields. This will allow to study the conformal group.

Given a coordinate transformation $x \to x' = x'(x)$ we can define its action on functions $\Phi(x)$ by defining the transformed functions

$$\Phi'(x') \equiv \Phi(x) \tag{2.14}$$

A infinitesimal coordinate transformation can always be written in the form

$$x'^{\mu} = x^{\mu} + \omega_a \frac{\delta x^{\mu}}{\delta \omega_a} \tag{2.15}$$

where ω_a is very small. We define the generator G_a of such transformation by

[3] One should be careful with the following subtlety. The inversion may not be a symmetry of our problem, even if full conformal transformations are. For instance, the inversion takes the origin to the point at infinity, which may not be part of the manifold on which the theory is defined. One can add the point at infinity, but has to be careful about subtleties.

$$\delta_\omega \Phi(x) = \Phi'(x) - \Phi(x) = -i\omega_a G_a \Phi(x) \tag{2.16}$$

From (2.14) we obtain

$$i\omega_a G_a \Phi(x) = \Phi(x) - \Phi'(x) = \Phi(x) - \Phi(x - \omega_a \frac{\delta x^\mu}{\delta \omega_a}) \tag{2.17}$$

So that

$$iG_a \Phi(x) = \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \Phi(x) \tag{2.18}$$

So far the discussion has been pretty general, let us know focus on conformal transformations. For instance, consider infinitesimal translations by ω^μ. In this case the a index in ω_a is just a space-time index and $\frac{\delta x^\mu}{\delta \omega^\nu} = \delta^\mu_\nu$. The generator of translations on functions, usually denoted by P_μ, then follows from (2.18) and takes the form

$$P_\mu = -i\partial_\mu \tag{2.19}$$

Infinitesimal Lorentz transformations take the form

$$x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu = x^\mu + \omega_{\rho\eta}\eta^{\rho\nu}x^\nu \tag{2.20}$$

where $\omega_{\rho\eta}$ is antisymmetric. Now the index $a = \rho\nu$ and

$$\frac{\delta x^\mu}{\delta \omega_{\rho\sigma}} = \frac{1}{2}\left(\eta^{\rho\mu}x^\nu - \eta^{\nu\mu}x^\rho\right) \tag{2.21}$$

from where the form of the Lorentz generators follows. Proceeding this way, we obtain a explicit representation for the generators of the conformal algebra as differential operators acting on functions:

- Translations: $P_\mu = -i\partial_\mu$
- Rigid rotations: $L_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right)$
- Dilatations: $D = -ix^\mu\partial_\mu$
- Special CT: $K_\mu = -i\left(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu\right)$

It is now straightforward (although quite tedious!) to compute the commutator among these operators. We obtain the conformal algebra

Conformal Algebra

$$[D, P_\mu] = iP_\mu, \quad [D, K_\mu] = -iK_\mu, \quad [K_\mu, P_\nu] = 2i\left(\eta_{\mu\nu}D - L_{\mu\nu}\right)$$
$$[L_{\mu\nu}, P_\rho] = -i\left(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\right), \quad [L_{\mu\nu}, K_\rho] = -i\left(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu\right)$$
$$[L_{\mu\nu}, L_{\rho\sigma}] = -i\left(L_{\mu\rho}\eta_{\nu\sigma} - L_{\mu\sigma}\eta_{\nu\rho} - L_{\nu\rho}\eta_{\mu\sigma} + L_{\nu\sigma}\eta_{\mu\rho}\right)$$
$$[D, L_{\mu\nu}] = 0, \quad [P_\mu, P_\nu] = 0, \quad [K_\mu, K_\nu] = 0, \quad [D, D] = 0$$

Defining

$$J_{\mu,\nu} = L_{\mu\nu}, \qquad J_{-1,\mu} = \frac{1}{2}\left(P_\mu - K_\mu\right) \tag{2.22}$$

$$J_{-1,0} = D, \qquad J_{0,\mu} = \frac{1}{2}\left(P_\mu + K_\mu\right) \tag{2.23}$$

and $J_{a,b} = -J_{b,a}$, for $a = -1, 0, 1, \cdots d$, one can check the generators $J_{a,b}$ satisfy the $SO(d+1,1)$ commutation relations (or $SO(d,2)$ in Minkowski space), which indeed has a total of $(d+1)(d+2)/2$ generators. Hence, the conformal Algebra in d dimensions is isomorphic to $SO(d+1,1)$ or $SO(d,2)$.

Before proceeding, note the following important point. $L_{\mu\nu}$ and P_μ form the Poincare group. On the other hand, $L_{\mu\nu}, P_\mu$ and D form also a subgroup. So mathematically full conformal symmetry is not necessarily implied by scale symmetry plus Poincare invariance.

2.3 Action on operators

In quantum field theory symmetries should be realised as operators acting on the Hilbert space (Schrodinger picture) or on local operators (Heisenberg picture). From now on we will follow this second view. Imagine we have a multicomponent operator $\phi_\alpha(x)$ (there is more than one component if the operator has spin). In the Heisenberg picture the x-dependence is given by

$$\phi_\alpha(x) = e^{-iPx}\phi_\alpha(0)e^{iPx} \tag{2.24}$$

taking a derivative with respect to x

$$\partial_\mu\phi_\alpha(x) = e^{-iPx}(-iP_\mu\phi_\alpha(0) + \phi_\alpha(0)iP_\mu)e^{iPx} = -i[P_\mu, \phi_\alpha(x)] \tag{2.25}$$

we obtain the action of the generator P_μ on the operator:

$$[P_\mu, \phi_\alpha(x)] = i\partial_\mu\phi_\alpha(x) \tag{2.26}$$

In order to find the action of the remaining generators of the conformal group we apply the following strategy. First we focus in the stability group, which is the subgroup that leaves the origin invariant. In the case of the conformal group this is spanned by Lorentz rotations $L_{\mu\nu}$, the dilatation operator D and special conformal transformations K_μ. Next we declare the following actions of these operators at the origin:

$$\begin{aligned} [D, \Phi_\alpha(0)] &= i\Delta\Phi_\alpha(0) & (2.27)\\ [L_{\mu\nu}, \Phi_\alpha(0)] &= i(S_{\mu\nu})_\alpha^\beta \Phi_\beta(0) & (2.28)\\ [K_\mu, \Phi_\alpha(0)] &= 0 & (2.29) \end{aligned}$$

where Δ is called the scaling dimension and $S_{\mu\nu}$ is a matrix that depends on the Lorentz spin of the field, and is zero for scalar fields. We take the transformations (2.27) - (2.29) as the definition of a *primary* operator of scaling dimension Δ. In particular, an operator is primary if it is annihilated by special conformal transformations at the origin. Actually, the Schur's lemma implies the right hand side of (2.29) has to be zero if the operator is in a irreducible representation of the Lorentz group, in other words, if it is not the derivative of another operator.

By using $\phi_\alpha(x) = e^{-iPx}\phi_\alpha(0)e^{iPx}$ together with the conformal algebra derived above we can then work out the action of the conformal generators on $\phi_\alpha(x)$. For instance:

$$\begin{aligned} [D, \phi_\alpha(x)] &= De^{-iPx}\phi_\alpha(0)e^{iPx} - e^{-iPx}\phi_\alpha(0)e^{iPx}D \\ &= e^{-iPx}\left(e^{iPx}De^{-iPx}\phi_\alpha(0) - \phi_\alpha(0)e^{iPx}De^{-iPx}\right)e^{iPx} \\ &= e^{-iPx}[\hat{D}, \phi_\alpha(0)]e^{iPx} \end{aligned}$$

where we have defined $\hat{D} = e^{iPx}De^{-iPx}$. $\hat{D}$ can be computed by using the conformal algebra

$$\begin{aligned} \hat{D} &= \left(1 + ixP - \frac{(xP)^2}{2} + \cdots\right) D \left(1 - ixP - \frac{(xP)^2}{2} + \cdots\right) \\ &= D + ix^\mu[P_\mu, D] - \frac{1}{2}x^\mu x^\nu[P_\mu, [P_\nu, D]] + \cdots \end{aligned}$$

but due to the structure of the conformal algebra $[P_\mu, [P_\nu, D]]$, as well as higher terms, vanishes. Actually, for any generator the above series truncates just after one or two terms! We obtain

$$\hat{D} = D + x^\mu P_\mu \tag{2.30}$$

the action $[\hat{D}, \phi_\alpha(0)]$ can then be obtained from the definition (2.27) and we obtain

$$[D, \phi_\alpha(x)] = i\left(\Delta + x^\mu\partial_\mu\right)\phi_\alpha(x) \tag{2.31}$$

The same trick can be applied to the rest of the generator. The complete list of the action of conformal generators on operators is

$$
\begin{aligned}
[P_\mu, \phi_\alpha(x)] &= i\partial_\mu \phi_\alpha(x) && (2.32)\\
[D, \phi_\alpha(x)] &= i\left(\Delta + x^\mu \partial_\mu\right)\phi_\alpha(x) && (2.33)\\
[L_{\mu\nu}, \phi_\alpha(x)] &= -i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right)\phi_\alpha(x) + i(S_{\mu\nu})_{\alpha\beta}\Phi_\beta(x) && (2.34)\\
[K_\mu, \phi_\alpha(x)] &= 2ix_\mu \Delta \phi_\alpha(x) + i\left(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu\right)\phi_\alpha(x) + 2ix^\rho (S_{\rho\mu})_{\alpha\beta}\Phi_\beta(x) && (2.35)
\end{aligned}
$$

From these transformations we can work out the finite transformations for primary fields[4]. For instance, for a primary scalar field (hence spinless) of scaling dimension Δ we find

$$\Phi(x) \to \Phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \Phi(x) \tag{2.36}$$

under conformal transformations $x \to x'$. $\left|\frac{\partial x'}{\partial x}\right|$ is the Jacobian of the transformation. Recalling

$$\left|\frac{\partial x'}{\partial x}\right| = \Lambda(x)^{-d/2} \tag{2.37}$$

we obtain

$$\Phi'(x') = \Lambda(x)^{\Delta/2}\Phi(x) \tag{2.38}$$

In the next section we will study the consequences of this fact. Let us close this section with the following interesting side remarks

In quantum mechanics one focuses on eigenstates of the Hamiltonian operator P_0. In CFT we rather focus on eigenstates of the dilation operator D, and their eigenvalues Δ. From the conformal algebra it follows that P will raise the dimension by one, while K will lower the dimension by one. The situation is very much as for the harmonic oscillator, with P and K playing the role of creating and annihilation operators. As for the harmonic oscillator, we cannot lower the dimension forever. At some point we reach a state $\mathcal{O}_\Delta$ such that further application of K annihilates it:

$$K \cdot \mathcal{O}_\Delta = 0$$

Such states are called primaries. Once we have identified the primaries, all other states in the CFT can be obtained by applying the "creation" operators P:

[4] We think of fields as fields operators, as usually done in second quantization.

$$P_\mu \cdots P_\nu \cdot \mathcal{O}_\Delta$$

These "descendant" operators are then simply derivatives of the primaries.

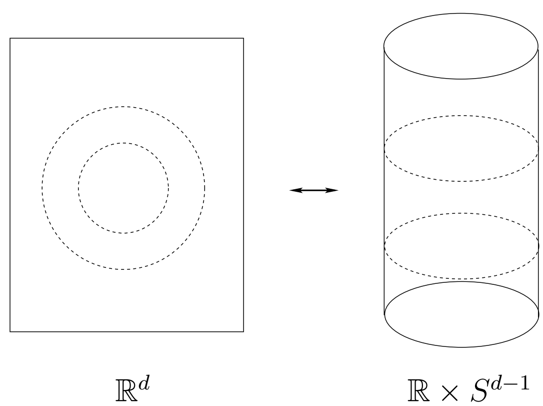
Actually, it turns out that the dilatation operator D can be though of as the Hamiltonian of another theory. Consider $\mathbb{R}^d$ in spherical coordinates

$$ds^2 = dr^2 + r^2 d\Omega_{d-1} = r^2 \left[\frac{dr^2}{r^2} + d\Omega_{d-1}\right]$$

Now let $t = \log r$ so that

$$\frac{dr^2}{r^2} + d\Omega_{d-1} = dt^2 + d\Omega_{d-1}$$

which is the metric on $\mathbb{R} \times S^{d-1}$. Now, if we are considering a CFT on $\mathbb{R}^d$, the theory should be invariant under rescaling of the metric! so that studying a CFT on $\mathbb{R}^d$ should be equivalent to study the theory on $\mathbb{R} \times S^{d-1}$:



A very interesting feature of this map is that it takes circles of constant radius in $\mathbb{R}^d$ to constant t slices on $\mathbb{R} \times S^{d-1}$. As a consequence, the dilatation operator on $\mathbb{R}^d$, which maps circles onto circles with different radius, corresponds to time translations on $\mathbb{R} \times S^{d-1}$, so it behaves as a Hamiltonian!

3 Consequences of conformal invariance

3.1 Classical symmetries in quantum field theory

Conformal transformations are continuous transformations of space-time and fields. Let us recall a few things about continuous transformations in QFT. Consider a general action

$$S = \int d^d x \mathcal{L}(\phi, \partial_\mu \phi) \tag{3.1}$$

depending on a collection of fields ϕ and their derivatives. A general transformation takes the form

$$\begin{aligned} x &\to x' = x'(x) && (3.2) \\ \phi(x) &\to \phi'(x') = \mathcal{F}(\phi(x)) && (3.3) \end{aligned}$$

Let us now consider the transformed action. We can show

$$\begin{aligned} S[\phi'] &= \int d^d x \mathcal{L}(\phi'(x), \partial_\mu \phi'(x)) \\ &= \int d^d x' \mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x')) \\ &= \int d^d x' \mathcal{L}(\mathcal{F}(\phi(x)), \partial'_\mu \mathcal{F}(\phi(x))) \\ &= \int d^d x \left| \frac{\partial x'}{\partial x} \right| \mathcal{L}(\mathcal{F}(\phi(x)), \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \mathcal{F}(\phi(x))) && (3.4) \end{aligned}$$

For a given transformation, we would like to understand under which conditions the classical action is invariant $S[\phi'] = S[\phi]$, so that it will lead to a theory with that symmetry at the classical level. Consider first translations:

$$x'^\mu = x^\mu + a^\mu \tag{3.5}$$

$$\phi'(x + a) = \phi(x) \tag{3.6}$$

so that $\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\mu_\nu$ and $\mathcal{F} = Id$, the identity. In this case the action is invariant unless it depends explicitly on x. Let us consider now Lorentz transformations

$$\begin{aligned} x'^\mu &= \Lambda^\mu{}_\nu x^\nu && (3.7) \\ \phi'(\Lambda \cdot x) &= L_\Lambda \phi(x) && (3.8) \end{aligned}$$

where $\eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}$ and the matrices L_Λ form a representation of the Lorentz group, depending on the spin of Φ [5]. For these transformations

$$\left|\frac{\partial x}{\partial x'}\right| = 1 \tag{3.9}$$

so that

$$S[\phi'] = \int d^d x \mathcal{L}(L_\Lambda \phi, \Lambda^{-1} \cdot \partial(L_\Lambda \phi)) \tag{3.10}$$

For instance, for scalar fields $L_\Lambda = 1$ and the action is invariant under Lorentz transformations provided the derivatives ∂_μ appear in a Lorentz invariant way (properly contracted).

Let us now analyse scale transformations

$$x' = \lambda x \quad \to \quad \left|\frac{\partial x}{\partial x'}\right| = \lambda^d \tag{3.11}$$

$$\phi'(\lambda x) = \lambda^{-\Delta}\phi(x) \tag{3.12}$$

where Δ is the scaling dimension of the field. Hence

$$S[\phi'] = \lambda^d \int d^d x \mathcal{L}(\lambda^{-\Delta}\phi, \lambda^{-1-\Delta}\partial_\mu \phi) \tag{3.13}$$

For instance, the action for the free scalar field

$$S[\varphi] = \int d^d \partial_\mu \varphi \partial^\mu \varphi \tag{3.14}$$

is invariant provided $\Delta = 1/2d - 1$ for the scalar field, in agreement with our expectations on dimensional grounds! Furthermore, in $d-$dimensions we can add a term:

$$S[\varphi]_{int} = \int d^d x \varphi^n \tag{3.15}$$

provided $\frac{\lambda^d}{\lambda^{n\Delta}} = 1$ so that $n = \frac{d}{\Delta} = \frac{2d}{d-2}$, for instance, we can add φ^4 in four dimensions and φ^3 in six.

[5] Infinitesimally $L_\Lambda = 1 - \frac{1}{2}\omega_{\rho\nu}S^{\rho\nu}$, where $S^{\rho\nu}$ are the matrices previously introduced.

3.2 Infinitesimal transformations and conserved quantities

Along the lines of the discussion above, let us consider infinitesimal transformations

$$
\begin{aligned}
x'^{\mu} &= x^{\mu} + \omega_a \frac{\delta x^{\mu}}{\delta \omega_a} \\
\phi'(x') &= \phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}
\end{aligned}
\tag{3.16}
$$

A very important result is the

Noether's theorem

To every continuous symmetry of the action one may associate a current which is classically conserved.

Let us assume (3.16) is a symmetry of the action. Namely, it leaves the action invariant, where ω_a is a rigid parameter, *i.e.* independent of x. Provided this is true one can promote $\omega_a \to \omega_a(x)$, an arbitrary function of x, and show that the infinitesimal variation of the action is given by

$$
\delta S = -\int d^d x j_a^{\mu} \partial_{\mu} \omega_a \tag{3.17}
$$

with the current associated to the transformation given by

$$
j_a^{\mu} = \left(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \partial_{\nu}\phi - \delta_{\nu}^{\nu} \mathcal{L} \right) \frac{\delta x^{\nu}}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \frac{\delta \mathcal{F}}{\delta \omega_a} \tag{3.18}
$$

Now, if the field configuration satisfies the classical equations of motion, then $\delta S = 0$, as against any variation. Then, integrating (3.17) by parts and since $\omega_a(x)$ is arbitrary we obtain

$$
\partial_{\mu} j_a^{\mu} = 0 \tag{3.19}
$$

which is the conservation equation! from this we can define a conserved charge

$$
Q_a = \int d^{d-1} x j_a^0 \tag{3.20}
$$

where the integral is over a constant time slice. This charge can be shown to be conserved provided the conservation equation holds and the fields decay at infinity sufficiently fast.

Let us make the following remark. Given a conserved current j_a^{μ} we could always redefine it

$$
j_a^{\mu} \to j_a^{\mu} + \partial_{\nu} B_a^{\mu\nu}
$$

with $B^{\mu\nu} = -B^{\nu\mu}$ and this new current would also be conserved. This extra term is called "improvement" term, since the new current may have nicer properties. We will use this freedom momentarily.

Energy momentum tensor

The conserved current associated with translational invariance is the energy momentum tensor. For an infinitesimal translation $x^\mu \to x^\mu + \epsilon^\mu$ we have $\frac{\delta x^\nu}{\delta \epsilon^\mu} = \delta^\nu_\mu$ and $\frac{\delta \mathcal{F}}{\delta \epsilon^\nu} = 0$ so that

$$T_c^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial^\nu \phi \tag{3.21}$$

where the subindex c stands for "canonical" energy momentum tensor. The conservation law is then

$$\partial_\mu T_c^{\mu\nu} = 0 \tag{3.22}$$

while the conserved charge is the momentum:

$$P^\nu = \int d^{d-1}x T_c^{0\nu} \tag{3.23}$$

For instance, the energy P^0 is given by

$$P^0 = \int d^{d-1}x \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\dot{\phi} - \mathcal{L} \right) \tag{3.24}$$

Which hopefully you all recognise!

It turns out that if the theory is Poincare invariant, namely, also invariant under Lorentz rotations, we can "improve" the stress tensor by using the freedom above:

$$T_c^{\mu\nu} \to T^{\mu\nu} \tag{3.25}$$

such that the new energy momentum tensor is symmetric $T^{\mu\nu} = T^{\nu\mu}$. This is called the Belinfante construction and is the stress tensor we will refer to.

The following feature will be important in what follows. We can consider a general infinitesimal coordinate transformation

$$x'^\mu = x^\mu + \epsilon^\mu(x) \tag{3.26}$$

as a translation with an $x-$dependent parameter $\epsilon^\mu(x)$. Hence from (3.17), with $j^\mu_a \to T^{\mu\nu}$, we obtain

$$\delta S = -\int d^d x T^{\mu\nu} \partial_\mu \epsilon_\nu = -\frac{1}{2}\int d^d x T^{\mu\nu} \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) \tag{3.27}$$

where we have assumed $T^{\mu\nu}$ is symmetric, which we can. Finally, let us mention that the stress tensor also generates the deformations under more general conformal transformations. Given ϵ^ν corresponding to an infinitesimal conformal transformation, the corresponding Noether current is $j_\mu = T_{\mu\nu}\epsilon^\nu$.

Stress tensor and conformal invariance

What are the consequences of scale and conformal symmetry for the stress tensor? As we have just discussed for a translation invariant QFT we can construct a conserved stress tensor $\partial_\mu T^{\mu\nu} = 0$. Furthermore, if the theory possesses Poincare invariance, then $T^{\mu\nu}$ can be chosen to be symmetric. Furthermore:

$$\delta S = -\frac{1}{2}\int d^dx T^{\mu\nu}\left(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu\right) \tag{3.28}$$

recall that for a conformal transformation $\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \eta_{\mu\nu}f(x)$ so that

$$\delta S = -\frac{1}{2}\int d^dx f(x) T^\mu_{\ \mu} \tag{3.29}$$

Scale invariance implies $\delta S = 0$ for $f(x) = \alpha$, so that for scale invariant theories $T^\mu_{\ \mu}$ is a total derivative:

$$T^\mu_{\ \mu} = \partial_\nu J^\nu \tag{3.30}$$

J^μ is usually called a virial current. Finally, if the stress tensor is traceless $T^\mu_{\ \mu} = 0$, then $\delta S = 0$ also for $f(x)$ corresponding to special conformal transformations, and the theory is conformally invariant. Note however, that this is a stronger condition and is in principle not implied by (3.30) [6]

In summary a conformal field theory must have a conserved stress tensor $\partial_\mu T^{\mu\nu} = 0$, which is symmetric $T^{\mu\nu} = T^{\nu\mu}$ and traceless $T^\mu_{\ \mu} = 0$.

Example: The free boson

Let us consider for example the Euclidean action for the free boson

$$S[\varphi] = \frac{1}{2}\int d^dx \partial_\mu\varphi\partial^\mu\varphi$$

The canonical stress tensor can be computed from (3.21) and we obtain

$$T^{\mu\nu} = -\frac{1}{2}\eta^{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi + \partial^\mu\varphi\partial^\nu\varphi$$

[6] Since $f(x)$ is not general it may seem that the requirement $T^\mu_{\ \mu} = 0$ is too strong. The precise condition is that it exists a rank two current such that $T^\mu_{\ \mu} = \partial_\nu\partial_\rho J^{\nu\rho}$. However, if such was the case, the stress tensor can always be improved as to make it traceless.

which is already symmetric. It is straightforward to check that $\partial_\mu T^{\mu\nu} = 0$, upon using the equations of motion $\partial_\mu \partial^\mu \varphi = 0$. On the other hand

$$T^\mu_{\ \mu} = -\frac{1}{2}(d-2)\partial_\rho\varphi\partial^\rho\varphi = -\frac{1}{2}(d-2)\partial_\rho(\varphi\partial^\rho\varphi)$$

In $d = 2$ this vanishes so that the theory is conformal invariant. In general dimensions it is possible to prove that there is an improvement term such that the new stress-tensor is traceless as well.

Often an alternative definition of the stress tensor is given, namely as measuring the response of the action under small variations of the metric. More precisely:

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}\bigg|_{g_{\mu\nu}=\eta_{\mu\nu}}$$

where $g = \det g_{\mu\nu}$. In order to compute this we need to couple our theory to gravity. In other words, we need to promote the metric to a dynamical field. For instance, for the free scalar field

$$S[\varphi, g_{\mu\nu}] = \frac{1}{2}\int d^dx\sqrt{g}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$$

Under a variation $g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$ we obtain

$$\delta\sqrt{g} = -\frac{1}{2\sqrt{g}}g_{\mu\nu}\delta g^{\mu\nu}$$

So that

$$T^{\mu\nu} = -\frac{1}{2}\eta^{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi + \partial^\mu\varphi\partial^\nu\varphi$$

which precisely agrees with what we obtained before. More generally the stress tensor obtained this way could differ from the canonical stress tensor by a improvement term.

3.3 Quantum conformal symmetry: implications for correlators

Once we have an action invariant under some symmetry, *i.e.* the theory has a classical symmetry, we focus on the quantum theory. At quantum level the natural object to study are correlation functions

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle = \frac{1}{Z}\int[d\phi]\phi(x_1)\cdots\phi(x_n)e^{-S[\phi]} \tag{3.31}$$

We will assume both the action and the functional integration measure $[d\phi]$ are invariant. The later is higher non-trivial for scale transformations, as we have seen that QFT comes endowed with a scale (the UV cut-off). If a symmetry of the classical theory is broken at the quantum level we say that there is an *anomaly*. Scale invariance is preserved at the quantum level at fixed points of the renormalization group flow, which requires a very precise tuning of the parameters of the theory. Provided a symmetry is present one can show:

$$\langle \phi(x_1') \cdots \phi(x_n') \rangle = \langle \phi'(x_1') \cdots \phi'(x_n') \rangle = \langle \mathcal{F}(\phi(x_1)) \cdots \mathcal{F}(\phi(x_n)) \rangle \tag{3.32}$$

Indeed

$$\begin{aligned} \int [d\phi] \phi(x_1') \cdots \phi(x_n') e^{-S[\phi]} &= \int [d\phi'] \phi'(x_1') \cdots \phi'(x_n') e^{-S[\phi']} \\ &= \int [d\phi] \mathcal{F}(\phi(x_1)) \cdots \mathcal{F}(\phi(x_n)) e^{-S[\phi]} \end{aligned} \tag{3.33}$$

In the first line we have renamed the dummy integration variable $\phi \to \phi'$. In the second line we have changed variables and assumed $[d\phi'] \to [d\phi]$ with Jacobian one, together with the invariance of the metric. Recall that $\phi'(x') = \mathcal{F}(\phi(x))$.

For instance, for translations (3.32) implies

$$\langle \phi(x_1 + \vec{a}) \cdots \phi(x_n + \vec{a}) \rangle = \langle \phi(x_1) \cdots \phi(x_n) \rangle \rangle \tag{3.34}$$

So that in a theory with translation invariance the correlator is only a function of relative positions, as well known! Next, consider Lorentz transformations on scalar operators, (3.32) implies

$$\langle \phi(\Lambda^\mu{}_\nu x_1^\nu) \cdots \phi(\Lambda^\mu{}_\nu x_n^\nu) \rangle = \langle \phi(x_1) \cdots \phi(x_n) \rangle \tag{3.35}$$

So that indices have to be contracted in a Lorentz invariant way.

Conformal invariance constraint on correlators

In addition to Poincare invariance, let us assume a theory has full conformal symmetry at the quantum level. As mentioned above operators are classified into primaries and descendants (which are derivatives of the primaries). For each primary there is a tower of descendants. Correlators involving descendants are fixed in terms of the correlators of their corresponding primaries, by taking derivatives. Hence we will focus on correlators of primaries. From now on we will focus on scalars, so that (3.32) implies

$$\langle \phi(x_1') \cdots \phi(x_n') \rangle = \left| \frac{\partial x'}{\partial x} \right|_{x=x_1}^{-\Delta_1/d} \cdots \left| \frac{\partial x'}{\partial x} \right|_{x=x_n}^{-\Delta_n/d} \langle \phi(x_1) \cdots \phi(x_n) \rangle \tag{3.36}$$

or equivalently

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \left| \frac{\partial x'}{\partial x} \right|_{x=x_1}^{\Delta_1/d} \cdots \left| \frac{\partial x'}{\partial x} \right|_{x=x_n}^{\Delta_n/d} \langle \phi(x_1') \cdots \phi(x_n') \rangle \tag{3.37}$$

this is a relation involving the same fields, but different points in space. Let us analyse the consequences of this relation case by case.

Two-point functions of scalars

Poincare invariance (translations plus rotations) implies

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = f(|x_1 - x_2|) \tag{3.38}$$

Scale transformations $x \to x' = \lambda x$ further imply

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle \phi_1(\lambda x_1)\phi_2(\lambda x_2)\rangle$$

so that $f(x) = \lambda^{\Delta_1+\Delta_2} f(\lambda x)$ and hence

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}$$

Finally, let us take into account special conformal transformations. These satisfy

$$\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{(1 - 2b\cdot x + b^2x^2)^d} \tag{3.39}$$

also, one can show the following very useful property

$$|x'_i - x'_j| = \frac{|x_i - x_j|}{\gamma_i^{1/2}\gamma_j^{1/2}}, \tag{3.40}$$

where we have introduced $\gamma_i = 1 - 2b\cdot x_i + b^2x_i^2$. Special conformal transformations then imply

$$\frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{1}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}}\frac{C_{12}}{|x'_1 - x'_2|^{\Delta_1+\Delta_2}} = \frac{(\gamma_1\gamma_2)^{\frac{\Delta_1+\Delta_2}{2}}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}}\frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{3.41}$$

Since γ_1 and γ_2 are independent this can only be satisfied provided $\Delta_1 = \Delta_2$. Hence, two primary fields are correlated only if they have the same scaling dimension:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \begin{cases} 0 & \Delta_1 \neq \Delta_2 \\ \frac{C_{12}}{|x_{12}|^{2\Delta}} & \Delta_1 = \Delta_2 \end{cases} \tag{3.42}$$

where we have introduced the notation $x_{12} = x_1 - x_2$ and we could have normalized the operators such that $C_{12} = 1$.

Three-point functions of scalars

In a Poincare invariant theory

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^a|x_{23}|^b|x_{13}|^c} \tag{3.43}$$

Scale invariance requires

$$a + b + c = \Delta_1 + \Delta_2 + \Delta_3 \tag{3.44}$$

However, this does not fix a, b, c completely, and in principle the answer could a sum over such allowed contributions. Finally, special conformal transformations require

$$\frac{C_{123}}{|x_{12}|^a |x_{23}|^b |x_{13}|^c} = \frac{(\gamma_1\gamma_2)^{a/2}(\gamma_2\gamma_3)^{b/2}(\gamma_3\gamma_1)^{c/2}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}} \frac{C_{123}}{|x_{12}|^a |x_{23}|^b |x_{13}|^c} \tag{3.45}$$

since all γ_i are independent, this constraint for all x_1, x_2, x_3 fixes a, b, c fully

$$\begin{aligned} a &= \Delta_1 + \Delta_2 - \Delta_3 \\ b &= \Delta_2 + \Delta_3 - \Delta_1 \\ c &= \Delta_3 + \Delta_1 - \Delta_2 \end{aligned} \tag{3.46}$$

So that

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{13}|^{\Delta_3+\Delta_1-\Delta_2}} \tag{3.47}$$

Note the following. Once we have normalized the operators such that $C_{ij} = 1$, we have no more freedom, so C_{ijk}, for three given primaries, has non-trivial physical content.

Fields with Lorentz indices

A natural question is what can conformal symmetry say about correlators of fields with Lorentz indices. Let us see some important examples. We start with the two point correlator of operators with one Lorentz index $J_\mu(x)$. Translational invariance allows us to write

$$\langle J_\mu(x) J_\nu(y)\rangle = \frac{\alpha_{\mu\nu}(x-y)}{|x-y|^{2\Delta}} \tag{3.48}$$

where $\alpha_{\mu\nu}(x-y)$, to be determined, should be invariant under scaling transformations. Furthermore, $\alpha_{\mu\nu}(x)$ should have the correct Lorentz structure: namely, it has two indices μ, ν, should be symmetric under $\mu \leftrightarrow \nu$ and should be built from the ingredients entering the problem. The most general form consistent with these constraints is

$$\alpha_{\mu\nu}(x) = \eta_{\mu\nu} + \alpha \frac{x_\mu x_\nu}{x^2} \tag{3.49}$$

where α is a constant to be fixed and we have chosen a normalization factor. Recall the general transformation rule for a vector field

$$J'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} J_\nu(x) \tag{3.50}$$

We can combine this with $\langle J_\mu(x')J_\nu(y')\rangle = \langle J'_\mu(x')J'_\nu(y')\rangle$ for the case of special conformal transformations. This fixes

$$\alpha_{\mu\nu}(x) = I_{\mu\nu}(x) = \eta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2} \tag{3.51}$$

I is called the inversion tensor. Consider again

$$\langle J_\mu(x)J_\nu(y)\rangle = \frac{I_{\mu\nu}(x-y)}{|x-y|^{2\Delta}} \tag{3.52}$$

And assume now that the vector J_μ is a conserved current $\partial^\mu J_\mu = 0$. This implies

$$\partial_x^\mu \frac{I_{\mu\nu}(x-y)}{|x-y|^{2\Delta}} = 0 \quad \rightarrow \quad \Delta = d-1 \tag{3.53}$$

so that a conserved current has dimension $d-1$. It turns out that two point functions of higher order tensors can also be constructed in terms of the inversion tensor. The most important example is the two point function of the stress tensor. Conformal symmetry implies

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(y)\rangle = \frac{c}{|x-y|^{2\Delta}}\left(\frac{1}{2}(I_{\mu\sigma}(x-y)I_{\nu\rho}(x-y) + I_{\mu\rho}(x-y)I_{\nu\sigma}(x-y)) - \frac{1}{d}\eta_{\mu\nu}\eta_{\sigma\rho}\right) \tag{3.54}$$

Note that the stress tensor has already been defined. As a result, the constant c is unambiguous. c is called the *central charge* and is the most important parameter characterising a conformal field theory. We will return to it repeatedly.

Finally, as for the case of conserved rank one currents, the conservation of the stress tensor, $\partial^\mu T_{\mu\nu} = 0$, fixes its dimension. Indeed

$$\partial_x^\mu \langle T_{\mu\nu}(x)T_{\rho\sigma}(y)\rangle \quad \rightarrow \quad \Delta = d. \tag{3.55}$$

so that the dimension of the stress tensor in d dimensions is always $\Delta = d$. If a conformal field theory has a Lagrangian, this also follows from the canonical definition (3.21), however, the result we just obtained is more general and does not rely on the theory having a Lagrangian.

In showing that a correlator has the correct properties under special conformal transformations, it is sometimes more convenient to show it has the correct properties under inversions:

$$x^\mu \rightarrow x'^\mu = \frac{x^\mu}{x^2}$$

hence, the correct properties un special conformal transformations follow. This is specially true for operators with Lorentz indices. In such proofs the following property

$$I_{\mu\alpha}(x)I_{\alpha\beta}(x-y)I_{\beta\nu}(y) = I_{\mu\nu}(x'-y')$$

can be very helpful. Here x', y' are the transforms of x, y under inversions.

four-point functions and higher

We have seen explicitly how conformal invariance fixes fully the spatial dependence of two and three point functions, which can hence be written in terms of scaling dimensions and constants C_{ijk}. Starting from four-point functions something new occurs. Under special conformal transformations:

$$x'^2_{ij} = \frac{x^2_{ij}}{\gamma_i \gamma_j} \tag{3.56}$$

where γ_i only depends on x_i. Hence, given four points x_1, x_2, x_3, x_3 we can construct the following cross-ratios[7]

$$u = \frac{x^2_{12} x^2_{34}}{x^2_{13} x^2_{24}}, \qquad v = \frac{x^2_{14} x^2_{23}}{x^2_{13} x^2_{24}} \tag{3.57}$$

which are invariant under special conformal transformations and actually the whole conformal group. Hence

$$\langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \phi_4(x_4) \rangle = \frac{F(u,v)}{\prod_{i<j} |x^2_{ij}|^{\delta_{ij}}} \tag{3.58}$$

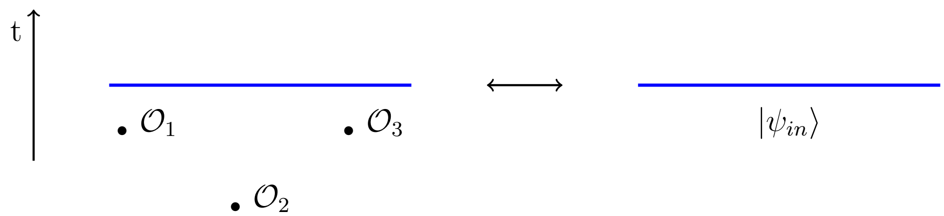
with $\sum_{j \neq i} \delta_{ij} = \Delta_i$, due to scale transformations, is the constraint from conformal symmetry. $F(u, v)$ could in principle be *any* function of the cross-ratios. In order to attack four point functions and higher, in the next section we will develop a new tool.

[7] Other cross ratios are not independent.

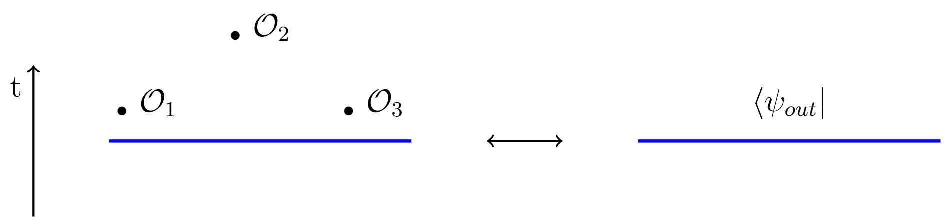
4 Radial quantization and the OPE

4.1 Radial quantization

In this section we will advocate a parallel view of correlators, from the point of view of Hilbert space and quantum mechanical evolution. In QFT in Minkowski space you can foliate the space-time by surfaces of equal time, or leafs, such that in each surface/time slice a Hilbert space is defined. ”in” states of a given Hilbert space can be created by inserting operators in the past of the corresponding surface:



equivalently an ”out” state is created by inserting operators in the future of the surface:



In this language the correlator among the operators that created the states is given by the overlap

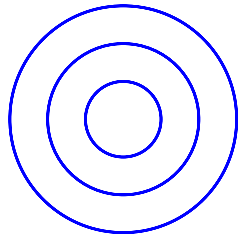
$$\langle\psi_{out}|\psi_{in}\rangle$$

If the states $|\psi_{in}\rangle$ and $\langle\psi_{out}|$ live on different leafs, we need to evolve them with the evolution operator $U = e^{-iP_0\Delta t}$ and then the correlator is

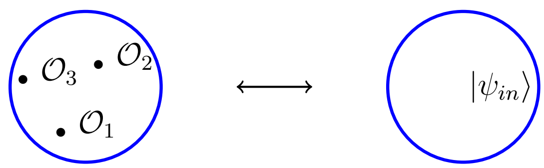
$$\langle\psi_{out}|U|\psi_{in}\rangle$$

Furthermore, in this language we characterize states living on these surfaces (belonging to the Hilbert space) by their momenta $P^\mu|k\rangle = k^\mu|k\rangle$, since the generators P^μ commute with P^0.

Consider now a CFT in Euclidean space. In CFT it is more convenient to foliate the space by sphere S^{d-1} with the origin at the center:



Note that in Euclidean signature, there is no difference between this choice and the equal time choice! Now in and out states are defined by inserting operators inside and outside the sphere, for instance



Now, the operator translating from a sphere to another sphere of different radius is the dilatation operators, hence $U = e^{iD\Delta\tau}$, with $\tau = \log r$. States are now classified according to their scale dimension and their $SO(d)$ spin ℓ (since only $L_{\mu\nu}$ commutes with D):

$$D|\Delta, \ell\rangle = i\Delta|\Delta, \ell\rangle \tag{4.1}$$

$$L_{\mu\nu}|\Delta, \ell\rangle_\alpha = i(S_{\mu\nu})_\alpha{}^\beta|\Delta, \ell\rangle_\beta \tag{4.2}$$

we will simply use $|\Delta\rangle$ to denote scalar states. This picture is called radial quantization.

4.2 State/operator correspondence

As seen above, we generate states living on the sphere $|\psi\rangle$ by inserting operators inside the sphere. In order to gain some intuition on how this works, let us see some examples.

Examples of states

- The vacuum state $|0\rangle$ corresponds to no insertion, has zero dilatation eigenvalue and is annihilated by all generators (P, K, L).
- Inserting a primary operator at the origin $\phi_\Delta(0)$ we get a state $|\Delta\rangle = \phi_\Delta(0)|0\rangle$ with eigenvalue Δ under dilatations. Indeed:

  $$D|\Delta\rangle = D\phi_\Delta(0)|0\rangle = [D, \phi_\Delta(0)]|0\rangle + \phi_\Delta(0)D|0\rangle = i\Delta\phi_\Delta(0)|0\rangle = i\Delta|\Delta\rangle$$

  which is primary, in the sense that it is annihilated by K, indeed:

  $$K|\Delta\rangle = K\phi_\Delta(0)|0\rangle = [K, \phi_\Delta(0)]|0\rangle = 0$$

- If we insert a primary operator not at the origin $\phi_\Delta(x)$, we get a state $|\psi\rangle = \phi_\Delta(x)|0\rangle$ which is *not* an eigenstate of the dilatation operator. Rather

  $$|\psi\rangle = \phi_\Delta(x)|0\rangle = e^{-ixP}\phi_\Delta(0)e^{ixP}|0\rangle = e^{-ixP}\phi_\Delta(0)|0\rangle = |\Delta\rangle - ix\cdot P|\Delta\rangle + \cdots$$

  where the eigenvalue of $P|\Delta\rangle$ under dilatations is given by $(\Delta+1)$, indeed:

  $$DP_\mu|\Delta\rangle = ([D,P_\mu] + P_\mu D)\,|\Delta\rangle = i(\Delta+1)P_\mu|\Delta\rangle$$

  Note that $P_\mu|\Delta\rangle$ is not a primary:

  $$K_\nu P_\mu|\Delta\rangle = ([K_\nu,P_\mu] + P_\mu K_\nu)\,|\Delta\rangle = [K_\nu,P_\mu]|\Delta\rangle \neq 0$$

  as expected, since it is just a descendant of $|\Delta\rangle$. Hence $|\psi\rangle$ is a linear combination of the primary $|\psi\rangle$ plus all its descendants.

Exactly as P_μ raises the dimension, it can be checked that K_μ lowers it by one unit. It turns out that there is a lower bound in the dimension of the states, so that at some point we will get to a state annihilated by K:

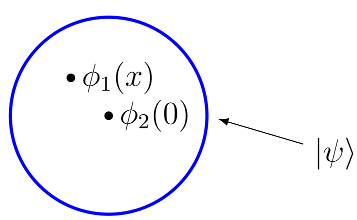
$$K_\mu|\psi\rangle = 0 \tag{4.3}$$

The state/operator correspondence says that such a state corresponds to the insertion at the origin of a local primary operator! of the form $\phi_\Delta(0)|0\rangle$. Furthermore, each eigenstate of the dilatation operator $|E_n\rangle$ is either a primary or a descendant (or linear combinations of those).

4.3 OPE in CFT

In QFT the operator product expansion (OPE) says that given two operators close to each other we can expand their product in terms of local operators at the middle point. In the following we will see how the OPE picture arises from radial quantization and why the structure of the OPE is more robust in CFT that in generic QFT.

Consider the insertion of two operators inside the sphere:



and consider the state they generate

$$|\psi\rangle = \phi_1(x)\phi_2(0)|0\rangle \tag{4.4}$$

We can expand $|\psi\rangle$ in a basis of eigenstates of the dilatation operator:

$$|\psi\rangle = \sum c_n |E_n\rangle \tag{4.5}$$

where c_n will in general depend on x, $c_n = c_n(x)$. However, due to the state/operator correspondence each $|E_n\rangle$ is a linear combination of primaries plus descendants! So that

$$\phi_1(x)\phi_2(0)|0\rangle = \sum_{primaries\ \phi} C_\Delta(x,\partial)\phi_\Delta(0)|0\rangle \tag{4.6}$$

which is exactly the statement of the OPE! Note that this works for any x inside the sphere, not necessarily close to the origin. In CFT the OPE is just the result of expanding states in a complete basis, and hence has an algebraic origin. Because of this it is also denoted *operator algebra*. Sometimes the OPE is written as

$$\phi_1(x)\phi_2(0) = \sum_{primaries\ \phi} C_\Delta(x,\partial)\phi_\Delta(0) \tag{4.7}$$

This is rigorously true only inside a correlator, and provided the other operators are sufficiently far. More precisely, there should exist a sphere including only these two operators.

What can we say about the function $C_\Delta(x,\partial)$? In order to answer this question let us focus on a single primary plus its tower of descendants:

$$\phi_1(x)\phi_2(0)|0\rangle = \frac{const}{|x|^k}\left(\phi_\Delta(0) + \cdots\right)|0\rangle \tag{4.8}$$

where $\phi_\Delta(0)$ is a primary operator and the dots stand for descendants of that operator (and we are disregarding contributions from other primaries). Let us act with the dilatation operator D on the l.h.s. of (4.8):

$$\begin{aligned} D\phi_1(x)\phi_2(0)|0\rangle &= i(\Delta_1 + x^\mu\partial_\mu)\phi_1(x)\phi_2(0)|0\rangle + i\Delta_2\phi_1(x)\phi_2(0)|0\rangle \\ &= i(\Delta_1+\Delta_2-k)\frac{const}{|x|^k}\left(\phi_\Delta(0)+\cdots\right)|0\rangle \end{aligned} \tag{4.9}$$

where in the second line we have used the functional dependence (4.8) itself. Now, acting on the r.h.s. of (4.8):

$$D\frac{const}{|x|^k}\left(\phi_\Delta(0)+\cdots\right)|0\rangle = i\Delta\frac{const}{|x|^k}\left(\phi_\Delta(0)+\cdots\right)|0\rangle \tag{4.10}$$

we are led to

$$k = \Delta_1 + \Delta_2 - \Delta \tag{4.11}$$

Fixing the small x behaviour in the above expansion! Let us go now one order higher:

$$\phi_1(x)\phi_2(0)|0\rangle = \frac{const}{|x|^{\Delta_1+\Delta_2-\Delta}}\left(\phi_\Delta(0) + \alpha x^\mu \partial_\mu \phi_\Delta(0) + \cdots\right)|0\rangle \tag{4.12}$$

Let us show how conformal symmetry can be used to fix α. The idea is to act on both sides of (4.12) with K_μ, recalling that for a scalar operator:

$$[K_\mu, \phi_\Delta(x)] = 2ix_\mu \Delta \phi_\Delta(x) + i\left(2x_\mu x^\nu \partial_\nu - x^2\partial_\mu\right)\phi_\Delta(x) \tag{4.13}$$

Acting on the l.h.s. of (4.12) we obtain

$$\begin{aligned} K_\mu \phi_1(x)\phi_2(0)|0\rangle &= \left(2ix_\mu\Delta_1 + i\left(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu\right)\right)\phi_1(x)\phi_2(0)|0\rangle && (4.14)\\ &= ix_\mu(\Delta_1 - \Delta_2 + \Delta)\frac{const}{|x|^{\Delta_1+\Delta_2-\Delta}}\left(\phi_\Delta(0) + \cdots\right)|0\rangle && (4.15)\end{aligned}$$

while acting on the r.h.s we obtain

$$\begin{aligned} K_\mu\left(\frac{const}{|x|^{\Delta_1+\Delta_2-\Delta}}\left(\phi_\Delta(0) - i\alpha x^\rho P_\rho \phi_\Delta(0) + \cdots\right)|0\rangle\right) &= && (4.16)\\ = \frac{const}{|x|^{\Delta_1+\Delta_2-\Delta}}\left(2i\alpha\Delta x_\mu \phi_\Delta(0) + \cdots\right)|0\rangle & && (4.17)\end{aligned}$$

so that as a consequence of conformal invariance

$$\alpha = \frac{\Delta_1 - \Delta_2 + \Delta}{2\Delta} \tag{4.18}$$

We could go to higher orders. Actually it turns out that conformal invariance fixes *fully* the functions $C_\Delta(x,\partial)$, up to an overall factor $C_{12\Delta}$. This would not be the case if the theory was scale but not conformal invariant. Note that $C_\Delta(x,\partial)$ depends only on the dimensions $\Delta_{1,2}$ and the dimension of the intermediate primary.

Practical method to compute $C_\Delta(x,\partial)$

Consider a three-point function of primaries and take the OPE of the first two operators:

$$\langle\phi_1(x)\phi_2(0)\phi_\Delta(z)\rangle = \sum_{primaries\ \Delta'} C_{12\Delta'}C_{\Delta'}(x,\partial)\,\langle\phi_{\Delta'}(y)\phi_\Delta(z)\rangle|_{y=0}$$

as we have seen conformal invariance implies that two primaries are correlated only if their dimensions agree, so that

$$\langle\phi_1(x)\phi_2(0)\phi_\Delta(z)\rangle = C_{12\Delta}C_\Delta(x,\partial)\,\langle\phi_\Delta(y)\phi_\Delta(z)\rangle|_{y=0}$$

where we have assumed there is a unique primary with dimension Δ. On the other hand, two and three-point functions are fixed by conformal invariance:

$$\langle \phi_\Delta(y)\phi_\Delta(z)\rangle = \frac{1}{|y-z|^{2\Delta}}$$

$$\langle \phi_1(x)\phi_2(0)\phi_\Delta(z)\rangle = \frac{C_{12\Delta}}{x^{\Delta_1+\Delta_2-\Delta} z^{\Delta_2+\Delta-\Delta_1}|x-z|^{\Delta_1+\Delta-\Delta_2}}$$

so that expanding around $x = 0$ we can fix all the coefficients in $C_\Delta(x, \partial)$. In particular, note that the coefficient appearing in the OPE expansion is exactly the same as the one appearing in the three-point function. Such coefficient is called OPE coefficient or structure constant.

4.4 Conformal blocks

Consider the four-point function of four scalar primaries, assumed to be identical for simplicity, of scaling dimension Δ_E. According to our discussion in section 3, conformal invariance implies

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{G(u,v)}{|x_{12}|^{2\Delta_E}|x_{34}|^{2\Delta_E}} \tag{4.19}$$

Now, we can use the OPE decompositions:

$$\phi(x_1)\phi(x_2) = \sum_\Delta c_\Delta C_\Delta(x_{12}, \partial_y)\ \phi_\Delta(y)|_{y=\frac{x_1+x_2}{2}} \tag{4.20}$$

$$\phi(x_3)\phi(x_4) = \sum_\Delta c_\Delta C_\Delta(x_{34}, \partial_z)\ \phi_\Delta(z)|_{z=\frac{x_3+x_4}{2}} \tag{4.21}$$

where the sum runs over the primary fields in the theory and c_Δ are the OPE coefficients [8]. Hence

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_\Delta c_\Delta^2 \left[C_\Delta(x_{12}, \partial_y)C_\Delta(x_{34}, \partial_z)\ \langle\phi_\Delta(y)\phi_\Delta(z)\rangle\right]|_{y=\frac{x_1+x_2}{2}, z=\frac{x_3+x_4}{2}} \tag{4.22}$$

since the functions $C_\Delta(x_{12}, \partial_y)$, as well as the two-point functions, are fixed by conformal symmetry, the whole object in bracket is! We define:

$$\left[C_\Delta(x_{12}, \partial_y)C_\Delta(x_{34}, \partial_z)\ \langle\phi_\Delta(y)\phi_\Delta(z)\rangle\right]|_{y=\frac{x_1+x_2}{2}, z=\frac{x_3+x_4}{2}} = \frac{G_{\Delta,\ell(u,v)}}{|x_{12}|^{2\Delta_E}|x_{34}|^{2\Delta_E}} \tag{4.23}$$

The functions $G_{\Delta,\ell}(u, v)$ are called conformal blocks: they depend on the dimension of the "external" operators, Δ_E and the dimension Δ and spin ℓ (whose index we have suppressed)

[8] Of course, only primary operators whose three-point functions with $\phi \times \phi$ is different from zero will appear.

of the intermediate primary. This allows to decompose the four-point correlator in *conformal partial waves.*

$$G(u,v) = \sum_{\Delta,\ell} c^2_{\Delta,\ell} G_{\Delta,\ell}(u,v). \tag{4.24}$$

This expansion gives the four-point function in terms of conformal blocks, which are fully fixed by conformal symmetry, together with the data appearing in lower order correlators. Actually, the same can be said about higher point correlators. Hence, we conclude the following remarkable fact: In a CFT, the dimensions of primaries, plus OPE coefficients, together with the structure of the OPE are enough to write any higher point correlation! (at least in principle).

5 Conformal invariance in two dimensions

In this section and the next, we will apply the general formalism above to two dimensions. As we will see, conformal symmetry becomes extremely powerful in two dimensions and this makes the subject beautiful, but also greatly developed.

5.1 Conformal algebra in two dimensions

For $d = 2$ and $g_{\mu\nu} = \eta_{\mu\nu}$, Euclidean flat metric, the condition

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} \partial_\rho \epsilon^\rho \tag{5.1}$$

becomes the Cauchy-Riemann equation

$$\partial_0 \epsilon_0 = \partial_1 \epsilon_1, \qquad \partial_0 \epsilon_1 = -\partial_1 \epsilon_0 \tag{5.2}$$

It is then natural to use complex coordinates

$$\begin{aligned} z = z^0 + iz^1, \quad \bar{z} = z^0 - iz^1 \\ \partial \equiv \partial_z = \frac{1}{2}(\partial_0 - i\partial_1) \\ \bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1) \end{aligned} \tag{5.3}$$

with metric $g_{zz} = g_{\bar{z}\bar{z}} = 0$ and $g_{z\bar{z}} = g_{\bar{z}z} = 1/2$ and inverse metric $g^{zz} = g^{\bar{z}\bar{z}} = 0$ and $g^{z\bar{z}} = g^{\bar{z}z} = 2$. Introducing $\epsilon = \epsilon^0 + i\epsilon^1$ and $\bar{\epsilon} = \epsilon^0 - i\epsilon^1$ the condition for a conformal infinitesimal transformation reduces to

$$\bar{\partial}\epsilon = 0 \;\rightarrow\; \epsilon = \epsilon(z) \tag{5.4}$$

$$\partial\bar{\epsilon} = 0 \;\rightarrow\; \bar{\epsilon} = \bar{\epsilon}(\bar{z}) \tag{5.5}$$

For arbitrary holomorphic and anti-holomorphic functions! Hence two dimensional conformal transformations coincide with analytic coordinate transformations

$$z \rightarrow z' = f(z), \quad \bar{z} \rightarrow \bar{z}' = \bar{f}(\bar{z}) \tag{5.6}$$

which generates an infinite dimensional algebra! to calculate the commutation relations of this conformal algebra we take as a basis:

$$z' = z + \epsilon(z), \qquad \epsilon_n(z) = \sum_n c_n z^{n+1} \tag{5.7}$$

so that a spinless and dimensionless field satisfying $\phi'(z',\bar{z}') = \phi(z,\bar{z})$ transforms as

$$\delta\phi = \sum_n \left(c_n \ell_n + \bar{c}_n \bar{\ell}_n\right) \phi(z,\bar{z}) \tag{5.8}$$

where we have introduced the generators

$$\ell_n = -z^{n+1}\partial_z, \quad \bar{\ell}_n = -\bar{z}^{n+1}\partial_z \bar{z} \tag{5.9}$$

It is easy to see these generators satisfy the algebras

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n}, \qquad [\bar{\ell}_m, \bar{\ell}_n] = (m-n)\bar{\ell}_{m+n} \tag{5.10}$$

together with $[\ell_m, \bar{\ell}_n] = 0$. Hence the conformal algebra is the direct sum of two infinite dimensional algebras, each called a Witt algebra. As we will see, quantum mechanically these algebras will get corrected by a tiny, but very important! contribution. Since two independent algebras naturally arise, it is usually convenient to regard z and $\bar{z}$ as independent coordinates. Then, we can impose the physical condition $\bar{z} = z^*$ whenever convenient.

The generators $\ell_n, \bar{\ell}_n$ generate a *local* conformal algebra. However, not all generators are *globally* well defined on the Riemmann sphere $S^2 = \mathbb{C} \cup \infty$. Indeed, holomorphic conformal transformations are generated by vector fields

$$v(z) = -\sum_n c_n \ell_n = \sum_n c_n z^{n+1} \partial_z \tag{5.11}$$

regularity of $v(z)$ as $z \to 0$ allows $c_n \neq 0$ only for $n \geq -1$. In order to investigate the behaviour at infinity we perform the transformation $z \to z' = -1/z$, under which:

$$v(z) \to \sum_n c_n \left(-\frac{1}{z'}\right)^{n-1} \partial_{z'}. \tag{5.12}$$

regularity at infinity allows $c_n \neq 0$ only for $n \leq 1$. Hence, only the conformal transformations generated by $\ell_0, \ell_{\pm 1}$ are globally defined. Exactly the same considerations apply to the anti-holomorphic part. In two dimensions the global conformal group is generated by conformal transformations that are well defined and invertible on the Riemann sphere, namely

$$\{\ell_1, \ell_0, \ell_1\} \cup \{\bar{\ell}_1, \bar{\ell}_0, \bar{\ell}_1\}$$

Following our discussion on section 2 we identify $\ell_{-1}, \bar{\ell}_{-1}$ as generators of translations, $\ell_0 + \bar{\ell}_0$ and $i(\ell_0 - \bar{\ell}_0)$ as generators of dilatations and rotations respectively and $\ell_1, \bar{\ell}_{-1}$ as generators of special conformal transformations. The finite form of these transformations is

$$z \to z' = \frac{az+b}{cz+d}, \qquad \bar{z} \to \bar{z}' = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}} \tag{5.13}$$

where $a,b,c,d \in \mathbb{C}$ and $ad - bc = 1$. This is the group $SL(2,\mathbb{C}) \simeq SO(3,1)$. Note that $SO(3,1)$ agrees with the conformal group we calculated in section 2 for $d = 2$, but in addition, there exist an infinite number of locally defined generators. This is a feature exclusively of two dimensions.

In 2d CFT it is convenient to work with a basis of eigenstates of the operators ℓ_0 and $\bar{\ell}_0$. We denote their eigenvalues by h and $\bar{h}$. These are known as the conformal weights of the state and we should think of them as independent real parameters. Since $\ell_0 + \bar{\ell}_0$ and $i(\ell_0 - \bar{\ell}_0)$ are identified with the generators of dilatations and rotations, the scaling dimension Δ and the spin s of the state are given by

$$\Delta = h + \bar{h}, \qquad s = h - \bar{h}. \tag{5.14}$$

Before proceeding, let us analyse the properties of the stress tensor in a 2d CFT. In complex coordinates the stress tensor has components $T_{zz}, T_{\bar{z}\bar{z}}$ and $T_{z\bar{z}} = T_{\bar{z}z}$. The condition of zero trace translates into

$$g^{\mu\nu} T_{\mu\nu} = 2(T_{z\bar{z}} + T_{\bar{z}z}) = 0 \quad \to \quad T_{z\bar{z}} = T_{\bar{z}z} = 0. \tag{5.15}$$

The conservation law $g^{\alpha\mu}\partial_\alpha T_{\mu\nu} = 0$ then implies

$$\partial_{\bar{z}} T_{zz} = 0, \qquad \partial_z T_{\bar{z}\bar{z}} = 0 \tag{5.16}$$

so that the non-vanishing components of the stress tensor have holomorphic and anti-holomorphic dependence

$$T(z) \equiv T_{zz}(z), \qquad \bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z}) \tag{5.17}$$

5.2 $2d$ Correlation functions

Primaries and quasi-primaries

In analogy with (2.36) we define a *primary* field of conformal weight $(h, \bar{h})$ as one which under conformal transformations $z \to z' = f(z)$ transforms as

$$\phi'(z', \bar{z}') = \left(\frac{\partial f}{\partial z}\right)^{-h} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}). \tag{5.18}$$

A field with the transformation property (5.18) under global conformal transformations is denoted *quasi-primary*, also denoted as $SL(2, \mathbb{C})$ primary. A primary field is also quasi-primary, but the converse is of course not true. This distinction arises only in two dimensions. A field which is not a primary is called a *secondary*. Under an infinitesimal variation $z \to z' = z + \epsilon(z)$ the variation of a (quasi-)primary field is

$$\delta_{\epsilon,\bar{\epsilon}}\phi(z, \bar{z}) = \phi'(z', \bar{z}') - \phi(z, \bar{z}) = -\left(h(\partial\epsilon) + \epsilon\partial\right)\phi(z, \bar{z}) - \left(\bar{h}(\bar{\partial}\bar{\epsilon}) + \bar{\epsilon}\bar{\partial}\right)\phi(z, \bar{z}) \tag{5.19}$$

Correlators

Consider the correlator of n primary fields of conformal weights $(h_i, \bar{h}_i)$. Relation (3.32) expressed in terms of holomorphic and anti-holomorphic coordinates becomes

$$\langle \phi_1(z'_1, \bar{z}'_1) \cdots \phi_n(z'_n, \bar{z}'_n) \rangle = \prod_{i=1}^{n} \left(\frac{\partial f}{\partial z} \right)^{-h_i}_{z=z_i} \left(\frac{\partial \bar{f}}{\partial \bar{z}} \right)^{-\bar{h}_i}_{z=z_i} \langle \phi_1(z_1, \bar{z}_1) \cdots \phi_n(z_n, \bar{z}_n) \rangle \tag{5.20}$$

Global conformal invariance fixes two and three point functions. For two point functions it is required that $h_1 = h_2 = h$ and $\bar{h}_1 = \bar{h}_2 = \bar{h}$ and the result is

$$\langle \phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2) \rangle = \frac{C_{12}}{z_{12}^{2h} \bar{z}_{12}^{2\bar{h}}} \tag{5.21}$$

the result for the three point function is

$$\langle \phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2) \phi_3(z_3, \bar{z}_3) \rangle = C_{123} \frac{1}{z_{12}^{h_1+h_2-h_3} z_{13}^{h_1+h_3-h_2} z_{23}^{h_2+h_3-h_1}} \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{13}^{\bar{h}_1+\bar{h}_3-\bar{h}_2} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1}} \tag{5.22}$$

These results are the analog of (3.42) and (3.47) but the novelty here is that the fields have spin (they are not necessarily scalars). Note the factorization between holomorphic and anti-holomorphic parts. This is a recurrent feature of observables in $2d$. As before, the four-point function is not fully determined. Given four complex coordinates $z_1, \cdots z_4$ we can construct the cross-ratio [9]

$$\eta = \frac{z_{12} z_{34}}{z_{13} z_{24}} \tag{5.23}$$

which is invariant under global conformal transformations. Hence, the four-point function takes the general form

$$\langle \phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2) \phi_3(z_3, \bar{z}_3) \phi_4(z_4, \bar{z}_4) \rangle = g(\eta, \bar{\eta}) \prod_{i<j} z_{ij}^{h/3-h_i-h_j} \bar{z}_{ij}^{\bar{h}/3-\bar{h}_i-\bar{h}_j} \tag{5.24}$$

where $h = \sum_{i=1}^{4} h_i$ and $\bar{h} = \sum_{i=1}^{4} \bar{h}_i$ and $g(\eta, \bar{\eta})$ is, as far as global conformal symmetry is concerned, a general function. The form of four-point functions can be understood as follows. Given three distinct points in the complex plane z_1, z_2, z_3, we can use global $SL(2, \mathbb{C})$ transformations to map them to the points $0, 1$ and ∞. Given four points, we can map them to $(0, \eta, 1, \infty)$, then a four point correlator would depend only on η. By global conformal transformations then we recover the full spatial dependence. This also explains why two and three point functions are fixed.

[9] In 2d, all other cross-ratios will depend on this one.

5.3 Radial quantization in 2d CFT

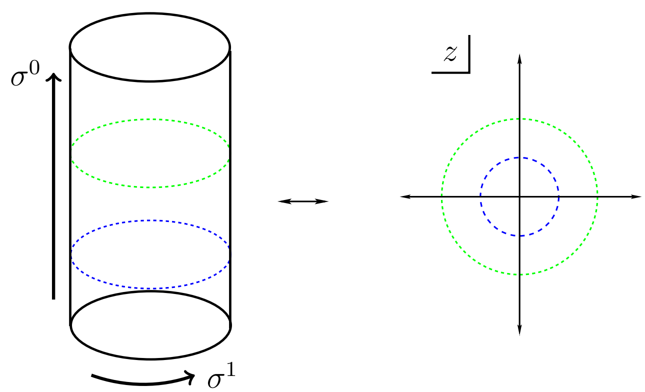
In order to study in more detail the consequences of conformal invariance in a two dimensional QFT we need to quantize the theory. Radial quantization is particularly well suited for 2d Euclidean CFT, as it allows to use the full power of complex analysis.

Consider a CFT in flat Euclidean space-time, with a space coordinate σ^1 and a "time" coordinate σ^0. We can introduce complex coordinates

$$\zeta = \sigma^0 + i\sigma^1 \tag{5.25}$$

$$\bar{\zeta} = \sigma^0 - i\sigma^1 \tag{5.26}$$

In order to eliminate long distance divergences from observables, it is customary to compactify the space coordinate $\sigma^1 \cong \sigma^1 + 2\pi$. This defines a cylinder in the (σ^0, σ^1) coordinates. Next consider the conformal map $\zeta \to z = e^{\zeta} = e^{\sigma^0 + i\sigma^1}$. This maps the cylinder to the complex plane, or more precisely the Riemann sphere, coordinatized by z. We call this plane the conformal plane.



Infinite past and future on the cylinder $\sigma^0 = \mp\infty$ are mapped to the origin and the point at infinity respectively. Equal time surfaces $\sigma^0 = const$ map to circles of constant radius on the $z-$plane. Finally, "time" translations $\sigma^0 \to \sigma^0 + a$ are mapped to dilatations $z \to e^a z$. Hence, the dilatation generator on the conformal plane can be regarded as the Hamiltonian of the system, and the Hilbert space is built up on surfaces of constant radius.

As seen in section 3 symmetry generators can be constructed via the Noether prescription. Recall that the corresponding charge was defined as the integral of the time-component along a constant time surface. In two dimensional radial quantization we should integrate along a contour of constant radius, schematically $\int j_r d\theta$. In the conformal plane local conformal transformations are generated by the non-zero components of the stress tensor, T and $\bar{T}$, so that the infinitesimal charge is given by

$$Q = \frac{1}{2\pi i} \oint \left(dz T(z)\epsilon(z) + d\bar{z}\bar{T}(\bar{z})\bar{\epsilon}(\bar{z}) \right) \tag{5.27}$$

where the integrals are performed over a circle of constant radius in the counter clockwise sense. In the quantum theory, the variation of a field under the corresponding transformation is given by the "equal-time" (or in this case "equal-radius") commutator with the charge[10]

$$\delta_\epsilon \psi(w) = -\frac{1}{2\pi i} \oint [dz T(z)\epsilon(z), \psi(w)] \tag{5.28}$$

where we have focused in a purely holomorphic variation, for simplicity.

Radial quantization and the OPE

Within radial quantization the time ordering that appears in the definition of correlation functions becomes *radial ordering*. We introduce the radial ordering operator, explicitly defined by

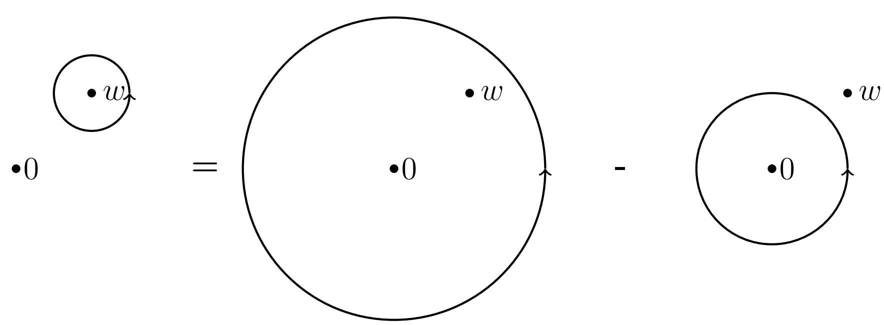
$$\mathcal{R}\left(\phi_1(z)\phi_2(w)\right) = \begin{cases} \phi_1(z)\phi_2(w) & \text{if } |z| > |w| \\ \phi_2(w)\phi_1(z) & \text{if } |z| < |w| \end{cases} \tag{5.29}$$

with a minus sign in front of the second relation if both fields are fermions. Since all field operators within correlation functions must be radially ordered, the l.h.s. of an OPE will always be assumed to be radially ordered (since remember, the OPE makes sense inside correlators).

There is a neat relation between the OPE and equal time commutation relations. Consider two holomorphic fields $a(z), b(z)$ and the following contour integral

$$\oint_{C_w} dz a(z) b(w) \tag{5.30}$$

where the integration contour is a small circle around w. Consider the above contour integral as an operator inside a correlator, so that the product $a(z)b(w)$ is really radially ordered $a(z)b(w) \to \mathcal{R}\left(a(z)b(w)\right)$. We then split the integration contour into two circles going on opposite directions, as shown in the figure



[10]We are following the conventions of the Di Francesco *et.al.* book, which have an opposite sign with respect to many others.

Our integral is now seen to be a commutator

$$\oint_{C_w} dz\mathcal{R}\left(a(z)b(w)\right) = \oint_{C_1} dza(z)b(w) - \oint_{C_2} dzb(w)a(z) = [Q_a, b(w)] \tag{5.31}$$

where C_1 and C_2 are two circles of constant radius just a little bigger and a little smaller than $|w|$ and we have defined $Q_a = \oint a(z)dz$. This can be interpreted as an equal time (or equal-radius) commutator.

Going back to (5.28) we can write the variation of the field as

$$\delta_\epsilon \psi(w) = -\frac{1}{2\pi i}\oint_{C_w} \epsilon(z)\mathcal{R}(T(z)\psi(w)) \tag{5.32}$$

where the contour of integration is a small circle around $z = w$. Now it comes a very important point. For a primary field this variation should be (5.19) so that

$$\frac{1}{2\pi i}\oint_{C_w} \epsilon(z)\mathcal{R}(T(z)\psi(w)) = \left(h(\partial\epsilon(w)) + \epsilon(w)\partial\right)\psi(w) \tag{5.33}$$

In order for this to be satisfied the product $\mathcal{R}(T(z)\psi(w))$ should have the correct short distance behavior:

$$\mathcal{R}(T(z)\psi(w)) = \frac{h}{(z-w)^2}\psi(w) + \frac{1}{z-w}\partial_w\psi(w) + \cdots \tag{5.34}$$

where the dots represent regular terms as $z \to w$. In concluding this, it was important that $\epsilon(z)$ is general, so that we are considering a primary operator (and not only quasi-primary). Including both, holomorphic and non-holomorphic parts, and suppressing the radial ordering operator (which is always to be understood in an OPE) we obtain

$$T(z)\psi(w,\bar{w}) = \frac{h}{(z-w)^2}\psi(w,\bar{w}) + \frac{1}{z-w}\partial_w\psi(w,\bar{w}) + \cdots \tag{5.35}$$

$$\bar{T}(\bar{z})\psi(w,\bar{w}) = \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\psi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\psi(w,\bar{w}) + \cdots \tag{5.36}$$

we have derived the OPE between a primary operator and the stress tensor! sometimes this serves as the *definition* of a primary field of weights $(h, \bar{h})$. Note that the singularity behaviour follows from our general discussion in section 4, given that the stress tensor has dimension 2.

Example: The free boson

Consider the action for a free scalar field in two dimensions

$$S = \int d^2z\partial\varphi\bar{\partial}\varphi$$

We can calculate the propagator and obtain

$$\langle \varphi(z,\bar{z})\varphi(w,\bar{w})\rangle = -\frac{1}{4\pi}\log|z-w|^2$$

For solutions to the equations of motion $\varphi(z,\bar{z}) = \varphi(z) + \varphi(\bar{z})$ we can split the field into a holomorphic and an anti-holomorphic piece. Let us focus in the holomorphic piece with propagator

$$\langle \varphi(z)\varphi(w)\rangle = -\frac{1}{4\pi}\log(z-w)$$

Note that the field $\varphi(z)$ doesn't transform nicely under conformal transformations. On the other hand

$$\langle \partial\varphi(z)\partial\varphi(w)\rangle = -\frac{1}{4\pi}\frac{1}{(z-w)^2}$$

So that the field $\partial\varphi$ has a chance to be a field of conformal weight $(1,0)$. Let us check that this is the case. Let us compute the holomorphic component of the stress tensor. Classically we obtain $T(z) = -2\pi\partial\varphi(z)\partial\varphi(z)$. However, in the quantum theory we need to be careful, because we get a divergence when the first operator $\partial\varphi$ gets close to the second. Hence we define the stress tensor $T(z)$ via the normal ordering prescription

$$T(z) = -2\pi :\partial\varphi(z)\partial\varphi(z): = -2\pi \lim_{w\to z}\left(\partial\varphi(z)\partial\varphi(w) + \frac{1}{4\pi}\frac{1}{(z-w)^2}\right)$$

which is finite and well defined in the limit. Using Wick contractions we can compute

$$\begin{aligned}
T(z)\partial\varphi(w) &= -2\pi :\partial\varphi(z)\partial\varphi(z): \partial\varphi(w) \\
&= -4\pi :\partial\varphi(z)\overbrace{\partial\varphi(z): \partial\varphi(w)} + \cdots \\
&= \frac{\partial\varphi(z)}{(z-w)^2} + \cdots
\end{aligned}$$

where we have disregarded terms of the form $:\partial\varphi^3:$, which are regular in the $z \to w$ limit. Finally, taylor expanding we get

$$T(z)\partial\varphi(w) = \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial_w^2\varphi(w)}{(z-w)} + \cdots$$

In perfect agreement with (5.35) for a primary of conformal weights $(1,0)$.

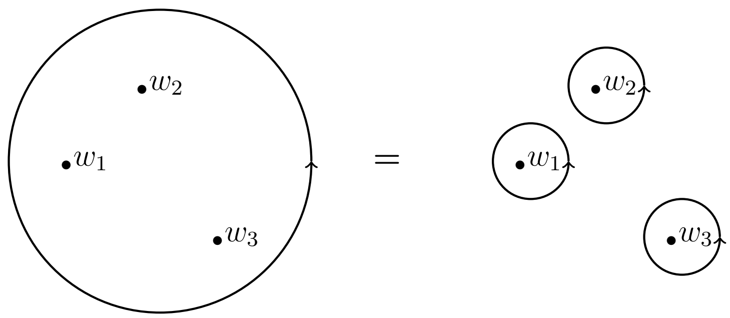
5.4 Conformal Ward identities

Ward identities are relations satisfied by correlation functions as a consequence of the symmetries of a theory. In the following we will derive conformal Ward identities satisfied by correlation functions of primary fields in a 2d CFT. Although it is possible (and not very difficult!) to derive Ward identities in general dimensions, in our derivation we will use the methods of 2d CFT.

Consider the correlator of n primary fields of conformal weights $(h_i, \bar{h}_i)$. By *global* conformal invariance these correlators satisfy (5.20). To get additional constraints from the *local* conformal algebra take operators $\phi_i(w, \bar{w})$ at points w_i and consider the following contour integral

$$\langle \oint \frac{dz}{2\pi i} \epsilon(z) T(z) \phi_1(w_1, \bar{w}_1) \cdots \phi_n(w_n, \bar{w}_n) \rangle \tag{5.37}$$

where the contour of integration is a great circle enclosing all operators. By analyticity the contour can be deformed to a sum over small contours, each encircling one operator, see figure.



So that

$$\begin{aligned} \langle \oint \frac{dz}{2\pi i} \epsilon(z) \quad T(z) \quad & \phi_1(w_1, \bar{w}_1) \cdots \phi_n(w_n, \bar{w}_n) \rangle = \\ &= \sum_{j=1}^{n} \langle \phi_1(w_1, \bar{w}_1) \cdots \left(\oint \frac{dz}{2\pi i} \epsilon(z) T(z) \phi_j(w_j, \bar{w}_j) \right) \cdots \phi_n(w_n, \bar{w}_n) \rangle \\ &= -\sum_{j=1}^{n} \langle \phi_1(w_1, \bar{w}_1) \cdots \delta_\epsilon \phi_j(w_j, \bar{w}_j) \cdots \phi_n(w_n, \bar{w}_n) \rangle \end{aligned} \tag{5.38}$$

But recall

$$\delta_\epsilon \phi(w, \bar{w}) = -(h(\partial \epsilon) + \epsilon \partial) \phi(w, \bar{w}) \tag{5.39}$$

So that the above equality can only work for general ϵ provided the following unintegrated relation holds true

$$\langle T(z) \phi_1(w_1, \bar{w}_1) \cdots \phi_n(w_n, \bar{w}_n) \rangle = \sum_{i=1}^{j} \left(\frac{h_j}{(z - w_j)^2} + \frac{1}{z - w_j} \frac{\partial}{\partial w_j} \right) \langle \phi_1(w_1, \bar{w}_1) \cdots \phi_n(w_n, \bar{w}_n) \rangle \tag{5.40}$$

This is called the conformal Ward identity, and it will be used later in the course.

6 The Virasoro algebra

6.1 Central charge and the Virasoro algebra

Central charge

Not all the fields satisfy the simple transformation property (5.18) under conformal transformations, or the corresponding OPE with the stress tensor (5.35). An example of field that does not satisfy (5.18) or (5.35) is the stress energy tensor itself! The OPE $T(z)T(w)$ is most easily computed by performing two conformal transformations in succession. We obtain

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots \tag{6.1}$$

The fourth order pole, with constant coefficient c, is allowed by analyticity, Bose symmetry and scale invariance. The constant c is known as the central charge and its value will depend on the CFT under consideration. As already mentioned, a field which is not primary is called secondary. Secondary fields are characterised by higher order poles in their OPE with the stress tensor, as for the case at hand. Identical considerations apply to $\bar{T}$, so that we can introduce another central charge $\bar{c}$, in principle independent of c.

The infinitesimal transformation law for $T(z)$ induced by (6.1) is

$$\begin{aligned} \delta_\epsilon T(w) &= -\frac{1}{2\pi i}\oint dz \epsilon(z) T(z) T(w) \\ &= -2T(w)\partial_w \epsilon(w) - \epsilon(w)\partial_w T(w) - \frac{c}{12}\partial_w^3 \epsilon(w) \end{aligned}$$

which under conformal transformations $z \to z' = f(z)$ can be integrated to

$$T'(z') = \left(\frac{\partial f}{\partial z}\right)^{-2} \left(T(z) - \frac{c}{12}\{f(z), z\}\right) \tag{6.2}$$

where the Schwartzian derivative

$$\{f(z), z\} = \frac{\partial_z f \partial_z^3 f - \frac{3}{2}(\partial_z^2 f)^2}{(\partial_z f)^2} \tag{6.3}$$

is the unique object of weight two which vanishes when restricted to the global subgroup of the two dimensional conformal group. The stress tensor is thus an example of a quasi-primary field which is not primary.

For the free boson we obtain

$$
\begin{aligned}
T(z)T(w) &= 4\pi^2 :\partial\varphi(z)\partial\varphi(z)::\partial\varphi(w)\partial\varphi(w): \\
&= \frac{1/2}{(z-w)^4} - 2\pi\frac{:\partial\varphi(z)\partial\varphi(w):}{(z-w)^2} + \cdots \\
&= \frac{1/2}{(z-w)^4} - 4\pi\frac{:\partial\varphi(w)\partial\varphi(w):}{(z-w)^2} - 2\pi\frac{:\partial^2\varphi(w)\partial\varphi(w):}{(z-w)} + \cdots \\
&= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots
\end{aligned}
$$

In agreement with (6.1). Furthermore, we conclude that for a free boson in 2d, $c = 1$.

Virasoro Algebra

As seen above, conformal transformations are generated by the holomorphic and anti-holomorphic components of the stress tensor, which in the quantum theory are operators. It is convenient to consider the Laurent expansion of the stress tensor

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n, \qquad L_n = \frac{1}{2\pi i}\oint dz z^{n+1}T(z) \tag{6.4}$$

$$\bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\bar{L}_n, \qquad \bar{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\bar{z}^{n+1}\bar{T}(\bar{z}) \tag{6.5}$$

In terms of modes $L_n, \bar{L}_n$ which are operators themselves. The contour of integration in the above integrals are circles around the origin. Consider the expansion of the infinitesimal variation $\epsilon(z)$

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} z^{n+1}\epsilon_n \tag{6.6}$$

Then the expression (5.27) for the corresponding conformal charge Q_ϵ becomes

$$Q_\epsilon = \sum_{n\in\mathbb{Z}} \epsilon_n L_n \tag{6.7}$$

so that the modes $L_n, \bar{L}_n$ generate all conformal transformations. In order to compute the algebra of the modes, we need to compute the commutator of two contour of integrations $[\oint dz, \oint dw]$. This is done as follows. First we fix w and deform the difference between the two z integrations to a single z contour which is a small circle around w, exactly as discussed in section 5.3. Once we have done this, we can use the OPE expansion and compute the integral over z by using residue theorem. Then the w integration is performed without further subtleties. This gives

$$
\begin{aligned}
[L_n, L_m] &= \frac{1}{(2\pi i)^2}\oint dw w^{m+1}\oint_{C_w} dz z^{n+1}\mathcal{R}\left(T(z)T(w)\right) \\
&= \frac{1}{(2\pi i)^2}\oint dw w^{m+1}\oint_{C_w} dz z^{n+1}\left(\frac{c/2}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w}+\text{reg.}\right) \\
&= \frac{1}{2\pi i}\oint dw w^{m+1}\left(\frac{c}{12}n(n^2-1)w^{n-2}+2(n+1)w^nT(w)+w^{n+1}\partial T(w)\right) \\
&= \frac{c}{12}n(n^2-1)\delta_{n+m,0}+2(n+1)L_{m+n}-\frac{1}{2\pi i}\oint dw w^{n+m+2}\partial T(w) \qquad (6.8)
\end{aligned}
$$

where C_w is a small circle around w while the integral over dw is around the origin. After integration by parts the last integral contributes to L_{m+n}. The same considerations apply to the anti-holomorphic modes and we obtain the

Virasoro Algebra

$$
\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m}+\frac{c}{12}n(n^2-1)\delta_{n+m,0} \\
[L_n, \bar{L}_m] &= 0 \\
[\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m}+\frac{\bar{c}}{12}n(n^2-1)\delta_{n+m,0}
\end{aligned}
$$

We find two copies of an infinite dimensional algebra called the Virasoro algebra. It depends on the central charge of the conformal field theory (usually $c=\bar{c}$). Note a very important point. The commutation relations among $L_{\pm 1}$ and L_0 do not depend on the central charge, and agree with the classical commutations among the $\ell_\pm$ and ℓ_0. The reason for this is that they generate the global transformations.

6.2 The Hilbert space

In the following we will discuss some generalities regarding the space of states of a two dimensional conformal field theory. Let us start with the vacuum state $|0\rangle$. The requirement that $T(z)|0\rangle$ and $\bar{T}(\bar{z})|0\rangle$ are well defined as $z, \bar{z}\to 0$ implies

$$
L_n|0\rangle = 0, \qquad \bar{L}_n|0\rangle = 0, \qquad \text{for } n\geq -1 \qquad (6.9)
$$

In particular this implies $L_0|0\rangle = L_{\pm 1}|0\rangle = 0$, namely, the vacuum is invariant under global conformal transformations, as it should in a conformal field theory. On the other hand, $L_{-n}|0\rangle$, for $n=2,3,\cdots$, will be in general different from zero[11].

[11]For instance $L_{-2}|0\rangle = \frac{1}{2\pi i}\oint dz z^{-1}T(z)|0\rangle = T(0)|0\rangle$.

Next consider a primary state with weights $(h, \bar{h})$ acting on the vaccum at the origin

$$|h, \bar{h}\rangle = \phi_{h,\bar{h}}(0,0)|0\rangle \tag{6.10}$$

going back to the theory on the cylinder, this maps to an incoming asymptotic state, since the origin is mapped to the infinite past, eigenstate of the Hamiltonian. From the OPE of the stress tensor with a primary, eq. (5.35), we see

$$T(z)|h, \bar{h}\rangle = \lim_{w,\bar{w}\to 0} T(z)\phi_{h,\bar{h}}(w,\bar{w})|0\rangle = \left(\frac{h}{z^2}\psi(0,\bar{0}) + \frac{1}{z}\partial\psi(0,\bar{0}) + \cdots\right)|0\rangle \tag{6.11}$$

but recall $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n$. Together with their non-holomorphic analogous relations this implies

$$L_0|h, \bar{h}\rangle = h|h, \bar{h}\rangle, \qquad \bar{L}_0|h, \bar{h}\rangle = \bar{h}|h, \bar{h}\rangle, \tag{6.12}$$

$$L_n|h, \bar{h}\rangle = 0, \qquad \bar{L}_n|h, \bar{h}\rangle = 0, \qquad \text{for} \quad n > 0. \tag{6.13}$$

Actually, this is commonly taken as the definition of a (Virasoro) primary in the Hilbert space. Once we have a primary we can obtain Virasoro descendants by acting with the raising operators $L_{-1}, L_{-2}, \cdots$. By virtue of the commutation relations $[L_0, L_{-m}] = mL_{-m}$, acting with these operators will raise the eigenvalue of L_0. Exactly the same considerations apply to the anti-holomorphic part. For example, for the lowest levels we have:

Level	dimension	states
0	h	$\|h\rangle$
1	$h+1$	$L_{-1}\|h\rangle$
2	$h+2$	$L_{-2}\|h\rangle,\ L_{-1}^2\|h\rangle$
3	$h+3$	$L_{-3}\|h\rangle,\ L_{-1}L_{-2}\|h\rangle,\ L_{-1}^3\|h\rangle$

The subset of the full Hilbert space generated by the primary state $|h\rangle$ and all its descendants is closed under the action of the Virasoro generators and thus forms a representation of the Virasoro algebra, with $|h\rangle$ the highest weight state. This subset is called a Verma module. Note that from (6.11) it also follows that $L_{-1}|h\rangle = \partial\phi(0)|0\rangle$, so that acting with L_{-1} corresponds to the usual derivative descendant (also present in higher dimensions). The Verma module, however, has a much richer structure!

This is a good place to introduce the concept of adjoint

$$\phi(z,\bar{z})^\dagger = \phi(\frac{1}{\bar{z}}, \frac{1}{z})\frac{1}{\bar{z}^{2h}}\frac{1}{z^{2\bar{h}}} \tag{6.14}$$

for a (quasi-)primary field of conformal weights $(h, \bar{h})$. This expression is better understood going back to the picture of the cylinder. If $\phi(z,\bar{z})$ defines an incoming state, from the infinite past, we would like $\phi(z,\bar{z})^\dagger$ to defined a outgoing state, to the infinite future. However, time reversal on the cylinder $\sigma_0 \to -\sigma_0$ maps to the inversion in the conformal plane $z \to \frac{z}{|z|^2}$, or $z \to 1/z^*$ on the real slice. We then understand (6.14) as the inverse acting on a quasi-primary field. The definition of an adjoint operation leads to a natural inner product. With this definition

$$\begin{aligned}
\langle \phi_{out}|\phi_{in}\rangle &= \lim_{z,\bar{z},w,\bar{w}\to 0} \langle 0|\phi(z,\bar{z})^\dagger \phi(w,\bar{w})|0\rangle \\
&= \lim_{z,\bar{z},w,\bar{w}\to 0} \bar{z}^{-2h} z^{-2\bar{h}} \langle 0|\phi(1/\bar{z},1/z)\phi(w,\bar{w})|0\rangle \\
&= \lim_{z,\bar{z}\to 0} \bar{z}^{-2h} z^{-2\bar{h}} \langle \phi(1/\bar{z},1/z)\phi(0,0)\rangle
\end{aligned} \tag{6.15}$$

which is finite and well defined, as a virtue of (5.21). In a theory with a Hermitian stress tensor the above relation implies the following two quantities are equal:

$$T(z)^\dagger = \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2} L_n^\dagger, \quad \frac{1}{\bar{z}^4} T(\frac{1}{\bar{z}}) = \sum_{n\in\mathbb{Z}} \frac{1}{\bar{z}^4} \bar{z}^{n+2} L_n \tag{6.16}$$

so that

$$L_n^\dagger = L_{-n} \tag{6.17}$$

This allows to define an inner product in the Hilbert space, and in particular the norm of states. More precisely, the inner product of two states $L_{-k_1}\cdots L_{-k_m}|h\rangle$ and $L_{-\ell_1}\cdots L_{-\ell_n}|k\rangle$ is given by

$$\langle h|L_{k_m}\cdots L_{k_1} L_{-\ell_1}\cdots L_{-\ell_n}|k\rangle$$

where the dual state satisfies $\langle h|L_j = 0$ for $j < 0$. This inner product can be computed by commuting the annihilation operators to the right and hitting $|k\rangle$.

Null vectors

Before going ahead, let us mention one important point. In constructing the Verma module we have considered a primary of generic dimension h in a CFT with generic central charge c. For special cases, it could happen that some linear combination of descendants, lets call it $|\chi\rangle$, satisfies itself the highest weight state condition:

$$L_n|\chi\rangle = 0, \quad \text{for } n \geq 1 \tag{6.18}$$

In this case $|\chi\rangle$ plus all its descendants will provide themselves a representation of the Virasoro algebra. Note furthermore that such a state is orthogonal to all descendants of $|h\rangle$, indeed:

$$\langle \chi|L_{-k_1}\cdots L_{-k_n}|h\rangle = 0 \tag{6.19}$$

Since $\langle\chi|L_{-k_1} = (L_{k_1}|\chi\rangle)^\dagger = 0$. In particular, $|\chi\rangle$ is orthogonal to itself, and hence has zero norm. We call such a state *null*. It also follows that all its decendants are also null. In order to obtain an irreducible representation of the Virasoro algebra we then need to take the original Verma module and quotient by such states. Namely, we identify two states that differ by a null one.

6.3 Correlators of descendants

Very much as for the higher dimensional case, it can be shown that correlators involving Virasoro descendants can be computed in terms of the correlators for the corresponding primary fields. Let us focus in the holomorphic part only and consider a primary operator $\phi(z)$ of weight h, which generates a state $|h\rangle$. A descendant for this state:

$$L_{-n}|h\rangle = L_{-n}\phi(0)|0\rangle = \frac{1}{2\pi i}\oint dz z^{1-n}T(z)\phi(0)|0\rangle \tag{6.20}$$

leads naturally to the descendant field

$$\phi^{(-n)}(w) = \left(L_{-n}\phi\right)(w) = \frac{1}{2\pi i}\oint_{C_w}\frac{dz}{(z-w)^{n-1}}T(z)\phi(w) \tag{6.21}$$

so that $\phi^{(-n)}(0)|0\rangle = L_{-n}|h\rangle$. Consider now the correlator of the descendant field with a chain of primary fields

$$\langle\phi^{(-n)}(w)\phi_{h_1}(w_1)\cdots\phi_{h_n}(w_n)\rangle = \frac{1}{2\pi i}\oint_{C_w}\frac{dz}{(z-w)^{n-1}}\langle T(z)\phi(w)\phi_{h_1}(w_1)\cdots\phi_{h_n}(w_n)\rangle \tag{6.22}$$

There are different way to relate this to the correlator of primary fields. The most direct one is to use the conformal Ward identities (5.40) to rewrite the correlator in the integrand in terms of $\langle\phi(w)\phi_{h_1}(w_1)\cdots\phi_{h_n}(w_n)\rangle$. Then it is straightforward to perform the contour integral (since the z dependence is explicit). We obtain

$$\langle\phi^{(-n)}(w)\phi_{h_1}(w_1)\cdots\phi_{h_n}(w_n)\rangle = \mathcal{L}_{-n}\langle\phi(w)\phi_{h_1}(w_1)\cdots\phi_{h_n}(w_n)\rangle \tag{6.23}$$

where the differential operator $\mathcal{L}_{-n}$ is given by

$$\mathcal{L}_{-n} = \sum_{i=1}^{n}\left(\frac{(n-1)h_i}{(w_i-w)^n} - \frac{1}{(w_i-w)^{n-1}}\partial_{w_i}\right) \tag{6.24}$$

Note in particular that $\mathcal{L}_{-1} = -\sum_{i=1}\partial_{w_i} = \partial_w$, since $\partial_w+\sum_{i=1}\partial_{w_i}$ annihilates any correlator translational invariant.

An important consequence of this discussion is the following. Given a primary field ϕ we define its conformal family $[\phi]$ as the set of the primary plus all its descendants. It follows that two members of a conformal family are correlated only if their respective primaries are correlated. Furthermore, they will only be correlated if they are descendants of the same level.

6.4 OPE and conformal blocks

Invariance under scaling transformations requires the operator algebra to have the following form

$$\phi_1(z,\bar{z})\phi_2(0,0) = \sum_p \sum_{k,\bar{k}} C_{12}^{p,\{k,\bar{k}\}} z^{h_p-h_1-h_2+|k|} \bar{z}^{\bar{h}_p-\bar{h}_1-\bar{h}_2+|\bar{k}|} \phi_p^{(k,\bar{k})}(0,0) \tag{6.25}$$

where the sum over p runs over primary fields and $\phi_p^{(k,\bar{k})}(0,0)$ are secondary fields belonging to the conformal family $[\phi_p(0,0)]$. $k,\bar{k}$ are indices denoting the precise descendant, and $|k|,|\bar{k}|$ are the levels in the holomorphic and anti-holomorphic sectors. Let us assume for simplicity that the spectrum of primary operators is non-degenerate. Taking the correlator of (6.25) with a third primary field $\phi_q(w,\bar{w})$, we learn that the OPE coefficient $C_{12}^{p,\{0,\bar{0}\}} \equiv C_{12p}$ is exactly the coefficient appearing in the three-point function, for canonically normalised two-point functions. Furthermore, by taking the correlator with descendants of $\phi_q(w,\bar{w})$ at a given level, and using (6.23), we can fix the coefficients $C_{12}^{p,\{k,\bar{k}\}}$ in terms of C_{12p}:

$$C_{12}^{p,\{k,\bar{k}\}} = C_{12p}\beta_{12}^{p,\{k\}}\bar{\beta}_{12}^{p,\{\bar{k}\}} \tag{6.26}$$

It is then customary to write

$$\phi_1(z,\bar{z})\phi_2(0,0) = \sum_p C_{12p} z^{h_p-h_1-h_2} \bar{z}^{\bar{h}_p-\bar{h}_1-\bar{h}_2} \Psi_p(z,\bar{z}|0,0) \tag{6.27}$$

where we have defined

$$\Psi_p(z,\bar{z}|0,0) = \sum_{\{k,\bar{k}\}} \beta_{12}^{p,\{k\}}\bar{\beta}_{12}^{p,\{\bar{k}\}} z^{|k|}\bar{z}^{|\bar{k}|}\phi_p^{(k,\bar{k})}(0,0) \tag{6.28}$$

Now we can consider a four-point function of primary operators, which we consider identical for simplicity, of weights $(h,\bar{h})$. We choose the following convenient locations

$$\lim_{z_1,\bar{z}_1\to\infty} z_1^{2h}\bar{z}_1^{2\bar{h}}\langle\phi(z_1,\bar{z}_1)\phi(1,1)\phi(z,\bar{z})\phi(0,0)\rangle = G(z,\bar{z}) \tag{6.29}$$

so that $G(z,\bar{z}) = \langle h,\bar{h}|\phi(1,1)\phi(z,\bar{z})|h,\bar{h}\rangle$. Note that in this limit the previously introduced cross-ratio $\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}}$ exactly coincides with z. By using (6.27) to write the OPE of the two fields on the right

$$\phi(z,\bar{z})\phi(0,0) = \sum_p C_p z^{h_p-2h}\bar{z}^{\bar{h}_p-2\bar{h}}\Psi_p(z,\bar{z}|0,0) \tag{6.30}$$

we can write

$$G(z,\bar{z}) = \sum_p C_p^2 A(p|z,\bar{z}) \tag{6.31}$$

where each of the partial waves is given by

$$A(p|z,\bar{z}) = C_p^{-1} z^{h_p-2h}\bar{z}^{\bar{h}_p-2\bar{h}}\langle h,\bar{h}|\phi(1,1)\Psi_p(z,\bar{z}|0,0)\rangle \tag{6.32}$$

We have rewritten the four-point function as a sum over intermediate conformal families, labeled by the index p, exactly as it was done in section 4.4. Since three point functions of primaries or descendants are fixed by the Virasoro algebra, this whole object is. Furthermore, it is clear that the partial waves have the factorized form

$$A(p|z,\bar{z}) = \mathcal{F}_h(p|z)\bar{\mathcal{F}}_h(p|\bar{z}) \tag{6.33}$$

where

$$\mathcal{F}_h(p|z) = z^{h_p-2h}\sum_{\{k\}}\beta^{p,\{k\}}z^{|k|}\frac{\langle h|\phi(1)L_{-k_1}\cdots L_{-k_n}|h_p\rangle}{\langle h|\phi(1)|h\rangle} \tag{6.34}$$

these functions are called conformal blocks, or Virasoro conformal blocks. They are much harder to compute that the conformal blocks in general dimensions,discussed in section 4.4, since the structure of the Virasoro algebra is much more intricate that its globally defined subgroup. On the other hand, as we will see, they are much more powerful.

7 Minimal models

7.1 Unitarity

So far we have been studying 2d CFT with arbitrary values of the central charge and where the spectrum of Virasoro primaries $\{h_i,\bar{h}_i\}$ can be arbitrary. It is of great interest to classify the possible 2d CFTs by requiring additional physical constraints. An important constraint on quantum theories is that of unitarity. In particular, a probabilistic interpretation of quantum mechanics requires states to have positive norm. In this section we will study the constraints of unitarity on 2d CFT. More precisely, for a given value of the central charge c we will ask which highest weight states $|h\rangle$ lead to unitary representations. Namely, representations which do not contain negative norm states.

Given a primary state $|h\rangle$, let us consider $L_{-n}|h\rangle$. Its norm is given by

$$\langle h|L_nL_{-n}|h\rangle = \langle h|\left(L_{-n}L_n + 2nL_0 + \frac{1}{12}cn(n^2-1)\right)|h\rangle = \left(2nh + \frac{1}{12}cn(n^2-1)\right)\langle h|h\rangle \tag{7.1}$$

Taking n sufficiently large, we see we should require $c > 0$. In other words, all representations are non-unitary if the central charge is negative. For $n = 1$ we obtain $h \geq 0$. $h = 0$ corresponds to the vacuum state, and we see that all representations with negative conformal dimension are also non-unitary. In summary, if we are after unitary representations we should focus on the positive quadrant in the (c, h) plane.

Let us now consider the descendants of $|h\rangle$ at level ℓ and choose a basis $\{s_1, s_2, \cdots\} = \{L_{-1}^{\ell}|h\rangle, \cdots, L_{-\ell}|h\rangle\}$ of linearly independent states. Let us call $p(\ell)$ the number of such states. The Gram matrix is defined as the $p(\ell) \times p(\ell)$ matrix given by the inner products

$$M_{ij}^{(\ell)} = \langle s_i | s_j \rangle, \quad i, j = 1, \cdots, p(\ell) \tag{7.2}$$

In this subspace there will be negative norm states if and only if $M^{(\ell)}$ has one or more negative eigenvalues. For instance, normalising $\langle h|h\rangle = 1$ we can compute

$$M^{(0)} = 1 \tag{7.3}$$

$$M^{(1)} = 2h \tag{7.4}$$

$$M^{(2)} = \begin{pmatrix} 4h(2h+1) & 6h \\ 6h & 4h + c/2 \end{pmatrix} \tag{7.5}$$

From $M^{(0)}$ and $M^{(1)}$ we don't recover any new conditions. For $M^{(2)}$ is it simpler to study its trace and determinant, both of which have to be non-negative. For the trace we obtain

$$\mathrm{Tr}\, M^{(2)} = 8h(h+1) + c/2 \tag{7.6}$$

so that no new condition arises. For the determinant we obtain

$$\det M^{(2)} = 32(h - h_{1,1})(h - h_{1,2})(h - h_{2,1}) \tag{7.7}$$

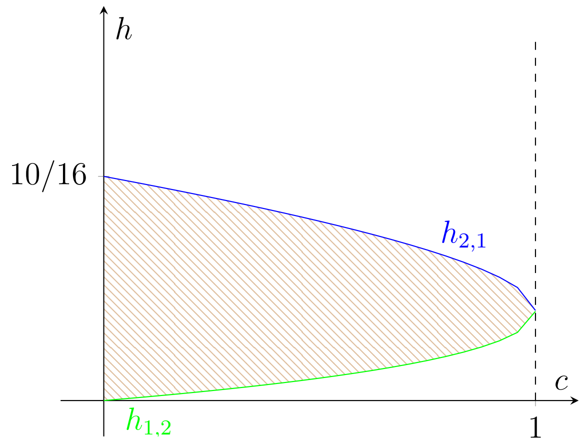
with

$$h_{1,1} = 0 \tag{7.8}$$

$$h_{1,2} = \frac{1}{16}\left(5 - c - \sqrt{(1-c)(25-c)}\right) \tag{7.9}$$

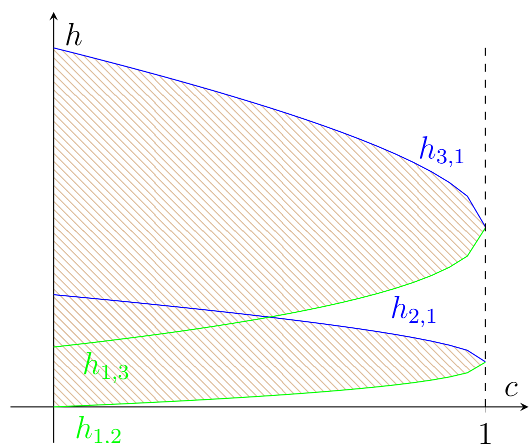
$$h_{2,1} = \frac{1}{16}\left(5 - c + \sqrt{(1-c)(25-c)}\right) \tag{7.10}$$

The first root is telling us that the Verma module contains a null state if $h = 0$. Indeed, the vacuum satisfies $L_{-1}|0\rangle = 0$. For $h > 0$ and $c > 1$ the determinant is always positive and hence consistent with unitarity. For $0 < c \leq 1$ something more interesting occurs. As a function of c, $h_{1,2}$ and $h_{2,1}$ describe two curves that join at $c = 1$:



The determinant is negative in the region between these two curves (shown as a shaded area). Furthermore, Verma modules associated to (c, h) lying on the curves are reducible, as there are null states at level two. We will come back to this later.

Considering $M^{(3)}$ excludes yet another region for $c < 1$ (while it does not impose constraints for $c \geq 1$). The determinant has two non-trivial roots, denoted by $h_{3,1}$ and $h_{1,3}$ and the region between these two curves again corresponds to non-unitary representations. See figure.



The same situation occurs at higher levels. There is a formula for the determinant of the Gram matrix, called the Kac determinant

$$\det M^{(\ell)} = \alpha_\ell \prod_{\substack{r,s\geq 1 \\ rs\leq \ell}} (h - h_{r,s}(c))^{p(\ell - rs)} \tag{7.11}$$

where α_ℓ is positive and the relation between $h_{r,s}(c)$ and the central charge is better written implicitly

$$c(m) = 1 - \frac{6}{m(m+1)} \tag{7.12}$$

$$h_{r,s}(m) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)} \tag{7.13}$$

where $m > 2$ is a real parameter. It turns out that all representations are unitary for $c > 1, h > 0$. For $c < 1$ the situation is much more restricted and all regions are excluded except for isolated points. The final result can be summarised as follows

- The central charge has to be of the form

$$c(m) = 1 - \frac{6}{m(m+1)}, \quad m = 2, 3, 4, \cdots$$

- For each m there is only a *finite* number of primaries leading to unitary representations, of dimensions

$$h_{r,s}(m) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}, \qquad 1 \leq r \leq m-1$$

$$1 \leq s \leq r$$

Conformal field theories with such a spectrum exist and are called unitary minimal models. They will be the subject of the rest of this section.

7.2 Correlators and OPE in unitary minimal models

In the following we will study in detail the models introduced above. Since the spectrum of primary operators for these models fall into vanishing curves $h = h_{r,s}$, the corresponding Verma modules will have null states. For a primary operator with $h = h_{r,s}$, this happens at level $r \times s$. The existence of null vectors imposes strong constraints on the structure of correlators and the structure of the operator algebra. In this course we will focus on a simple example which will show all the main ingredients.

Let us study a simple example of reducible Verma module. Consider a primary $|h\rangle$ and the following descendant at level two:

$$|\chi\rangle = \left(L_{-2} + \eta L_{-1}^2\right)|h\rangle. \tag{7.14}$$

We want to choose η and h in such a way that $|\chi\rangle$ is null. It will suffice to require $L_1|\chi\rangle = L_2|\chi\rangle = 0$, as $L_n|\chi\rangle = 0$ for $n > 2$ will then follow from the Virasoro algebra. We obtain

$$L_1|\chi\rangle = (3 + 2\eta + 4h\eta)L_{-1}|h\rangle \tag{7.15}$$

$$L_2|\chi\rangle = (\frac{c}{2} + 4h + 6hn)L_{-1}|h\rangle \tag{7.16}$$

Hence $|\chi\rangle$ is null provided

$$\eta = -\frac{3}{2(2h+1)} \tag{7.17}$$

$$h = \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right) \tag{7.18}$$

As expected, the condition on h is either $h = h_{1,2}$ or $h = h_{2,1}$. Let us denote by $\phi(z)$ the field corresponding to the primary operator. To $|\chi\rangle$ we can associate a descendant null field $\chi(z)$ given by

$$\chi(z) = \phi^{(-2)}(z) - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\phi(z) \tag{7.19}$$

As already mentioned, the null state is orthogonal to the whole Verma module, or to any state in the theory. In terms of correlators this translate into

$$\langle \chi(z)\phi_{h_1}(z_1)\cdots\phi_{h_n}(z_n)\rangle = 0 \tag{7.20}$$

however, according to (6.23) this implies

$$\left(\mathcal{L}_{-2} - \frac{3}{2(2h+1)}\mathcal{L}_{-1}^2\right)\langle \phi(z)\phi_{h_1}(z_1)\cdots\phi_{h_n}(z_n)\rangle = 0 \tag{7.21}$$

where the operators $\mathcal{L}_{-n}$ have been defined in (6.24). More explicitly

$$\left(\sum_{i=1}^{n}\left[\frac{h_i}{(z-z_i)^2} + \frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right] - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\right)\langle \phi(z)\phi_{h_1}(z_1)\cdots\phi_{h_n}(z_n)\rangle = 0 \tag{7.22}$$

This is a beautiful equation! For two point functions it does not add new information. However, let us consider its effect on a three-point function of $\phi(z)$ with two other primaries:

$$\langle \phi(z)\phi_{h_1}(z_1)\phi_{h_2}(z_2)\rangle = \frac{C_{h,h_1,h_2}}{(z-z_1)^{h+h_1-h_2}(z_1-z_2)^{h_1+h_2-h}(z-z_2)^{h+h_2-h_1}} \tag{7.23}$$

where we have focused in the holomorphic dependence only and C_{h,h_1,h_2} is a constant not fixed by global conformal invariance. As discussed in the previous section, it appears in the operator algebra, and is the coefficient with which the primary ϕ_{h_2} appears in the OPE $\phi \times \phi_{h_1}$. Applying (7.22) to this three-point function we see that C_{h,h_1,h_2} vanishes unless the following constraint holds

$$2(2h+1)(h+2h_2-h_1) = 3(h-h_1+h_2)(h-h_1+h_2+1) \tag{7.24}$$

This can be seen as a quadratic equation for h_2. We then arrive to the conclusion that in the OPE of the degenerate field $\phi(z)$ with another primary there are only two conformal families! (*i.e.* two primaries plus all its descendants). Assume now that $\phi_{h_1}(z)$ is one of the primaries of a minimal model, so that $h_1 = h_{r,s}$, given in (7.12), we obtain

$$h = h_{1,2}, \quad h_1 = h_{r,s} \;\rightarrow\; h_2 = h_{r,s-1} \text{ or } h_2 = h_{r,s+1} \tag{7.25}$$
$$h = h_{2,1}, \quad h_1 = h_{r,s} \;\rightarrow\; h_2 = h_{r-1,s} \text{ or } h_2 = h_{r+1,s}$$

Denoting by $\phi_{(r,s)}$ the field corresponding to $|h_{r,s}\rangle$, we can write these relations in a symbolic form

$$\begin{aligned} \phi_{(1,2)} \times \phi_{(r,s)} &= \phi_{r,s-1} + \phi_{r,s+1} \\ \phi_{(2,1)} \times \phi_{(r,s)} &= \phi_{(r-1,s)} + \phi_{(r+1,s)} \end{aligned} \tag{7.26}$$

Telling us which conformal families can appear in the OPE of the left hand side. The conditions under which a given conformal family occurs in the short-distance product of two conformal fields are called the *fusion rules* of the theory. Be aware that there are implicit coefficients (the OPE coefficients) which may even be zero. One can derive similar fusion rules for more general OPE $\phi_{r,s} \times \phi_{r',s'}$. A very important property, already manifest in (7.26), is that the conformal families $[\phi_{(r,s)}]$ associated with reducible modules form a closed set under the operator algebra. The final result for minimal models is a finite set of conformal families (as claimed above) which closes under fusion. This last property allows minimal models to be consistent CFTs. Let us make the following two remarks. First, the field $\phi_{1,1}(z)$ has a level one null descendant. But at level one the only descendant is $(L_{-1}\phi_{1,1})$ so that, $\partial_z \phi_{1,1}(z) = 0$ inside any correlator. We hence identify $\phi_{1,1}(z)$ with the identity operator $\phi_{1,1}(z) = \mathbb{I}$. This also fits with $h_{1,1}$ being zero, since $\phi_{1,1}(0)|0\rangle = \mathbb{I}|0\rangle = |0\rangle$. The fusion relation involving the identity operator is of course

$$\phi_{(r,s)} \times \phi_{(1,1)} = \phi_{r,s} \tag{7.27}$$

Finally, when writing fusion relations it sometimes happens that the conformal families on the r.h.s fall outside the range defining a minimal model. It is sometimes convenient to take conformal families $\phi_{(r,s)}$ with $1 \leq r \leq m-1$ and $1 \leq s \leq m$ with the identification

$$\phi_{(r,s)} = \phi_{(m-r,m+1-s)} \tag{7.28}$$

So far our discussions were restricted to the holomorphic sector. The Hilbert space of a physical theory is in fact constructed out of tensor products of holomorphic and anti-holomorphic modules. In the example below we will restrict ourselves to a "diagonal" choice where we associate to each holomorphic module $M(c, h_{r,s})$ the corresponding anti-holomorphic module $\bar{M}(c, h_{r,s})$ (and $\bar{c} = c$). The Hilbert space of the theory will then be of the form

$$\mathcal{H} = \bigoplus_{r,s} M(c, h_{r,s}) \otimes \bar{M}(c, h_{r,s}) \tag{7.29}$$

7.3 Example: The Ising model

The simplest non-trivial unitary CFT is called the critical Ising model. In addition to the identity it contains two fields, $\sigma(z,\bar{z})$ and $\epsilon(z,\bar{z})$, of conformal dimensions

$$(h,\bar{h})_\sigma = \left(\frac{1}{16},\frac{1}{16}\right), \qquad (h,\bar{h})_\epsilon = \left(\frac{1}{2},\frac{1}{2}\right) \tag{7.30}$$

This allows to identify this CFT with the unitary minimal model with $m=3$. Furthermore, if we focus in the holomorphic part of the theory we can make the following identification

$$\begin{aligned} \mathbb{I} &\Leftrightarrow \phi_{(1,1)} = \phi_{(2,3)} \\ \sigma &\Leftrightarrow \phi_{(2,2)} = \phi_{(1,2)} \\ \epsilon &\Leftrightarrow \phi_{(2,1)} = \phi_{(1,3)} \end{aligned} \tag{7.31}$$

Then the fusion rules (7.26) lead to

$$\sigma \times \sigma = \mathbb{I} + \epsilon \tag{7.32}$$

$$\sigma \times \epsilon = \sigma \tag{7.33}$$

$$\epsilon \times \epsilon = \mathbb{I} \tag{7.34}$$

In order to see the power of (7.22) into action, let us consider the four point correlator of identical fields $\sigma(z,\bar{z})$. Focusing for the moment only in the holomorphic dependence conformal invariance allows to write

$$\langle \sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle = \frac{g(\eta)}{z_{12}^{2h} z_{34}^{2h}}, \qquad \eta = \frac{z_{12}z_{34}}{z_{13}z_{24}} \tag{7.35}$$

With $h = 1/16$. The equation (7.22) written in this case:

$$\left(\sum_{i=2}^{4} \left[\frac{h}{(z_1-z_i)^2} + \frac{1}{z_1-z_i}\frac{\partial}{\partial z_i} \right] - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z_1^2} \right) \langle \sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle = 0 \tag{7.36}$$

implies a differential equation on $g(\eta)$. Precisely:

$$g''(\eta) + \frac{-4\eta + 4\eta h - 8h + 2}{3\eta - 3\eta^2} g'(\eta) - \frac{2h(2h+1)}{3(1-\eta)^2} g(\eta) = 0 \tag{7.37}$$

where remember $h = 1/16$. This has two linearly independent solutions

$$g_\pm(z) = \frac{\sqrt{1 \pm \sqrt{\eta}}}{(1-\eta)^{1/8}} \tag{7.38}$$

Building up the solution

Let us reinsert back the anti-holomorphic dependence on the full correlator

$$\langle\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\sigma(z_3,\bar{z}_3)\sigma(z_4,\bar{z}_4)\rangle = \frac{g(\eta,\bar{\eta})}{|z_{12}|^{4h}|z_{34}|^{4h}} \tag{7.39}$$

Our previous result, plus its anti-holomorphic analog, implies

$$g(\eta,\bar{\eta}) = \kappa_{++}g_+(z)g_+(\bar{z}) + \kappa_{+-}g_+(z)g_-(\bar{z}) + \kappa_{-+}g_-(z)g_+(\bar{z}) + \kappa_{--}g_-(z)g_-(\bar{z}) \tag{7.40}$$

So that we have fixed the correlator up to four constants! can we do better than this? which other requirements should this correlator satisfy?

- Single valued: On the real section $\bar{\eta} = \eta^*$, $g(\eta,\bar{\eta})$ should be single-valued as we move η around the complex plane. The non-trivial points are $\eta = 0, 1$. We define the monodromy transformations around $\eta = 0$ and $\eta = 1$ as

$$\mathcal{M}_0\left(g(\eta,\bar{\eta})\right) = \lim_{t\to 1^-} g(\eta e^{2\pi i t}, \bar{\eta}e^{-2\pi i t}) \tag{7.41}$$

$$\mathcal{M}_1\left(g(\eta,\bar{\eta})\right) = \lim_{t\to 1^-} g(1+(\eta-1)e^{2\pi i t}, 1+(\bar{\eta}-1)e^{-2\pi i t}) \tag{7.42}$$

  $g(\eta,\bar{\eta})$ should be invariant under both transformations.

- Crossing relations: since the operators are identical, the correlator should be invariant under the exchanges of any two of them:

$$1\leftrightarrow 2: \quad \eta \leftrightarrow \frac{\eta}{\eta-1} \quad \rightarrow \quad g(\eta,\bar{\eta}) = g\left(\frac{\eta}{\eta-1}, \frac{\bar{\eta}}{\bar{\eta}-1}\right) \tag{7.43}$$

$$1\leftrightarrow 3: \quad \eta \leftrightarrow 1-\eta \quad \rightarrow \quad \frac{g(\eta,\bar{\eta})}{|z_{12}|^{4h}|z_{34}|^{4h}} = \frac{g(1-\eta,1-\bar{\eta})}{|z_{23}|^{4h}|z_{14}|^{4h}} \tag{7.44}$$

  In particular this last condition reads

$$g(\eta,\bar{\eta}) = \left|\frac{\eta}{1-\eta}\right|^{4h} g(1-\eta,1-\bar{\eta}) \tag{7.45}$$

- Consistency with the OPE of $\sigma\times\sigma$. In particular, we know that the lowest dimension field in the OPE is the identity field, so that (see (6.25))

$$\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2) = \frac{1}{z_{12}^{1/8}\bar{z}_{12}^{1/8}} + \dots$$

  with exact coefficient one. Where we have assumed canonical normalization for σ and used $C_{\sigma\sigma}^{\mathbb{I}} = 1$. This fixes the small $\eta,\bar{\eta}$ behaviour of the answer to be:

$$g(\eta,\bar{\eta}) = 1 + \cdots$$

Let us start with single-valuedness. The monodromy transformations around zero and one act as

$$\mathcal{M}_0 : \begin{cases} \sqrt{1+\sqrt{\eta}} \to \sqrt{1-\sqrt{\eta}}, & \sqrt{1+\sqrt{\bar\eta}} \to \sqrt{1-\sqrt{\bar\eta}} \\ \sqrt{1-\sqrt{\eta}} \to \sqrt{1+\sqrt{\eta}}, & \sqrt{1-\sqrt{\bar\eta}} \to \sqrt{1+\sqrt{\bar\eta}} \end{cases}$$
$$\mathcal{M}_1 : \begin{cases} \sqrt{1+\sqrt{\eta}} \to \sqrt{1+\sqrt{\eta}}, & \sqrt{1+\sqrt{\bar\eta}} \to \sqrt{1+\sqrt{\bar\eta}} \\ \sqrt{1-\sqrt{\eta}} \to -\sqrt{1-\sqrt{\eta}}, & \sqrt{1-\sqrt{\bar\eta}} \to -\sqrt{1-\sqrt{\bar\eta}} \end{cases} \tag{7.46}$$

Or at the level of the basis of solutions

$$\begin{aligned} \mathcal{M}_0 : &\quad g_+ \leftrightarrow g_- \\ \mathcal{M}_1 : &\quad g_- \leftrightarrow -g_- \end{aligned} \tag{7.47}$$

Invariance under both transformations fixes the solution up to an overall constant! This constant is then fixed by requiring the correct behaviour for small $\eta, \bar\eta$. We obtain the final answer:

$$g(\eta, \bar\eta) = \frac{|1+\sqrt{\eta}| + |1-\sqrt{\eta}|}{2|1-\eta|^{1/4}} \tag{7.48}$$

We have chosen the prefactors in defining $g(\eta, \bar\eta)$ in such a way that the partial decomposition in terms of conformal blocks takes the form

$$g(\eta, \bar\eta) = \sum_p C_p^2 \mathcal{F}(p|\eta)\bar{\mathcal{F}}(p|\bar\eta) \tag{7.49}$$

where now

$$\mathcal{F}(p|\eta) = \eta^{h_p}\left(1 + a\eta + \cdots\right) \tag{7.50}$$

starts with η to the dimension of the intermediate operator, and differs from (6.34) just by the overall power η^{-2h}. An important point is that the normalization is fixed and all subsequent powers of η are integer powers times η^{h_p}. The explicit answer can be decomposed as

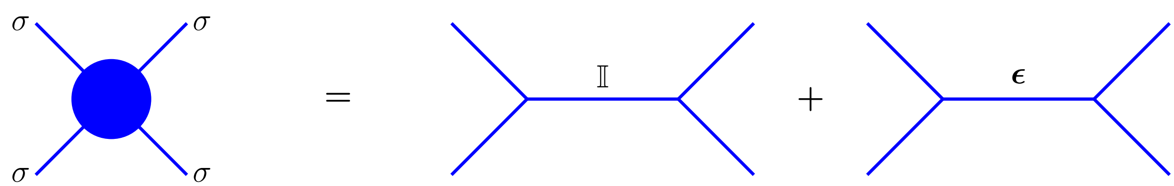
$$g(\eta, \bar\eta) = \mathcal{F}(0|\eta)\bar{\mathcal{F}}(0|\bar\eta) + C_\epsilon^2 \mathcal{F}(1/2|\eta)\bar{\mathcal{F}}(1/2|\bar\eta) \tag{7.51}$$

where $C_\epsilon = 1/2$ and

$$\mathcal{F}(0|\eta) = \frac{\sqrt{1+\sqrt{\eta}} + \sqrt{1-\sqrt{\eta}}}{2(1-\eta)^{1/8}} = \eta^0\left(1 + \frac{\eta^2}{64} + \cdots\right) \tag{7.52}$$

$$\mathcal{F}(\frac{1}{2}|\eta) = \frac{\sqrt{1+\sqrt{\eta}} - \sqrt{1-\sqrt{\eta}}}{(1-\eta)^{1/8}} = \eta^{1/2}\left(1 + \frac{\eta}{4} + \cdots\right) \tag{7.53}$$

correspond to the conformal blocks of the identity operator and the ϵ operator! This is consistent with the fusion rule $\sigma \times \sigma = \mathbb{I} + \epsilon$. Graphically, we represent the decomposition as



Let us close this section with a discussion of the crossing relations. Under $\eta \to \frac{\eta}{\eta-1}$ each conformal block transforms to itself up to a phase, which cancels between holomorphic and anti-holomorphic pieces. The symmetry $\eta \to 1-\eta$ is more interesting:

$$\frac{\eta^{1/8}}{(1-\eta)^{1/8}} \begin{pmatrix} \mathcal{F}(0|1-\eta) \\ \\ \mathcal{F}(\frac{1}{2}|1-\eta) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \\ \sqrt{2} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \mathcal{F}(0|\eta) \\ \\ \mathcal{F}(\frac{1}{2}|\eta) \end{pmatrix} \tag{7.54}$$

This transformation properties are such that $g(\eta,\bar{\eta})$ has the correct symmetry! Note the following feature: the conformal blocks transform to a finite linear combination of conformal blocks under $\eta \to 1-\eta$, this is very peculiar! and is specific to these models in two dimensions. Finally, note that we could have considered

$$g(\eta,\bar{\eta}) = \mathcal{F}(0|\eta)\bar{\mathcal{F}}(0|\bar{\eta}) + C_\epsilon^2 \mathcal{F}(1/2|\eta)\bar{\mathcal{F}}(1/2|\bar{\eta}) \tag{7.55}$$

with arbitrary OPE coefficient C_ϵ. Crossing symmetry, plus the explicit transformations for the conformal blocks, would have fixed it! In this way we have fixed a dynamical (not fixed by conformal symmetry) quantity, such as C_ϵ, by considering a higher point function, and resorting to the structure of the OPE and crossing symmetry. This is the spirit of the conformal bootstrap.

8 Conformal bootstrap in $d > 2$

After our discussions in two dimensions, let us return to the less gentle world of $d > 2$. Over this course we have seen that a conformal field theory in d dimensions is characterised by

- The spectrum of primary operators: a list of their scaling dimensions Δ_i and the $SO(d)$ representations under which they transform, labelled by ℓ.
- The constants C_{ijk} that appear in the three-point functions of primaries and the OPE of two primaries.
- Some additional data, such as the central charge. Some theories also admit exactly marginal deformations, so that one can introduce a coupling τ, such that the theory is conformal for all values of this coupling. The spectrum and OPE coefficients will in general depend on these parameters.

We have seen that four-point functions are determined by conformal symmetry, and the structure of the operator algebra, once this *CFT data* is known, and this is also true for higher point functions. A natural question arises: does any CFT data define a consistent CFT? while we haven't defined consistent yet, the answer to this question is clearly no! we have seen already in two dimensions that by requiring the correct properties of a given four-point function we were able to fix the OPE coefficients in the critical Ising model! In this section we will attempt to pursue a similar program in $d > 2$, and try to determine which CFT data defines a consistent CFT.

8.1 Crossing symmetry and the bootstrap equation

Consider the correlator of four identical scalar operators of dimension Δ_ϕ. Due to conformal invariance

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{G(u,v)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \tag{8.1}$$

where we have introduced the cross-ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \tag{8.2}$$

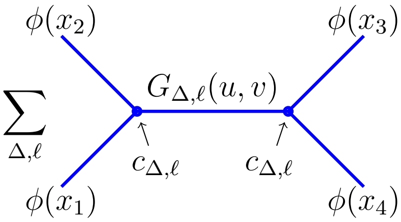
By considering the OPE $\phi(x_1)\phi(x_2)$ and $\phi(x_3)\phi(x_4)$ we have seen we can expand the above correlator in conformal partial waves

$$G(u,v) = \sum_{\Delta,\ell} c_{\Delta,\ell}^2 G_{\Delta,\ell}(u,v). \tag{8.3}$$

where the sum runs over intermediate conformal primaries, in other words, the conformal primaries present in the OPE $\phi(x_1)\phi(x_2)$. $c_{\Delta,\ell}^2$ are the square of the OPE coefficients and

$G_{\Delta,\ell}(u,v)$ denotes the conformal block of the corresponding primary. We can express this decomposition diagrammatically

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{\Delta,\ell}$$



It is convenient to single out the contribution from the identity operator: it is always present in the OPE of two identical operators, has OPE coefficient $c_{0,0} = 1$ and has no descendants,[12] so that we can write:

$$G(u,v) = 1 + \sum_{\Delta,\ell} c_{\Delta,\ell}^2 G_{\Delta,\ell}(u,v) \tag{8.4}$$

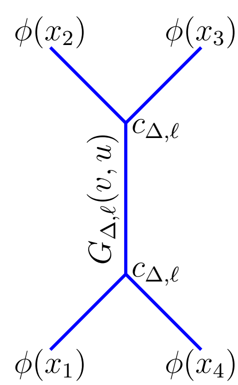
In case you are curious, in four-dimensions the conformal blocks for external scalar operators have a closed expression and for the case at hand take the form

$$G_{\Delta,\ell}(z,\bar{z}) = \frac{1}{z-\bar{z}}\left(z^{\ell+1}k_{\Delta+\ell}(z)k_{\Delta-\ell-2}(\bar{z}) - (z \leftrightarrow \bar{z})\right)$$
$$k_\beta(z) = {}_2F_1\left(\frac{\beta}{2},\frac{\beta}{2},\beta;z\right)$$

where we have introduced $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$ not to be confused with complex coordinates.

Of course, we could have considered instead the OPE $\phi(x_2)\phi(x_3)$ and $\phi(x_1)\phi(x_4)$, this would have led to an expansion of the form

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{\Delta,\ell}$$



[12]As we have seen, the situation is very different in two dimensions, since we have many more "creation" operators $L_{-2}, L_{-3}, \cdots$ in addition to the globally well defined one L_{-1}, which annihilates the identity operator.

The attentive reader may have noted that the cross ratios are exchanged $u \leftrightarrow v$, as we are effectively exchanging points $x_1 \leftrightarrow x_3$. However, the answer of course should be exactly the same! so that both conformal wave expansions should be equivalent. More precisely, crossing symmetry implies

$$\begin{aligned} \frac{G(u,v)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} &= \frac{G(v,u)}{x_{23}^{2\Delta_\phi} x_{14}^{2\Delta_\phi}} \\ &\Downarrow \\ v^{\Delta_\phi}\left(1+\sum_{\Delta,\ell} c_{\Delta,\ell}^2 G_{\Delta,\ell}(u,v)\right) &= u^{\Delta_\phi}\left(1+\sum_{\Delta,\ell} c_{\Delta,\ell}^2 G_{\Delta,\ell}(v,u)\right) \end{aligned} \tag{8.5}$$

which can be rearranged into the

Conformal bootstrap equation

$$\sum_{\Delta,\ell} c_{\Delta,\ell}^2 \left(\frac{v^{\Delta_\phi} G_{\Delta,\ell}(u,v) - u^{\Delta_\phi} G_{\Delta,\ell}(v,u)}{u^{\Delta_\phi} - v^{\Delta_\phi}}\right) = 1$$

This is a remarkable equation. Unlike in two dimension, there is an infinite number of primaries on the OPE $\phi \times \phi$. This equation is telling us, that whatever the spectrum Δ_i, ℓ_i is and whatever the OPE coefficients $c_{\Delta,\ell}$ are, the combination above has to be exactly one! and this should be true for all values of u, v and any value of extra parameters (such as the central charge).

The idea

The bootstrap equation turns out to be particularly powerful when supplemented by unitarity. In general dimensions, unitarity imposes the following constraints:

- Lower bounds on the spectrum. For a field of spin ℓ, in the symmetric traceless representation:

$$\begin{aligned} \Delta &\geq \ell + d - 2 \quad \text{if} \quad \ell = 1, 2, 3, \cdots \\ \Delta &\geq d/2 - 1 \quad \text{if} \quad \ell = 0 \end{aligned}$$

- The OPE coefficients are real, so that $c_{\Delta,\ell}^2 \geq 0$.

Let us write the bootstrap equation as follows

$$\sum_{\Delta,\ell} c^2_{\Delta,\ell} F_{\Delta,\ell}(u,v) = 1 \tag{8.6}$$

How do we extract information from this? The procedure is simple: first assume a spectrum consistent with unitarity bounds, namely, a list of Δ_i's for each spin, for instance:

Putative spectrum:

- For $\ell = 0$, $\Delta_i = \{1, 3/2, 2, \cdots\}$
- For $\ell = 1$, $\Delta_i = \{3, 4, 5, \cdots\}$
- For $\ell = 2$, $\Delta_i = \{5, 5 + 3/2, 5 + 3, \cdots\}$

and so on. Now it comes the interesting part. Immagine you can find a linear operator Φ, which acts on functions of u, v, such that

$$\begin{aligned} \Phi\left(F_{\Delta,\ell}(u,v)\right) &\geq 0, \quad \text{for all the putative spectrum} && (8.7) \\ \Phi(1) &< 0 && (8.8) \end{aligned}$$

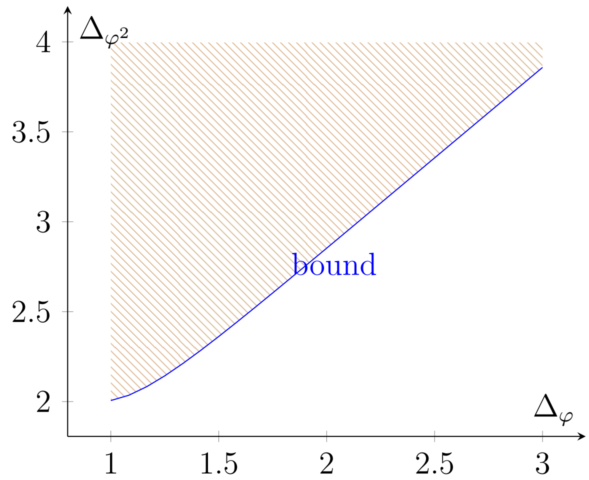
Since $c^2_{\Delta,\ell} > 0$ due to unitarity, this means the spectrum you have chosen is no good! since it would imply some of the OPE coefficients are negative. As a result, you can rule the spectrum out. From a more mathematical standpoint, we can understand this problem as follows: choosing a spectrum is equivalent to choose a basis of function $F_{\Delta_i,\ell_i}(u,v)$. The bootstrap equation (8.6) can then be seen as an expansion of the function 1 in terms of our basis. If one of the coefficients in that expansion is negative, then this means that the basis we have chosen (namely the spectrum) is not consistent with unitarity. Following this procedure, one can rule out entire families of spectra!

Results

Let us discuss the kind of results that the conformal bootstrap gives, for a particular example. Imagine a generic four-dimensional CFT, with a scalar operator φ of dimension Δ_φ. Its OPE with itself will be of the form

$$\varphi(x) \times \varphi(0) = \frac{1}{|x|^{2\Delta_\varphi}} \left(1 + C|x|^{\Delta_{\varphi^2}} \varphi^2(0) + \cdots\right) \tag{8.9}$$

where by φ^2 we denote the lowest twist operator (after the identity) in the OPE above. Of course, in a free theory this would be $:\varphi\varphi:$ but here we are taking about a generic CFT, which may not even have a Lagrangian description. Let us say that the dimension of this operator is Δ_{φ^2}. The conformal bootstrap gives upper bounds for Δ_{φ^2} for fixed values of Δ_φ (See for instance Rychkov lectures):



These bounds are very robust, and have not even assumed the theory has a Lagrangian! Although not as powerful as in two dimensions, consistency conditions in higher dimensions have led to a variety of very interesting results, but this will take us outside the scope of this course.