0. BACKGROUND MATERIAL

There is a good deal of background material in this chapter. It is hoped, given the course's pre-requisites and recommedations, that most students will have met some, perhaps most, of this material but very few may have met it all. At no single point in this course will all this material be simultaneously necessary, but it will be helpful to either do preparatory reading on a topic ahead of the relevant lectures or revisit the material if you find yourself rustier than you expected.

As a guide:

- *arc length, curvature* and *surface area* will prove useful ahead of Chapter 3.
- *the real projective plane*'s topology will also appear in Chapters 1 and 2.
- the real and complex projective planes will appear in Chapter 7.
- *holomorphic branches* will prove useful in Chapter 7.
- *multivariable differentiability* will prove somewhat helpful in Chapter 3.
- *identification spaces* will be important ahead of Chapter 2.

0.1 Arc Length and Curvature

This is largely material from Prelims *Introductory Calculus.*

Definition 0.1 *A* ***smooth parameterized curve*** *in* $\mathbb{R}^3$ *is a map* $\gamma\colon I \to \mathbb{R}^3$ *from an open interval* $I \subseteq \mathbb{R}$ *such that*

- γ *is smooth, i.e.* γ *has derivatives of all orders;*
- $\gamma\colon I \to \gamma(I)$ *is a homeomorphism;*
- $\gamma'(t) \neq \mathbf{0}$ *for all* $t \in I$.

The requirement that γ be a homeomorphism onto its image is somewhat unusual here. Some authors will omit this requirement which allows the possibility of self-intersections, for the curve crossing itself. Defining a smooth parameterized curve as above means that the curve has no singular points and also mirrors the later definition of a smooth parameterized surface (see Definition 1.7).

A smooth parameterized curve γ is a curve in $\mathbb{R}^3$ with a preferred parameterization. The image of γ is also the image of other smooth parameterized curves. It's important to check that our definitions relating to curves and surfaces are independent of the choice of parameterization. For example, a simple application of the chain rule shows that the tangent line to a curve and arc length on a curve (as defined below) are independent of the choice of parameter. Arc length is an 'intrinsic' parameter for a curve.

Definition 0.2 *Let $\gamma \colon I \to \mathbb{R}^3$ be a smooth parameterized curve with $t_0 \in I$. Then the* ***arc length*** *$s(t)$ from $\gamma(t_0)$ to a point $\gamma(t)$ is defined by the integral*

$$s(t) = \int_{t_0}^{t} |\gamma'(u)| \, \mathrm{d}u.$$

As $\gamma'(t) \neq \mathbf{0}$ for all t then there is a well defined *tangent line* at each point of $\gamma(I)$.

Definition 0.3 *Let $\gamma \colon I \to \mathbb{R}^3$ be a smooth parameterized curve with $t_0 \in I$.*

(a) The ***tangent line*** *to γ at $\gamma(t_0)$ is the line containing the point $\gamma(t_0)$ and parallel to $\gamma'(t_0)$.*

(b) The ***unit tangent vector*** *$\mathbf{t}(s)$ is the tangent vector*

$$\mathbf{t}(s) = \frac{\mathrm{d}\gamma}{\mathrm{d}s},$$

when γ is parameterized by arc length s.

(c) The ***curvature*** *$\kappa(s)$ of γ at $\gamma(s)$ is defined to be*

$$\kappa(s) = \left| \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} \right| = \left| \frac{\mathrm{d}^2\gamma}{\mathrm{d}s^2} \right|.$$

Example 0.4 *(**Logarithmic spiral**) Let $\gamma(t) = \left(ae^{bt}\cos t, ae^{bt}\sin t\right)$ for $t > 0$ and real constants $a > 0 > b$. Show that γ has finite arc length.*

Solution. The tangent vector $\gamma'(t)$ equals

$$(ae^{bt}(b\cos t - \sin t), ae^{bt}(b\sin t + \cos t)),$$

and has magnitude

$$ae^{bt}\sqrt{((b\cos t - \sin t)^2 + (b\sin t + \cos t)^2)} = ae^{bt}\sqrt{b^2+1}.$$

So the arc length from $\gamma(0) = (a, 0)$ to $\lim_{t\to\infty}\gamma(t) = (0,0)$ equals

$$a\sqrt{1+b^2}\int_0^\infty e^{bu}\mathrm{d}u = a\sqrt{1+b^{-2}}.$$

■

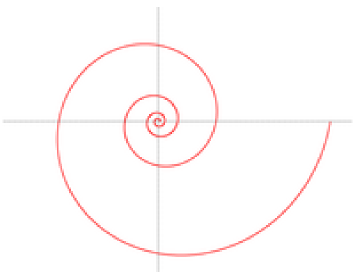

Figure 0.1 – The logarithmic spiral

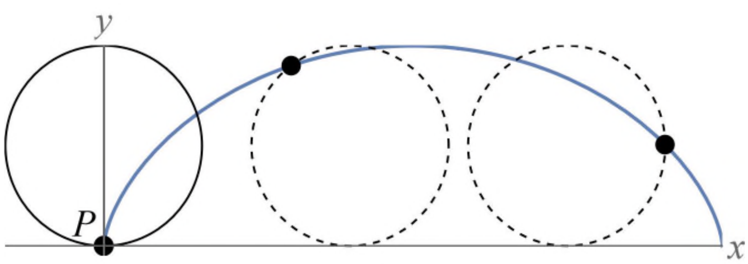

Figure 0.2 – The cycloid

Example 0.5 *The **tractrix** is the curve given by*

$$\gamma(t) = \left(-\cos t + \log\tan\left(\frac{t}{2}\right), \sin t\right), \qquad 0 < t < \frac{\pi}{2}.$$

Show that the length of the tangent line from a point $\gamma(t)$, to the point where the tangent meets the x-axis, is always 1 (see Figure 3.5).

Solution. Differentiating we find that $\gamma'(t)$ equals

$$\left(\frac{-\cos^2 t}{\sin t}, \cos t\right), \qquad 0 < t < \frac{\pi}{2}.$$

So the tangent from the curve at $\gamma(t)$ meets the x-axis at

$$\gamma(t) + (\cos t, -\sin t),$$

a point distance 1 away. ■

Example 0.6 *A circular disc of radius r in the xy-plane rolls without slipping along the x-axis. The locus described by a point of the circumference of the disc is called a **cycloid** (see Figure 0.2). Determine the arc length of a section of the cycloid which corresponds to a complete rotation of the disc.*

Solution. Assume that the disc begins with its centre at $(0, r)$. Consider the curve described by the point $(0, 0)$ as the disc rolls. After the disc has rolled distance $r\theta$ then the point $(0, 0)$ has moved on to

$$(x(\theta), y(\theta)) = (r(\theta - \sin\theta), r(1 - \cos\theta)).$$

Thus $(x')^2 + (y')^2 = r^2\left[(1-\cos\theta)^2 + \sin^2\theta\right] = 2r^2(1-\cos\theta)$ and so

$$s = \sqrt{2}\,r\int_0^{2\pi}\sqrt{1-\cos\theta}\,\mathrm{d}\theta = 2r\int_0^{2\pi}\left|\sin\frac{1}{2}\theta\right|\mathrm{d}\theta = 8r.$$

■

Example 0.7 *Show that the curvature of a curve is identically zero if and only if the curve is part of a line.*

Solution. For a curve that is part of a line, $\mathbf{t}$ is constant and so $\kappa = |\mathrm{d}\mathbf{t}/\mathrm{d}s| = 0$. Conversely if κ is identically zero, then $\ddot{\gamma}(s) = \mathbf{0}$ and hence $\gamma(s) = \mathbf{a}s + \mathbf{b}$ for constant vectors $\mathbf{a}, \mathbf{b}$. This is the parameterization of a line. ■

Example 0.8 *(a) Show that a circle of radius a has constant curvature $\kappa = a^{-1}$.*

(b) Conversely let γ be a curve in the xy-plane which has constant positive curvature κ. Show that γ is part of a circle. (There are non-planar curves with constant curvature, such as helices.)

Proof. (a) Without loss of generality we can take the circle's centre to be the origin in the xy-plane. A parameterization by arc length is

$$\gamma(s) = \left(a \cos\left(\frac{s}{a}\right), a \sin\left(\frac{s}{a}\right)\right).$$

Then

$$\kappa(s) = \ddot{\gamma}(s) = \left|\left(-\frac{1}{a}\cos\left(\frac{s}{a}\right), -\frac{1}{a}\sin\left(\frac{s}{a}\right)\right)\right| = \frac{1}{a}.$$

(b) Asumme now that the curvature κ is constant. We can write

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \kappa \mathbf{n},$$

where $\mathbf{n}$ is a unit vector in the same plane. As $\mathbf{t}$ is a unit vector, then $\mathbf{t}$ and $\mathbf{n}$ are perpendicular. As $\mathbf{n}$ is a unit vector then $\mathrm{d}\mathbf{n}/\mathrm{d}s$ is perpendicular to $\mathbf{n}$ and so parallel to $\mathbf{t}$. Further, differentiating $\mathbf{t} \cdot \mathbf{n} = 0$ gives

$$0 = \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} \cdot \mathbf{n} + \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} \cdot \mathbf{t} = \boldsymbol{\kappa} + \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} \cdot \mathbf{t},$$

showing $\mathrm{d}\mathbf{n}/\mathrm{d}s = -\kappa \mathbf{t}$.

Now consider the vector

$$\mathbf{c} = \gamma + \frac{1}{\kappa}\mathbf{n}.$$

Note

$$\frac{\mathrm{d}\mathbf{c}}{\mathrm{d}s} = \mathbf{t} + \frac{1}{\kappa}(-\kappa \mathbf{t}) = \mathbf{0}.$$

So $\mathbf{c}$ is constant and $|\gamma - \mathbf{c}| = 1/\kappa$, showing γ is a circular arc, with centre $\mathbf{c}$ and radius κ^{-1}. ■

Example 0.9 *Let γ be a smooth curve in $\mathbb{R}^3$ parameterized by t, which need not be arc length. Show that*

$$\kappa = \frac{|\gamma' \wedge \gamma''|}{|\gamma'|^3}.$$

Solution. The is left as Exercise 1 on Sheet 0. ■

0.2 Surface Area

This is largely material from Prelims *Geometry.*

Let $\mathbf{r} \colon U \to \mathbb{R}^3$ be a smooth parameterized surface with

$$\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$$

and consider the small rectangle of the plane that is bounded by the co-ordinate lines $u = u_0$ and $u = u_0 + \delta u$ and $v = v_0$ and $v = v_0 + \delta v$. Then $\mathbf{r}$ maps this to a small region of the surface $\mathbf{r}(U)$ and we are interested in calculating the surface area of this small region, which is approximately that of a parallelogram. Note

$$\begin{aligned} \mathbf{r}(u+\delta u, v) - \mathbf{r}(u,v) &\approx \frac{\partial \mathbf{r}}{\partial u}(u,v)\,\delta u, \\ \mathbf{r}(u, v+\delta v) - \mathbf{r}(u,v) &\approx \frac{\partial \mathbf{r}}{\partial v}(u,v)\,\delta v. \end{aligned}$$

Recall that the area of a parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a}\wedge\mathbf{b}|$ where $\wedge$ denotes the vector product. So the element of surface area we are considering is approximately

$$\left|\frac{\partial\mathbf{r}}{\partial u}\,\delta u\wedge\frac{\partial\mathbf{r}}{\partial v}\,\delta v\right| = \left|\frac{\partial\mathbf{r}}{\partial u}\wedge\frac{\partial\mathbf{r}}{\partial v}\right|\,\delta u\,\delta v.$$

This motivates the following definitions.

Definition 0.10 *Let $\mathbf{r}\colon U\to\mathbb{R}^3$ be a smooth parameterized surface. Then the* ***surface area*** *(or simply* ***area****) of $\mathbf{r}(U)$ is defined to be*

$$\iint\limits_U \left|\frac{\partial\mathbf{r}}{\partial u}\wedge\frac{\partial\mathbf{r}}{\partial v}\right|\,\mathrm{d}u\,\mathrm{d}v.$$

We will often write

$$\mathrm{d}S = \left|\frac{\partial\mathbf{r}}{\partial u}\wedge\frac{\partial\mathbf{r}}{\partial v}\right|\,\mathrm{d}u\,\mathrm{d}v$$

to denote an infinitesimal part of surface area.

Proposition 0.11 *The surface area of $\mathbf{r}(U)$ is independent of the choice of parameterization.*

Proof. Let $\Sigma=\mathbf{r}(U)=\mathbf{s}(W)$ be two different parameterizations of a surface X; take u,v as the co-ordinates on U and p,q as the co-ordinates on W. Let $f=(f_1,f_2):U\to W$ be the co-ordinate change map; that is for any $(u,v)\in U$ we have

$$\mathbf{r}(u,v)=\mathbf{s}(f(u,v))=\mathbf{s}(f_1(u,v),f_2(u,v)).$$

By the chain rule

$$\frac{\partial\mathbf{r}}{\partial u}=\frac{\partial\mathbf{s}}{\partial p}\frac{\partial f_1}{\partial u}+\frac{\partial\mathbf{s}}{\partial q}\frac{\partial f_2}{\partial u},\qquad \frac{\partial\mathbf{r}}{\partial v}=\frac{\partial\mathbf{s}}{\partial p}\frac{\partial f_1}{\partial v}+\frac{\partial\mathbf{s}}{\partial q}\frac{\partial f_2}{\partial v},$$

giving

$$\begin{aligned}\frac{\partial\mathbf{r}}{\partial u}\wedge\frac{\partial\mathbf{r}}{\partial v} &= \frac{\partial\mathbf{s}}{\partial p}\frac{\partial f_1}{\partial u}\wedge\frac{\partial\mathbf{s}}{\partial q}\frac{\partial f_2}{\partial v}+\frac{\partial\mathbf{s}}{\partial q}\frac{\partial f_2}{\partial u}\wedge\frac{\partial\mathbf{s}}{\partial p}\frac{\partial f_1}{\partial v}\\ &= \left(\frac{\partial f_1}{\partial u}\frac{\partial f_2}{\partial v}-\frac{\partial f_1}{\partial v}\frac{\partial f_2}{\partial u}\right)\frac{\partial\mathbf{s}}{\partial p}\wedge\frac{\partial\mathbf{s}}{\partial q}\\ &= \frac{\partial(p,q)}{\partial(u,v)}\frac{\partial\mathbf{s}}{\partial p}\wedge\frac{\partial\mathbf{s}}{\partial q}.\end{aligned}$$

Finally

$$\begin{aligned}\iint\limits_U\left|\frac{\partial\mathbf{r}}{\partial u}\wedge\frac{\partial\mathbf{r}}{\partial v}\right|\mathrm{d}u\,\mathrm{d}v &= \iint\limits_U\left|\frac{\partial(p,q)}{\partial(u,v)}\frac{\partial\mathbf{s}}{\partial p}\wedge\frac{\partial\mathbf{s}}{\partial q}\right|\mathrm{d}u\,\mathrm{d}v\\ &= \iint\limits_U\left|\frac{\partial\mathbf{s}}{\partial p}\wedge\frac{\partial\mathbf{s}}{\partial q}\right|\left|\frac{\partial(p,q)}{\partial(u,v)}\right|\mathrm{d}u\,\mathrm{d}v\\ &= \iint\limits_W\left|\frac{\partial\mathbf{s}}{\partial p}\wedge\frac{\partial\mathbf{s}}{\partial q}\right|\mathrm{d}p\,\mathrm{d}q\end{aligned}$$

by the two-dimensional substitution rule (Apostol, *Mathematical Analysis* p.421). ■

Example 0.12 *Find the surface area of the cone*

$$x^2+y^2=z^2\cot^2\alpha \qquad 0\leqslant z\leqslant h.$$

Solution. We can parameterize the cone as

$$\mathbf{r}(z,\theta)=(z\cot\alpha\cos\theta, z\cot\alpha\sin\theta, z), \qquad 0<\theta<2\pi,\ 0<z<h.$$

We have

$$\mathbf{r}_z=(\cot\alpha\cos\theta,\cot\alpha\sin\theta,1)\,, \qquad \mathbf{r}_\theta=(-z\cot\alpha\sin\theta, z\cot\alpha\cos\theta,0),$$

giving

$$\mathbf{r}_z\wedge\mathbf{r}_\theta=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cot\alpha\cos\theta & \cot\alpha\sin\theta & 1\\ -z\cot\alpha\sin\theta & z\cot\alpha\cos\theta & 0\end{vmatrix}=\begin{pmatrix}-z\cot\alpha\cos\theta\\ -z\cot\alpha\sin\theta\\ z\cot^2\alpha\end{pmatrix}.$$

Thus the cone has surface area

$$\begin{aligned}
& \int_{\theta=0}^{2\pi}\int_{z=0}^{h}\sqrt{z^2\cot^2\alpha\cos^2\theta+z^2\cot^2\alpha\sin^2\theta+z^2\cot^4\alpha}\,\mathrm{d}z\,\mathrm{d}\theta\\
= & \int_{\theta=0}^{2\pi}\int_{z=0}^{h} z\cot\alpha\sqrt{1+\cot^2\alpha}\,\mathrm{d}z\,\mathrm{d}\theta\\
= & \ 2\pi\int_{z=0}^{h} z\cot\alpha\csc\alpha\,\mathrm{d}z\\
= & \ 2\pi\times\frac{\cos\alpha}{\sin^2\alpha}\times\left[\frac{z^2}{2}\right]_0^h\\
= & \ \frac{\pi h^2\cos\alpha}{\sin^2\alpha}.
\end{aligned}$$

Note that as $\alpha\to 0$ this area tends to infinity as the cone transforms into the plane and the area tends to zero as $\alpha\to\pi/2$. ■

Proposition 0.13 *(**Surface area of a graph**) Let $z=f(x,y)$ denote the graph of a function f defined on a subset S of the xy-plane. Show that the graph has surface area*

$$\iint_S\sqrt{1+(f_x)^2+(f_y)^2}\,\mathrm{d}x\,\mathrm{d}y.$$

Proof. We can parameterize the surface as

$$\mathbf{r}(x,y)=(x,y,f(x,y)) \qquad (x,y)\in S.$$

Then

$$\mathbf{r}_x\wedge\mathbf{r}_y=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & f_x\\ 0 & 1 & f_y\end{vmatrix}=(-f_x,-f_y,1)\,.$$

Hence the graph has surface area

$$\iint_S |\mathbf{r}_x \wedge \mathbf{r}_y| \, \mathrm{d}x \, \mathrm{d}y = \iint_S \sqrt{1 + (f_x)^2 + (f_y)^2} \, \mathrm{d}x \, \mathrm{d}y.$$

■

Example 0.14 *Find the area of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 4$.*

Solution. By Proposition 0.13 the desired area equals

$$A = \iint_R \sqrt{1 + (2x)^2 + (2y)^2} \, \mathrm{d}A$$

where R is the disc $x^2 + y^2 \leqslant 4$ in the xy-plane. We can parameterize R using polar co-ordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad 0 < r < 2, \quad 0 < \theta < 2\pi,$$

and then we have that

$$\begin{aligned} A &= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \sqrt{1 + (2r\cos\theta)^2 + (2r\sin\theta)^2} \, r \, \mathrm{d}r \, \mathrm{d}\theta \\ &= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \sqrt{1 + 4r^2} \, r \, \mathrm{d}r \, \mathrm{d}\theta \\ &= 2\pi \int_{r=0}^{2} \sqrt{1 + 4r^2} \, r \, \mathrm{d}r \\ &= 2\pi \times \frac{1}{8} \times \frac{2}{3} \times \left[\left(1 + 4r^2\right)^{3/2} \right]_{r=0}^{2} \\ &= \frac{\pi}{6} \left[17^{3/2} - 1 \right]. \end{aligned}$$

■

Proposition 0.15 *(**Surfaces of revolution**) A surface S is formed by rotating the graph of*

$$y = f(x) \qquad a < x < b,$$

about the x-axis. Here $f(x) > 0$ for all x. The surface area of S equals

$$\text{Area}(S) = 2\pi \int_{x=a}^{x=b} f(x) \, \frac{\mathrm{d}s}{\mathrm{d}x} \, \mathrm{d}x.$$

Proof. Using the parameterization

$$\mathbf{r}(x, \theta) = (x, f(x)\cos\theta, f(x)\sin\theta) \qquad -\pi < \theta < \pi, \; a < x < b$$

we have

$$\mathbf{r}_x \wedge \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(x)\cos\theta & f'(x)\sin\theta \\ 0 & -f(x)\sin\theta & f(x)\cos\theta \end{vmatrix} = \begin{pmatrix} f'(x)f(x) \\ -f(x)\cos\theta \\ -f(x)\sin\theta \end{pmatrix}.$$

So

$$|\mathbf{r}_x \wedge \mathbf{r}_\theta|^2 = f(x)^2 f'(x)^2 + f(x)^2 = f(x)^2(1 + f'(x)^2) = f(x)^2 \left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)^2.$$

The result follows. ■

Example 0.16 *Rederive the area of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 4$, by thinking of the paraboloid as a surface of revolution.*

Solution. We can consider the paraboloid as a rotation of the curve $x = \sqrt{z}$ about the z-axis where $0 < z < 4$. We then have

$$\left(\frac{\mathrm{d}s}{\mathrm{d}z}\right)^2 = 1 + \left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 = 1 + \left(\frac{1}{2\sqrt{z}}\right)^2 = 1 + \frac{1}{4z}.$$

Hence

$$\begin{aligned}
A &= 2\pi \int_{z=0}^{4} x \frac{\mathrm{d}s}{\mathrm{d}z}\, \mathrm{d}z \\
&= 2\pi \int_{z=0}^{4} \sqrt{z}\sqrt{1 + \frac{1}{4z}}\, \mathrm{d}z \\
&= 2\pi \int_{z=0}^{4} \sqrt{z + \frac{1}{4}}\, \mathrm{d}z \\
&= 2\pi \left[\frac{2}{3}\left(z + \frac{1}{4}\right)^{3/2}\right]_0^4 \\
&= \frac{4\pi}{3}\left[\left(\frac{17}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right] \\
&= \frac{\pi}{6}\left[17^{3/2} - 1\right].
\end{aligned}$$

■

Proposition 0.17 *Isometries preserve area.*

Proof. An isometry is a bijection between surfaces which preserves the lengths of curves. Say that $\mathbf{r}\colon U \to \mathbb{R}^3$ is a parameterization of a smooth surface $X = \mathbf{r}(U)$ and $f\colon \mathbf{r}(U) \to Y$ is an isometry from X to another smooth surface Y. Then the map

$$\mathbf{s} = f \circ \mathbf{r} : U \to Y$$

is a parameterization of Y also using co-ordinates from U.

Consider a curve

$$\gamma(t) = \mathbf{r}(u(t), v(t)) \qquad a \leqslant t \leqslant b$$

in X. By the chain rule

$$\gamma' = u'\mathbf{r}_u + v'\mathbf{r}_v$$

and

$$|\gamma'|^2 = E(u')^2 + 2Fu'v' + G(v')^2$$

where

$$E = \mathbf{r}_u \cdot \mathbf{r}_u, \qquad F = \mathbf{r}_u \cdot \mathbf{r}_v, \qquad G = \mathbf{r}_v \cdot \mathbf{r}_v.$$

The length of γ equals is

$$\mathcal{L}(\gamma) = \int_{t=a}^{t=b} |\gamma'(t)| \, \mathrm{d}t = \int_{t=a}^{t=b} \sqrt{E(u')^2 + 2Fu'v' + G(v')^2} \, \mathrm{d}t.$$

In a similar fashion the length of the curve $f(\gamma)$ equals

$$\mathcal{L}(f(\gamma)) = \int_{t=a}^{t=b} \sqrt{\tilde{E}(u')^2 + 2\tilde{F}u'v' + \tilde{G}(v')^2} \, \mathrm{d}t$$

where

$$\tilde{E} = \mathbf{s}_u \cdot \mathbf{s}_u, \qquad \tilde{F} = \mathbf{s}_u \cdot \mathbf{s}_v, \qquad \tilde{G} = \mathbf{s}_v \cdot \mathbf{s}_v.$$

As f is an isometry then

$$\int_{t=a}^{t=b} \sqrt{E(u')^2 + 2Fu'v' + G(v')^2} \, \mathrm{d}t = \int_{t=a}^{t=b} \sqrt{\tilde{E}(u')^2 + 2\tilde{F}u'v' + \tilde{G}(v')^2} \, \mathrm{d}t.$$

This is true for all b, so it must follow that

$$E(u')^2 + 2Fu'v' + G(v')^2 = \tilde{E}(u')^2 + 2\tilde{F}u'v' + \tilde{G}(v')^2$$

for all values of t and all functions u, v. By choosing $u = t, v = 0$, we find $E = \tilde{E}$ and we also obtain $G = \tilde{G}$ by setting $u = 0, v = t$. It follows then that $F = \tilde{F}$ as well.

Now the area of a subset $\mathbf{r}(V)$ of X is given by

$$\iint_V |\mathbf{r}_u \wedge \mathbf{r}_v| \, \mathrm{d}u \, \mathrm{d}v.$$

However, by the quadruple scalar product

$$|\mathbf{r}_u \wedge \mathbf{r}_v|^2 = (\mathbf{r}_u \cdot \mathbf{r}_u)(\mathbf{r}_v \cdot \mathbf{r}_v) - (\mathbf{r}_u \cdot \mathbf{r}_v)(\mathbf{r}_v \cdot \mathbf{r}_u) = EG - F^2.$$

As

$$|\mathbf{s}_u \wedge \mathbf{s}_v| = \sqrt{\tilde{E}\tilde{G} - \tilde{F}^2} = \sqrt{EG - F^2} = |\mathbf{r}_u \wedge \mathbf{r}_v|$$

then the area of $f(\mathbf{r}(V))$ equals

$$\iint_V |\mathbf{s}_u \wedge \mathbf{s}_v| \, \mathrm{d}u \, \mathrm{d}v = \iint_V |\mathbf{r}_u \wedge \mathbf{r}_v| \, \mathrm{d}u \, \mathrm{d}v$$

and we see that isometries preserve areas. ■

Remark 0.18 *As angles between curves can similarly be written in terms of E, F, G, then isometries also preserve angles.*

0.3 The Real Projective Plane

This is material from Part A *Projective Geometry.*

The following theorem, *Bézout's theorem,* is not actually part of this course, but it is a clean and general result which readily motivates the worth of projective geometry. For those interested, the theorem is part of B3.3 *Algebraic Curves.*

Bézout's theorem is a first significant result in *algebraic geometry*, which is unsurprisingly interested in geometric objects that can be described using the language of algebra, and proved using the theorems of algebra.

So, for example, curves defined by polynomials such as $x^2 + xy + y^2 = 1$ are of interest to an algebraic geometer whereas the curve with equation $y = e^x$ would not be. Bézout's theorem addresses a natural first question: how many times do two curves, defined by polynomials of degrees m and n, intersect?

If we begin with $m = n = 1$ then we are talking about two lines. These typically meet in a point but we recognize that this wouldn't be the case if the lines are parallel. If $m = 1$ and $n = 2$, so that we're considering a line and, say, a parabola, then there can be as many as two intersections. We appreciate that there may be no intersections – with $y = 0$ and $y = x^2 + 1$ – but that can be circumvented by working with complex numbers, and we can see that the answer might be just one – with $y = 0$ and $y = x^2$ – but we could think of this as a double contact or repeated root in some sense. But we are still left with cases like $y = x$ and $(y - x)^2 = 1$ which appear to have no intersection, or $y = 0$ and $y^2 = x$ which has one 'single contact' intersection. Think about the $m = n = 2$ case and you'll find the number of intersections can be $0, 1, 2, 3, 4$.

Perhaps, then, the best we can do is to say that the two curves meet in at most mn points. Even the use of complex numbers and appreciation of multiple contacts cannot completely resolve the issue. It turns out, though, that all we are missing is the notion of *points at infinity.* Once we properly introduce the notion of parallel lines meeting at a point at infinity then Bézout's theorem states that the two curves have mn intersections, counting multiple contacts, using complex numbers, and including points at infinity.

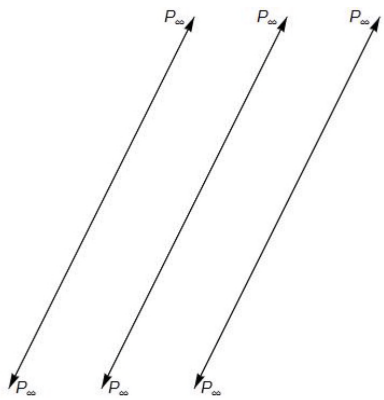

Figure 0.3 – parallel lines meeting at infinity

So given two parallel lines, we will agree that they meet at some idealized point at infinity (Figure 0.3). As lines should only meet once, this point at infinity lies in both directions. Given a third parallel line, it will meet each of these two lines in a point at infinity, and so in fact at the same point at infinity. So to each family of parallel lines there is a single point at infinity.

Put another way there is a point at infinity for each gradient m, that is the lines $y = mx + c$ all meet in the same point at infinity. And we need to remember to allow $m = \infty$ as a possible gradient, relating to the family of parallel vertical lines. These points at infinity make the *line at infinity.*

Note though that these 'points at infinity' aren't special in any way, or rather we've only made them special by our choice of where to put our affine xy-axes. The family of parallel lines passing through a point at infinity, properly judged from infinity, would look the same as the family of lines passing through the origin.

If we return to our earlier examples when Bézout's theorem appeared not to hold:

- $y = 0$, $y^2 = x$. The parabola and line meet a second time at the point at infinity at the 'end' of the x-axis.
- $y = x$, $(y - x)^2 = 1$. The two lines $y = x \pm 1$ both meet $y = x$ at a point at infinity in the same way that $y = 0$ and $y^2 = x^2$ meet at the origin.

We need, then, a rigorous, formal way of introducing these points at infinity if we are to prove geometric results involving them. For fixed m the lines $y = mx + c$ all meet at a point at infinity. This point at infinity is where the points (x, mx) move to as $x \to \pm\infty$. So it's the ratio of x and y that is important here. Somehow we want to include all the points (x, y) of the standard affine plane $\mathbb{R}^2$ and a line at infinity including the points $(\infty, m\infty)$ where $m \in \mathbb{R} \cup \{\infty\}$.

We cannot make easy meaning of $(\infty, m\infty)$ but if we recognize this ∞ as the consequence of some erroneous division by zero, then we can describe our 'extended' plane with the introduction of **homogeneous co-ordinates.**

Definition 0.19 *Given real x_0, x_1, x_2, not all zero, then we write $[x_0 \colon x_1 \colon x_2]$ for the equivalence class of $(x_0, x_1, x_2) \in \mathbb{R}^3 \backslash \{\mathbf{0}\}$ under the equivalence relation*

$$(x_0, x_1, x_2) \sim (\lambda x_0, \lambda x_1, \lambda x_2) \qquad \textit{where } \lambda \neq 0.$$

How does this help us with the previous discussion? Well if $x_0 \neq 0$ then we may divide by x_0 (i.e. set $\lambda = 1/x_0$) to see that such equivalence classes can be represented as $[1 \colon x \colon y]$ where $x = x_1/x_0$ and $y = x_2/x_0$. These are 'most' of the equivalence classes and $[1 \colon x \colon y]$ can be identified with the point $(x, y) \in \mathbb{R}^2$. And the remaining equivalence classes, when $x_0 = 0$ are $[0 \colon 1 \colon m]$ when $x_1 \neq 0$ which corresponds to the point at infinity $(\infty, m\infty)$, and finally $[0 \colon 0 \colon 1]$ which corresponds to '$m = \infty$' the point at infinity of the vertical lines.

Whilst here, and remembering that $x = x_1/x_0$ and $y = x_2/x_0$, we can see that the affine lines $y = mx + c$ would become

$$x_2 = mx_1 + cx_0$$

and that each passes through the point at infinity $[0 \colon 1 \colon m]$. Further the parabola $y^2 = x$ would become $x_2^2 = x_0 x_1$. The variables x_1/x_0 and x_2/x_0 are known as **inhomogeneous co-ordinates.**

So the earlier 'problematic' examples we see now that

- $y = 0$, $y^2 = x$ homogeneously become $x_2 = 0$ and $x_2^2 = x_0 x_1$ so each passes through the point at infinity $[0 \colon 1 \colon 0]$.

- $y = x$, $(y-x)^2 = 1$ homogeneously become $x_2 = x_1$ and $(x_2 - x_1)^2 = x_0^2$ so each passes includes the points at infinity at $[0\colon 1\colon 1]$. Indeed these two curves in a like manner to how $y = 0$ meets with $y^2 = x^2$ at the origin.

0.4 Holomorphic branches

This is material from A2 *Metric Spaces and Complex analysis* we know:

Proposition 0.20 *Let $z \in \mathbb{C}\backslash(-\infty, 0]$.*
(a) Then z can be written as $z = re^{i\theta}$ where $r > 0, \theta \in (-\pi, \pi)$ in a unique fashion.
(b) The function

$$\sqrt{z} = \sqrt{r}e^{i\theta/2}$$

is a holomorphic function on the cut plane $\mathbb{C}\backslash(-\infty, 0]$ with a sign discontinuity over the cut.

Remark 0.21 *If we were to take points z_+ and z_-, respectively just above and below the cut $(-\infty, 0]$ then we would have*

$$z_+ = re^{i\theta_+} \quad \textit{where } \theta_+ \approx \pi; \qquad z_- = re^{i\theta_-} \quad \textit{where } \theta_- \approx -\pi.$$

So with $\sqrt{z}$ as defined above we see

$$\sqrt{z_+} \approx \sqrt{r}e^{i\pi/2} = i\sqrt{r}; \qquad \sqrt{z_-} \approx \sqrt{r}e^{-i\pi/2} = -i\sqrt{r}.$$

We see this time that there is a sign change as we cross the cut.

The only other holomorphic function on $\mathbb{C}\backslash(-\infty, 0]$ which satisfies $w^2 = z$ is $w = -\sqrt{z}$ and these two functions, $\sqrt{z}$ and $-\sqrt{z}$ are the two holomorphic branches of $\sqrt{z}$ on this cut plane. We see that as we cross the cut we move from one branch's values to the other's values.

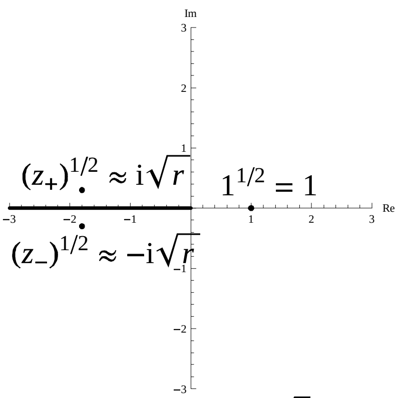

Figure 0.4a: $\sqrt{z}$

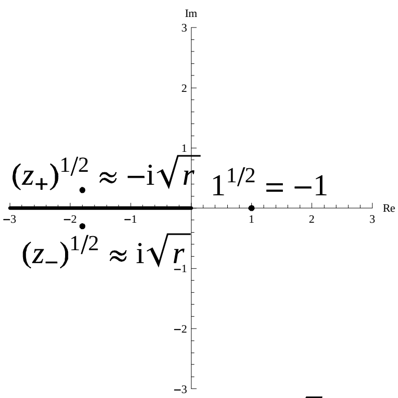

Figure 04.b: $-\sqrt{z}$

Example 0.22 *For z in the cut plane $\mathbb{C}\backslash(-\infty, 1]$ we will let*

$$\begin{aligned} &\theta_1 \quad \textit{denote the value of } \arg(z+1) \quad \textit{in the range} \quad (-\pi, \pi), \\ &\theta_2 \quad \textit{denote the value of } \arg(z-1) \quad \textit{in the range} \quad (-\pi, \pi), \end{aligned}$$

as in the diagram below.

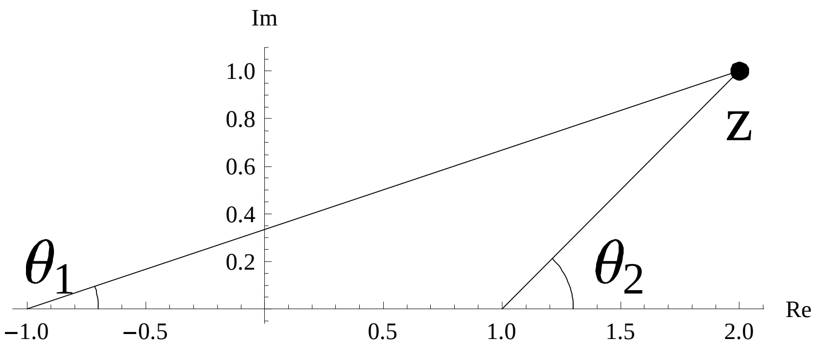

Figure 0.5

So we have

$$(z+1)(z-1) = |z+1|\, e^{i\theta_1}\, |z-1|\, e^{i\theta_2}$$

and

$$w = \sqrt{|z+1|\,|z-1|}e^{i(\theta_1+\theta_2)/2}$$

is a holomorphic function on $\mathbb{C}\backslash(-\infty, 1]$ *which satisfies*

$$w^2 = z^2 - 1.$$

What about the continuity, or otherwise, of w *over the cut? Firstly let* r *be a real number in the range* $-1 < r < 1$ *and let* r_+ *and* r_- *be complex numbers just above and just below* r *in the complex plane. Then*

$$\begin{aligned} \textit{for } r_+ \textit{ we have } \theta_1 &\approx 0 \textit{ and } \theta_2 \approx \pi; \\ \textit{for } r_+ \textit{ we have } \theta_1 &\approx 0 \textit{ and } \theta_2 \approx -\pi. \end{aligned}$$

So

$$\begin{aligned} w_+ &\approx \sqrt{1-r^2}e^{i(0+\pi)/2} = i\sqrt{1-r^2}; \\ w_- &\approx \sqrt{1-r^2}e^{i(0-\pi)/2} = -i\sqrt{1-r^2}. \end{aligned}$$

So we see that we have a sign discontinuity across $(-1, 1)$.

However if we take r *be a real number in the range* $r < -1$ *and let* r_+ *and* r_- *be complex numbers just above and just below* r *in the complex plane. Then*

$$\begin{aligned} \textit{for } r_+ \textit{ we have } \theta_1 &\approx \pi \textit{ and } \theta_2 \approx \pi; \\ \textit{for } r_+ \textit{ we have } \theta_1 &\approx -\pi \textit{ and } \theta_2 \approx -\pi. \end{aligned}$$

So

$$\begin{aligned} w_+ &\approx \sqrt{r^2-1}e^{i(\pi+\pi)/2} = -\sqrt{r^2-1}; \\ w_- &\approx \sqrt{r^2-1}e^{i(-\pi-\pi)/2} = -\sqrt{r^2-1}. \end{aligned}$$

We see that w *is actually continuous across* $(-\infty, -1)$ *and we can in fact extend* w *to a holomorphic function on all of* $\mathbb{C}\backslash\,[-1, 1]$.

Note the behaviour of w *near the points* -1 *and* 1. *If* $z \approx -1$ *then* $w \approx \sqrt{2}i\sqrt{z+1}$ *where* $\sqrt{z+1}$ *is a standard branch of* $\sqrt{z+1}$ *on the cut plane* $\mathbb{C}\backslash[-1, \infty)$. *If* $z \approx 1$ *then* $w \approx \sqrt{2}\sqrt{z-1}$ *where* $\sqrt{z-1}$ *is a standard branch of* $\sqrt{z-1}$ *on the cut plane* $\mathbb{C}\backslash(-\infty, 1]$.

Remark 0.23 *To properly consider the* ***multifunction*** $\sqrt{z^2-1}$ *(or any similar multi-valued function) it helps to consider its* ***Riemann surface****. In this case the (affine) Riemann surface is the set of points*

$$\Sigma = \left\{(w,z) \in \mathbb{C}^2 : w^2 = z^2 - 1\right\}.$$

Firstly consider the situation in $\mathbb{R}^2$*. The curve* $y^2 = x^2 - 1$ *is a hyperbola. Above* $(1,\infty)$ *and* $(-\infty,-1)$ *sit branches* $y = \pm\sqrt{x^2-1}$ *and these two branches meet at* $(\pm 1, 0)$*. So most of the curve is in one or other of the sets*

$$C_+ = \left\{\left(x, \sqrt{x^2-1}\right) \mid |x| > 1\right\}; \qquad C_- = \left\{\left(x, -\sqrt{x^2-1}\right) \mid |x| > 1\right\}.$$

In fact $C_+ \cup C_-$ *excludes only the branch points* $(\pm 1, 0)$ *and we also see that as we cross the branch points we move from* C_+ *to* C_- *(or vice versa).*

In the complex case, for $z \notin [-1,1]$ *there are two values* $\pm w$*. For* $z = \pm 1$ *the only value of* w *is* 0*. The points* (z,w) *and* $(z,-w)$ *have already been described as two different branches of* $\sqrt{z^2-1}$ *but we need to take some care to see how these branches fit together as subset of* Σ*. If we set as above*

$$\Sigma_+ = \{(z,w) \mid z \notin [-1,1]\} \qquad \text{and} \qquad \Sigma_- = \{(z,-w) \mid z \notin [-1,1]\}.$$

Then $\Sigma_+ \cup \Sigma_-$ *is most of* Σ *missing only those points associated with* $z \in [-1,1]$*. We can note, as with previous branches, that as* z *crosses the cut* $[-1,1]$ *then* (z,w) *moves continuously to the other branch* Σ_- *and likewise* $(z,-w)$ *moves continuously to the other branch* Σ_+*.*

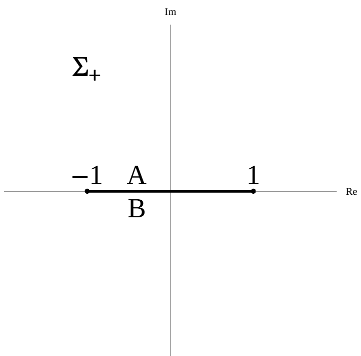

Figure 0.6a

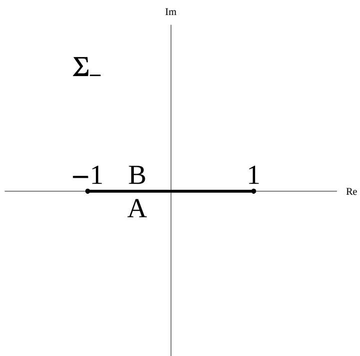

Figure 0.6b

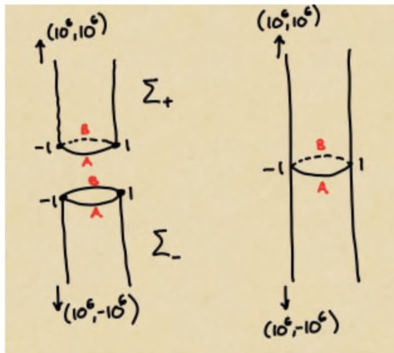

Figure 0.6c

So Σ_+ *and* Σ_- *fit together on* Σ *by gluing either side of* $[-1,1]$ *as labelled in Figures 0.6a/b. We can then see that topologically* Σ *is a cylinder in* $\mathbb{C}^2$ *(Figure 0.6c).*

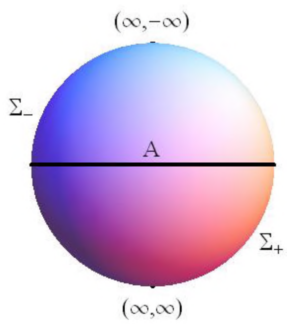

Figure 0.7

Whilst there is only one ∞ in the extended complex plane, note that, as z becomes large, then (z,w) and $(z,-w)$ are diverging points in $\mathbb{C}^2$. So we should introduce two points at infinity to Σ which at either end of the cylinder to reflect this behaviour. Topologically, with these points included, Σ is a sphere (in what is called complex projective space).

More rigorously, considering instead Σ as a subset of the complex projective plane $\mathbb{CP}^2$, the projectivized version of Σ is

$$\Sigma = \left\{[z_0\colon z_1\colon z_2] \in \mathbb{CP}^2 : z_1^2 = z_2^2 - z_0^2\right\}.$$

The line at infinity has equation $z_0 = 0$ and so the two points at infinity are $[0\colon 1\colon 1]$ and $[0\colon 1\colon -1]$.

0.5 Differentiability in $\mathbb{R}^n$

This is material from Part A *Multidimenstional Analysis and Geometry.*

Definition 0.24 *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map, (i.e. all partial derivatives of f of all orders exist everywhere.) Let $\mathbf{p}, \mathbf{v} \in \mathbb{R}^n$ and let $\gamma\colon (-\varepsilon,\varepsilon) \to \mathbb{R}^n$ be a smooth curve in $\mathbb{R}^n$ such that*

$$\gamma(0) = \mathbf{p} \quad \text{and} \quad \gamma'(0) = \mathbf{v}.$$

*Then $f\circ\gamma$ is a smooth curve in $\mathbb{R}^m$. The **differential** of f at p is the linear map $\mathrm{d}f_p\colon \mathbb{R}^n \to \mathbb{R}^m$ defined by*

$$\mathrm{d}f_p(\mathbf{v}) = \mathrm{d}f_p(\gamma'(0)) = (f\circ\gamma)'(0).$$

Proposition 0.25 *$\mathrm{d}f_p(\mathbf{v})$ is independent of the choice of curve γ.*

Proof. For ease of notation we shall consider the case when $m = n = 2$. Write $f = \binom{f_1}{f_2}$ and $\gamma = \binom{\gamma_1}{\gamma_2}$. Then

$$\begin{aligned}
(f\circ\gamma)'(0) &= \begin{pmatrix} (f_1\circ\gamma)'(0) \\ (f_2\circ\gamma)'(0) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial f_1}{\partial x}\gamma_1'(0) + \frac{\partial f_1}{\partial y}\gamma_2'(0) \\ \frac{\partial f_2}{\partial x}\gamma_1'(0) + \frac{\partial f_2}{\partial y}\gamma_2'(0) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.
\end{aligned}$$

As the partial derivatives in the above matrix depend only on the function f and the point p then $\mathrm{d}f_p$ (which we see has the Jacobian as its matrix) is independent of the choice of γ. ■

For those meeting multivariable differentials for the first time, this definition contrasts markedly with the usual notion of a differential $\mathrm{d}f/\mathrm{d}x$. Clearly when $m = n = 1$ then the two definitions agree, but the general differential cannot simply be visualized as a gradient. Rather $\mathrm{d}f_p$ is a first, linear approximation of the function f at p. Here are two examples to help motivate this appreciation.

Example 0.26 *By Taylor's theorem for a smooth function $f = (u, v) : \mathbb{R}^2 \to \mathbb{R}^2$ we have*

$$\begin{aligned} f\begin{pmatrix} x+h \\ y+k \end{pmatrix} &= \begin{pmatrix} u(x+h, y+k) \\ v(x+y, y+k) \end{pmatrix} \\ &= \begin{pmatrix} u(x,y) + hu_x(x,y) + ku_y(x,y) + \cdots \\ v(x,y) + hv_x(x,y) + kv_y(x,y) + \cdots \end{pmatrix} \\ &= \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} + \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} + O\left(|(h,k)|^2\right). \end{aligned}$$

This result generalizes naturally to the general m, n case.

Example 0.27 *For a holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ and $p \in \mathbb{C}$, then*

$$\mathrm{d}f_p = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

where $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$. By the Cauchy-Riemann equations

$$\mathrm{d}f_p = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix} = \lambda \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $\lambda = \sqrt{u_x^2 + v_x^2} = |f'(p)|$ and $\theta = \arg z$. This shows that when $f'(p) \neq 0$, then f is approximately enlarging by $|f'(p)|$ and rotating by $\arg z$.

This can be more easily seen using Taylor's theorem for a holomorphic function in one complex variable. We then have

$$f(p+h) = f(p) + f'(p)h + O\left(|h|^2\right).$$

At the zeroth degree of approximation then p maps to $f(p)$. When we consider nearby points $p+h$ to p, then the first degree approximation is the map to $f(p) + f'(p)h$. The effect of multiplying by $f'(p)$ is a scaling by $|f'(p)|$ and rotation by $\arg f'(p)$.

On occasion we will also find the following result useful.

Theorem 0.28 *(**Inverse function theorem**) Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map defined near p. If $\mathrm{d}f_p$ is invertible then f is a local diffeomorphism. That is there is a smooth map g defined near $f(p)$ such that $g(f(x)) = x$ and $f(g(y)) = y$ for x near p and y near $f(p)$.*

Note that when $f = u + iv$ is holomorphic the determinant $|\mathrm{d}f_p| = u_x^2 + v_x^2 = |f'(p)|^2$ and so f will be a local diffeomorphism if and only if it is conformal at p.

0.6 Identification Spaces

This is material from Part A *Topology.*

Definition 0.29 *Let $(X, \mathcal{T})$ be a topological space and $f\colon X \to Y$ be a map onto a set Y. Then the **quotient topology** on Y is the collection*

$$\tau = \left\{ U \subseteq Y \mid f^{-1}(U) \in \mathcal{T} \right\},$$

and (Y, τ) is called a ***quotient space.***

As pre-image respects unions and intersections then τ is closed under arbitrary unions and finite intersections. Further $f^{-1}(\varnothing) = \varnothing \in \mathcal{T}$ and $f^{-1}(Y) = X \in \mathcal{T}$. Thus τ is a topology.

By definition, $f\colon (X, \mathcal{T}) \to (Y, \tau)$ is continuous. Indeed τ is the finest topology on Y such that f is continuous.

Definition 0.30 *Given an equivalence relation $\sim$ on a topological space $(X, \mathcal{T})$ then there is a natural surjective map*

$$\pi\colon X \to X/\sim \qquad \text{given by} \qquad x \mapsto [x]$$

which sends an element x to its equivalence class $[x]$. In this case $(X/\sim, \tau)$ is referred to as an ***identification space.***

Example 0.31 *The quotient space of any compact (resp. connected) space is compact (resp. connected). This is because the continuous image of a compact (resp. connected) space is compact (resp. connected).*

Example 0.32 *Define $\sim$ on $\mathbb{R}$ by $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Show that $\mathbb{R}/\sim$, which is also written $\mathbb{R}/\mathbb{Z}$, is homeomorphic to the circle S^1.*

Solution. The bijection $\mathbb{R}/\mathbb{Z} \to S^1$ defined by $[x] \mapsto e^{2\pi i x}$ is a homeomorphism. It is an easy check that basic open subsets in the circle correspond to open subsets of $\mathbb{R}$ which are unions of equivalence classes. ■

Example 0.33 *Define $\sim$ on $\mathbb{R}$ by $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Show that $\mathbb{R}/\sim$, which is also written $\mathbb{R}/\mathbb{Q}$, has the trivial topology.*

Solution. Let U be a non-empty open set in $\mathbb{R}/\mathbb{Q}$. Then $U + \mathbb{Q}$ is open in $\mathbb{R}$ and is a union of equivalence classes. But as a non-empty open subset of $\mathbb{R}$ contains a representative of each equivalence class we have $U + \mathbb{Q} = \mathbb{R}$ and hence $U = \mathbb{R}/\mathbb{Q}$. ■

Example 0.34 *Define $\sim$ on $\mathbb{C}$ by $z_1 \sim z_2$ if and only if there exists $\lambda > 0$ such that $z_1 = \lambda z_2$. Show that $\mathbb{C}/\sim$ is not Hausdorff.*

Solution. In a Hausdorff space singleton points are closed. But in $\mathbb{C}/\sim$ the only closed point is $[0]$. Note the closure of $[1]$ is $[1] \cup [0]$. ■