1. INTRODUCTION

Definition 1.1 *A **topological surface**, or **topological 2-manifold**, is a Hausdorff topological space S such that for every $p \in S$ there is an open set $U \subseteq S$ and a homeomorphism $\varphi\colon U \to V$ where V is an open subset of $\mathbb{R}^2$. Such a surface is referred to as an **abstract** topological surface, the term 'abstract' refers to the fact that the surface is not situated (or 'embedded') in some Euclidean space.*

*The map φ is called a **chart** or **patch** and a collection $\{\varphi_i\colon U_i \to V_i\}$ such that*

$$\bigcup_i U_i = S$$

*is known as an **atlas**.*

*The surface S is called **closed** if it is compact.*

Remark 1.2 *In Definition 1.1 we have defined an 'abstract' topological surface. The surface has not been situated in any Euclidean space; the surface's topology is part of the definition, rather than being inherited as a subspace of some ambient space. This may contrast with most previous examples you have of surfaces, especially compared with parameterized surfaces discussed in Prelims Geometry.*

*As a consequence of Whitney's embedding theorem, every (separable) topological surface can be embedded in $\mathbb{R}^3$ or $\mathbb{R}^4$, so the benefit of the above definition may be even less clear. Here an **embedding** is a continuous, injective map which is a homeomorphism between the surface and its image. However these embeddings can often be complicated functions, in which case it's easier to work with an abstract definition. For example the Klein bottle, which we will introduce soon, cannot be embedded in $\mathbb{R}^3$; the hyperbolic plane, which is topologically just $\mathbb{R}^2$, cannot be isometrically embedded in $\mathbb{R}^3$.*

With an atlas we can therefore parameterize the surface S. At this point the atlas provides no further structure to S, which already has a Hausdorff topology. However these parameters provide a useful means with which to define functions on S. But $\mathbb{R}^2$ has (or can have) further structures – smooth, metric, orientable, complex – and we will in due course see how we can use atlases to consistently transfer these structures to surfaces.

Example 1.3 ***(Atlas for the sphere)*** *Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$. As $\mathbb{R}^3$ is Hausdorff then so is S^2, and as S^2 is closed and bounded then it is compact.*

The following six maps form an atlas for S^2.

$$\begin{aligned}
U_1 &= \left\{(x,y,z) \in S^2 \mid z > 0\right\}, \quad \varphi_1(x,y,z) = (x,y);\\
U_2 &= \left\{(x,y,z) \in S^2 \mid z < 0\right\}, \quad \varphi_2(x,y,z) = (x,y);\\
U_3 &= \left\{(x,y,z) \in S^2 \mid x > 0\right\}, \quad \varphi_3(x,y,z) = (y,z);\\
U_4 &= \left\{(x,y,z) \in S^2 \mid x < 0\right\}, \quad \varphi_4(x,y,z) = (y,z);\\
U_5 &= \left\{(x,y,z) \in S^2 \mid y > 0\right\}, \quad \varphi_5(x,y,z) = (x,z);\\
U_6 &= \left\{(x,y,z) \in S^2 \mid y < 0\right\}, \quad \varphi_6(x,y,z) = (x,z).
\end{aligned}$$

In each case $V_i = \varphi_i(U_i)$ is the open unit disc in $\mathbb{R}^2$. As $x^2 + y^2 + z^2 = 1$ for any $(x, y, z) \in S^2$ then at least one of the co-ordinates is non-zero, meaning every point of S lies in at least one patch.

We have thus shown S^2 to be a topological surface. Note an atlas for S^2 cannot consist of a single chart $\varphi\colon S^2 \to V$ as S^2 is compact and V is not, but it's not hard to find an atlas consisting of two charts.

Example 1.4 ***(Bug-eyed plane)*** *The following example shows the necessity of the requirement that S be Hausdorff. Consider $S = X/\sim$ where $X = \mathbb{R}^2 \times \{\pm 1\}$ and every point $(x, y, -1)$ is identified with $(x, y, 1)$ except when $x = y = 0$. The space S is then not Hausdorff as the two origins $(0, 0, \pm 1)$ cannot be separated but the two charts $\varphi_{\pm 1}(x, y, \pm 1) = (x, y)$ form an atlas for S.*

Proposition 1.5 *Let S be a topological surface with atlas $\{\varphi_i\colon U_i \to V_i\}$. Let $f\colon S \to T$ be a map to a topological space T. Then f is continuous if and only if each $f \circ \varphi_i^{-1}\colon V_i \to T$ is continuous.*

Proof. If f is continuous then $f \circ \varphi_i^{-1}$ is the composition of two continuous maps and therefore continuous. Conversely suppose all these maps are continuous and take $p \in S$. As we have an atlas then $p \in U_i$ for some i and then $f = f \circ \varphi_i^{-1} \circ \varphi_i$ is continuous at p. ■

Example 1.6 *The* ***real projective plane*** *$\mathbb{P} = S^2/\{\pm 1\}$ is the space formed by identifying antipodal points of the sphere. Find an atlas for $\mathbb{P}$.*

Solution. Each equivalence class of points in $\mathbb{P} = S^2/\{\pm 1\}$ has a representative in one (or more) of the domains U_1, U_3, U_5 previously used in Example 1.3 to cover the sphere. Given a point where $z \neq 0$, we can assume in fact that $z > 0$ without loss of generality. Then the maps

$$\psi_1(x, y, z) = (x, y), \quad \psi_3(x, y, z) = (y, z), \quad \psi_5(x, y, z) = (x, z),$$

form an atlas for $\mathbb{P}$. ■

In the Prelims Geometry course, the definition of a parameterized surface was as follows.

Definition 1.7 *A* ***smooth parameterized surface*** *is a map,*

$$\mathbf{r}\colon U \to \mathbb{R}^3 \qquad (u, v) \mapsto (x(u, v), y(u, v), z(u, v))$$

from an open subset $U \subseteq \mathbb{R}^2$ to $\mathbb{R}^3$ such that

- $\mathbf{r}$ *is smooth i.e. x, y, z have continuous partial derivatives of all orders,*
- $\mathbf{r}\colon U \to \mathbf{r}(U)$ *is a homeomorphism,*
- ***(smoothness condition)*** *at each point of $\mathbf{r}(U)$ the vectors*

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} \quad \text{and} \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$$

  are linearly independent.

Comparing this with our earlier definition of a topological surface, we note that $\mathbf{r}^{-1}\colon \mathbf{r}(U) \to U$ is a chart forming an atlas by itself. So parameterized surfaces in $\mathbb{R}^3$ are examples of topological surfaces. However the adjective *smooth* suggests that we have more structure now than a topological surface generally has. The independence of the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ means that the surface has a well-defined *tangent plane* and *normal* at each point. But it's currently unclear how we might generalize this notion to a topological surface S that is not situated in Euclidean space. Around each point $p \in S$ we can assign co-ordinates via a chart $\varphi\colon U \to V$ and so it might seem reasonable to say that a function $f\colon S \to \mathbb{R}$ is smooth at p if

$$f \circ \varphi^{-1}\colon V \to \mathbb{R}$$

is smooth. Recall that V is an open subset of $\mathbb{R}^2$ so this would just mean that $f \circ \varphi^{-1}$ has partial derivatives of all orders. The catch is that, when p is in the domain of more than one chart, f might be deemed to be smooth at p using one chart and not smooth using another chart. We need to ensure we have consistency across the surface.

Definition 1.8 *Given an atlas $\{\varphi_i\colon U_i \to V_i\}$ for a topological surface S, if $U_i \cap U_j \neq \varnothing$ then*

$$\varphi_i \circ \varphi_j^{-1}\colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

*is known as a **transition map**.*

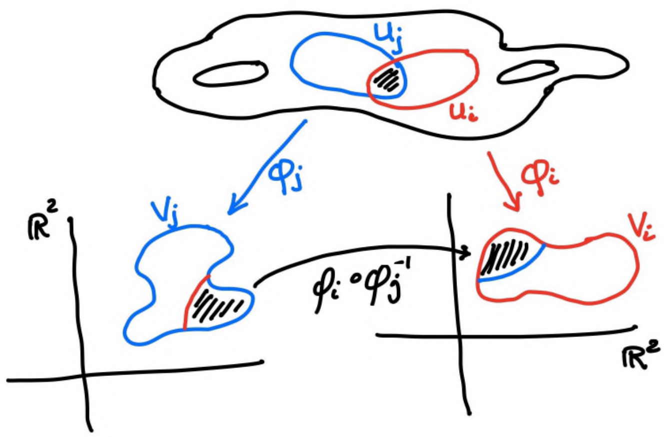

Figure 1.1 – a transition map

For a topological surface the transition maps are always homeomorphisms. So if $f \circ \varphi_i^{-1}$ is continuous then $f \circ \varphi_j^{-1}$ automatically is too. But we need to further require the transition maps to be smooth, to have a consistent notion of smoothness across a surface. Hence we define:

Definition 1.9 *A **differentiable surface**, or **differentiable 2-manifold**, is a topological surface S with an atlas $\{\varphi_i\colon U_i \to V_i\}$ such that all the transition maps*

$$\varphi_i \circ \varphi_j^{-1}\colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

*are smooth. Such an atlas is called a **differentiable structure** on S.*

Definition 1.10 *(a) Let S be a differentiable surface with atlas $\{\varphi_i \colon U_i \to V_i\}$. We define $f \colon S \to \mathbb{R}$ to be* ***smooth*** *at $p \in U_i$ if*

$$f \circ \varphi_i^{-1} \colon V_i \to \mathbb{R}$$

is smooth. A quick check shows there is no possibility of inconsistency.

(b) Let Σ be a second differentiable surface with atlas $\{\psi_i \colon A_i \to B_i\}$ and let $f \colon S \to \Sigma$ be a map between the surfaces. Let $p \in S$, so that $p \in U_i$ for some i, and then $f(p) \in B_j$ for some j. We define f to be ***smooth*** *at p if*

$$\psi_j \circ f \circ \varphi_i^{-1}$$

is smooth at $\varphi_i(p)$. As the transition maps are smooth there is again no chance of inconsistency.

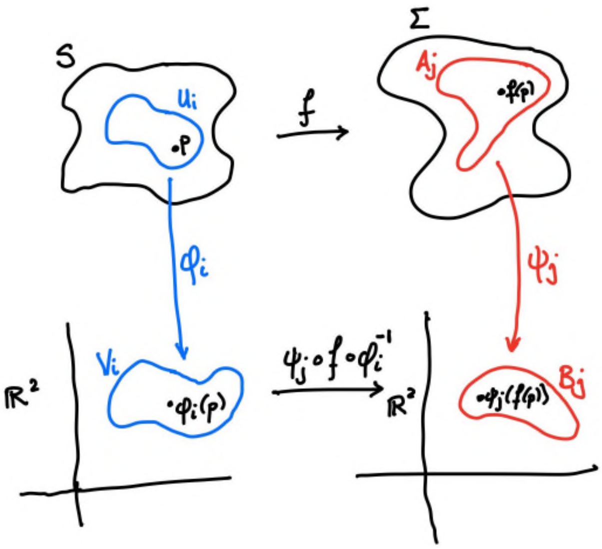

Figure 1.2 – a map between surfaces

Remark 1.11 *(**Consistency and other structures**) Note that a transition map ofr a differentiable surface is bijective, is smooth, and its inverse – another transition map – is also smooth. That is to say that the transition maps are* ***diffeomorphisms****.*

We can now see how the previous definitions can be generalized to higher dimensions to define topological manifolds and smooth manifolds. Despite surfaces being the focus of much study in the eighteenth and nineteenth centuries – by Euler, Lagrange, Monge, Gauss, Riemann, Möbius, et al. – a formal definition of surfaces (and manifolds) did not arise until the 1930s, variously due to Whitehead, Whitney and Veblen.

The transition maps are the key to assigning structures to a surface beyond the purely topological. Requiring the transition maps to be smooth means we can consistently define a smooth structure on the whole surface. But $\mathbb{R}^2$ naturally has other structures:

- *metric structure – we would then need the transition maps to be isometries;*
- *orientability – we would then need the transition maps to be orientation-preserving;*

- *complex structure – we can identify $\mathbb{R}^2$ with $\mathbb{C}$ and would then need the transition maps to be biholomorphic (that is, conformally equivalent).*

Note that a single patch of surface can be assigned any of these structures. However for a general topological surface, it may not be possible to endow a surface globally with certain structures precisely because of its topology. The real projective plane cannot be consistently oriented; the sphere cannot be given a metric structure with everywhere 'negative curvature'. When we meet Riemann surfaces later we will see there are a great deal of differences between complex structures and real smooth ones. In higher dimensions, these problems are yet more complicated and subtle.

We say a little now about how a metric structure can be assigned to a co-ordinate patch of a surface. We will revisit these ideas in detail in Chapter 3. We have already noted that $\mathbf{r}_u(p)$ and $\mathbf{r}_v(p)$ are independent tangent vectors of a point p in a co-ordinate patch $\mathbf{r}(U) \subseteq \mathbb{R}^3$. Given a curve $\gamma(t) = \mathbf{r}(u(t), v(t))$ where $a \leqslant t \leqslant b$ then, by the chain rule,

$$\dot{\gamma}(t) = \dot{u}\mathbf{r}_u + \dot{v}\mathbf{r}_v$$

and

$$|\dot{\gamma}(t)|^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$$

where

$$E = \mathbf{r}_u \cdot \mathbf{r}_u, \qquad F = \mathbf{r}_u \cdot \mathbf{r}_v, \qquad G = \mathbf{r}_v \cdot \mathbf{r}_v.$$

Definition 1.12 *The quadratic form $I_p \colon T_p \to \mathbb{R}$,*

$$\alpha\mathbf{r}_u + \beta\mathbf{r}_v \mapsto |\alpha\mathbf{r}_u + \beta\mathbf{r}_v|^2 = E\alpha^2 + 2F\alpha\beta + G\beta^2$$

on the tangent space $T_p = \langle \mathbf{r}_u, \mathbf{r}_v \rangle$ is known as the ***first fundamental form****. Any property of a surface that can be expressed in terms of the first fundamental form is said to be* ***intrinsic****.*

The first fundamental form expresses how the co-ordinate domain has been curved on to the surface. All metric properties of the surface can be expressed in terms of the first fundamental form. We will need to consider quite what we mean by tangent spaces when we have an abstract surface, rather than one situated in $\mathbb{R}^3$, but we will deal with that in Chapter 3. In the meantime note that lengths and areas can be expressed in terms of the first fundamental form; importantly these definitions apply whatever Euclidean space the surface is situated in.

The **length** of the above curve γ equals

$$\mathcal{L}(\gamma) = \int_a^b |\dot{\gamma}(t)| \, \mathrm{d}t = \int_a^b \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2} \, \mathrm{d}t.$$

We have also previously defined the **area** of $\mathbf{r}(U)$ by

$$\mathcal{A} = \iint_U |\mathbf{r}_u \wedge \mathbf{r}_v| \, \mathrm{d}u \, \mathrm{d}v.$$

One issue with this definition is that the vector product $\wedge$ is defined in $\mathbb{R}^3$ but not generally in higher dimensions. However, the scalar quadruple product gives

$$\begin{aligned} |\mathbf{r}_u \wedge \mathbf{r}_v|^{\,2} &= (\mathbf{r}_u \wedge \mathbf{r}_v) \cdot (\mathbf{r}_u \wedge \mathbf{r}_v) \\ &= (\mathbf{r}_u \cdot \mathbf{r}_u)(\mathbf{r}_v \cdot \mathbf{r}_v) - (\mathbf{r}_u \cdot \mathbf{r}_v)(\mathbf{r}_v \cdot \mathbf{r}_u) \\ &= EG - F^2. \end{aligned}$$

Hence we can instead define the area of $\mathbf{r}(U)$ as

$$\mathcal{A} = \iint_U \sqrt{EG - F^2}\, \mathrm{d}u\, \mathrm{d}v,$$

a definition which is well-defined whatever $\mathbb{R}^n$ the surface is situated in.

Let's conclude this introduction by considering the transition maps for the atlases we previously defined for the sphere and real projective plane.

Example 1.13 *(The sphere reprised.) Consider the two charts*

$$\begin{aligned} U_1 &= \left\{(x,y,z) \in S^2 \mid z > 0\right\}, \quad \varphi_1(x,y,z) = (x,y); \\ U_3 &= \left\{(x,y,z) \in S^2 \mid x > 0\right\}, \quad \varphi_3(x,y,z) = (y,z). \end{aligned}$$

So $U_1 \cap U_3 = \{(x,y,z) \in S^2 \mid x, z > 0\}$ *is an open quarter of the sphere and*

$$\begin{aligned} \varphi_1(U_1 \cap U_3) &= \left\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1, x > 0\right\} \\ \varphi_3(U_1 \cap U_3) &= \left\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1, y > 0\right\} \end{aligned}$$

and

$$(u(x,y), v(x,y)) = \varphi_1 \circ \varphi_3^{-1}(x,y) = \left(\sqrt{1 - x^2 - y^2}, x\right).$$

Note that the Jacobian of this map equals

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{--x}{\sqrt{1-x^2-y^2}} & \frac{-y}{\sqrt{1-x^2-y^2}} \\ 1 & 0 \end{vmatrix} = \frac{y}{\sqrt{1-x^2-y^2}} > 0.$$

That this is non-zero means that the transition map is smooth. That it is positive means that the transition map is orientation preserving. As this is true of the other transition maps too, then we have given the sphere the structure of an oriented differentiable surface.

Example 1.14 *(The real projective plane revisited.) Recall the charts*

$$\begin{aligned} \psi_1(x,y,z) &= (x,y), \quad \textit{where } z > 0 \textit{ by assumption WLOG;} \\ \psi_5(x,y,z) &= (x,z), \quad \textit{where } y > 0 \textit{ by assumption WLOG.} \end{aligned}$$

So $U_1 \cap U_5$ *consists of those* $[x\colon y\colon z]$ *where* $y \neq 0 \neq z$ *and* $x^2 + y^2 + z^2 = 1$. *Then*

$$\begin{aligned} \psi_1(U_1 \cap U_5) &= \left\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1, y \neq 0\right\}, \\ \psi_5(U_1 \cap U_5) &= \left\{(x,z) \in \mathbb{R}^2 \mid x^2 + z^2 < 1, z \neq 0\right\}, \end{aligned}$$

so in fact $\psi_1(U_1 \cap U_5) = \psi_5(U_1 \cap U_5)$. *Then*

$$(u(x,y), v(x,y)) = \psi_1 \circ \psi_5^{-1}(x,y) = (x,y),$$

as

$$(x,y) \overset{\psi_5^{-1}}{\mapsto} \left[x \colon \sqrt{1-x^2-y^2} \colon y\right] \overset{\psi_1}{\mapsto} \begin{cases} \left(x, \sqrt{1-x^2-y^2}\right) & \text{when } y > 0; \\ \left(-x, -\sqrt{1-x^2-y^2}\right) & \text{when } y < 0. \end{cases}$$

The Jacobian of this map when $y > 0$ *equals*

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & \frac{-x}{\sqrt{1-x^2-y^2}} \\ 0 & \frac{-y}{\sqrt{1-x^2-y^2}} \end{vmatrix} = \frac{-y}{\sqrt{1-x^2-y^2}} < 0.$$

That this is non-zero means that the transition map is smooth. That it is negative means that the transition map is orientation-reversing. As the transition maps are all smooth then we have endowed $\mathbb{P}$ *with a differentiable structure. As this particular transition map is orientation-reversing then we have not endowed* $\mathbb{P}$ *with an oriented structure. It is then a somewhat harder matter to show that* ***no*** *oriented atlas exists.*