2. TOPOLOGICAL SURFACES

Recall from the introductory lecture the definition of a topological surface.

A **topological surface**, or **topological 2-manifold**, is a Hausdorff topological space S such that for every $p \in S$ there is an open set $U \subseteq S$ and a homeomorphism $\varphi\colon U \to V$ where V is an open subset of $\mathbb{R}^2$.

A surface S is called **closed** if it is compact. In this chapter we discuss the *classification of closed topological surfaces up to homeomorphism*. So two topological surfaces are to be considered the same if they are homeomorphic; the 'classification' then means providing a comprehensive list of the different homeomorphism classes with no omissions and no duplications.

This material was discussed at some length in the A5 topology course. The closed surfaces there were created as *identification spaces* (or quotient spaces) from closed polygons. Two examples are given below.

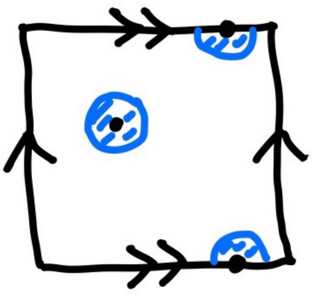


Figure 2.1 – torus

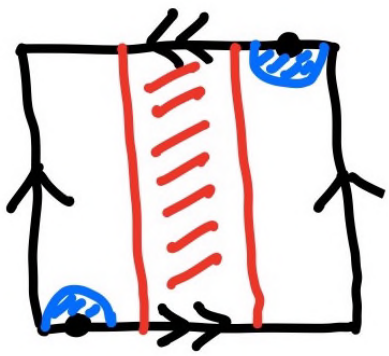


Figure 2.2 – Klein bottle

In Figure 2.1 a torus is formed by pairwise identifying the edges of the square $[0,1]^2$ as described by the arrows. So $(0,y)$ and $(1,y)$ are identified for $0 \leqslant y \leqslant 1$ (the single arrows) and $(x,0)$ and $(x,1)$ are identified for $0 \leqslant x \leqslant 1$ (the double arrows). The square is compact and so the resulting identification space also is. Around each interior point of the square we can associate an open disc U; points on the square's boundary can be associated with a disc split as two semi-discs as sketched in Figure 2.1.

Similarly, in Figure 2.2 a Klein bottle is formed by pairwise identifying the edges of the square $[0,1]^2$ as described by the arrows. So $(0,y)$ and $(1,y)$ are identified for $0 \leqslant y \leqslant 1$ (the single arrows) and $(x,0)$ and $(1-x,1)$ are identified for $0 \leqslant x \leqslant 1$ (the double arrows). The square is compact and so the resulting identification space also is. Around each interior point of the square we can associate an open disc U; points on the square's boundary can be associated with a disc split as two semi-discs as sketched in Figure 2.2. Note in the case of a boundary point on the bottom/top edges the semi-discs are not directly opposite one another because of the reverse identification. Note further, because of this reversed identification, the shaded rectangle is in fact a Möbius strip, rather than a cylinder.

Further examples include the

- **torus with** $g \geqslant 0$ **holes** or, equally, sphere with g handles (Figure 1.3);
- **sphere with** $k \geqslant 1$ **cross-caps** (Figure 1.4).

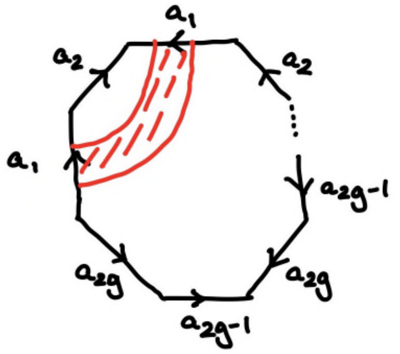


Figure 2,3 – torus with g holes

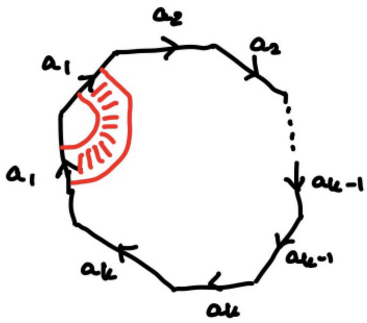


Figure 2.4 – sphere with k cross-caps

The torus with zero holes is the sphere. The torus with $g \geqslant 1$ can be formed by pairwise identifying the edges of a $4g$-gon as shown in Figure 2.3. Note that, in each case, the shaded region connecting identified edges is a cylinder. Consequently the torus with g holes is an orientable surface. This canonical identification is can be denoted

$$a_1 a_2 a_1^{-1} a_2^{-1} a_3 a_4 a_3^{-1} a_4^{-1} \cdots a_{2g-1} a_{2g} a_{2g-1}^{-1} a_{2g}^{-1}.$$

Each string $a_i a_{i+1} a_i^{-1} a_{i+1}^{-1}$ represents a further hole or handle being attached to the surface. See Proposition 2.9.

The sphere with $k \geqslant 1$ cross-caps can be formed by pairwise identifying the edges of a $2k$-gon as shown in Figure 2.4. Note that, in each case, the shaded bar connecting identified edges is a Möbius strip. Consequently the sphere with $k \geqslant 1$ cross-caps is an non-orientable surface. This canonical identification is denoted

$$a_1 a_1 a_2 a_2 \cdots a_k a_k.$$

A cross-cap is formed in the sphere by making a cut and identified the cut's two sides in reverse orientation. This is the equivalent of sewing a Möbius strip into the sphere, which is what each string $a_i a_i$ represents. See Proposition 2.9.

Example 2.1 *The Klein bottle* $\mathbb{K}$ *is homeomorphic to the sphere with 2 cross-caps.*

Solution. These two versions of the Klein bottle can be transformed into one another as shown below.

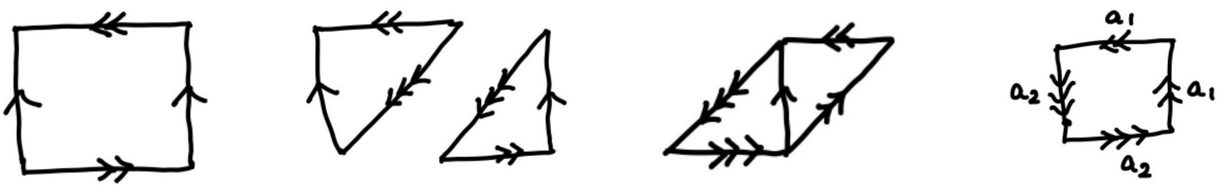


Figure 2.5 – equivalent Klein bottles

This transforms $\mathbb{K}$ from the surface in Figure 2.2 to the surface $a_1 a_1 a_2 a_2$, the sphere with two cross-caps. ■

In the A5 topology course it was rigorously shown that:

- Every closed topological surface is homeomorphic to one of (a) a torus with $g \geqslant 0$ holes or (b) a sphere with $k \geqslant 1$ cross-caps.

This is half the classification theorem. The above is a comprehensive list of all closed topological surfaces up to homeomorphisms. There are no omissions but there may yet be duplications. We need one or more topological invariants which can be used to distinguish between the homeomorphism classes listed above. The two invariants we shall use are *orientability* and the *Euler characteristic.*

We already introduced the notion of orientability in the introductory lecture; a differentiable surface was orientable if it had an atlas with orientation-preserving transition map. We shall use, in this chapter, an equivalent criterion for orientability. This second definition of orientability is due to Klein.

Proposition 2.2 *A differentiable surface is non-orientable if and only if it contains a Möbius strip.*

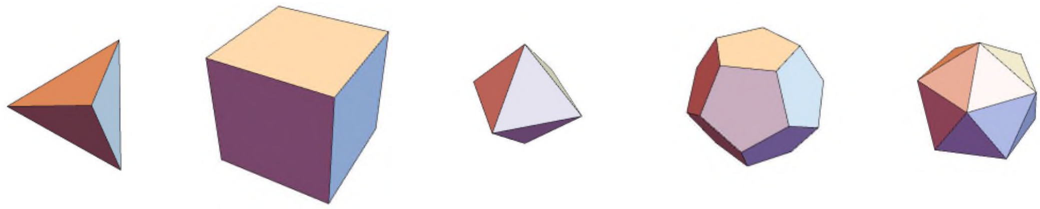
Proof. Say that a surface includes a Möbius strip. Then we can take an orientation-reversing curve along the Möbius strip and consider the co-ordinate patches it passes through (which can be taken to be finite by compactness). Each transition map between patches cannot be orientation-preserving or else the curve would not be orientation-reversing.

Conversely, suppose that the surface contains no Möbius strip and so no orientation-reversing curve. Make a choice of orientation at a fixed point. Any other point can be connected by a path to the fixed point and the chosen orientation can be extended consistently to the second point. Thus the surface is orientable. ■

So the Klein bottle, and more generally, the spheres with k cross-caps are therefore non-orientable. As the tori with n holes can be embedded in $\mathbb{R}^3$ then they are orientable; we can consistently associate an outward-pointing normal on the entirety of such a surface. Thus orientability separates out the closed surfaces into two families, but we need a further invariant to separate the orientable surfaces from one another and likewise separate out the non-orientable surfaces. This invariant is the *Euler characteristic.*

You may well be aware that for the Platonic solids $V - E + F = 2$ where V, E, F respectively denote the number of vertices, edges and faces on the solid.

surface	V	E	F
tetrahedron	4	6	4
cube	8	12	6
octahedron	6	12	8
dodecahedron	20	30	12
icosahedron	12	30	20



tetrahedron cube octahedron dodecahedron icosahedron

Figure 2.6 – the Platonic solids

Indeed this relation is true for any polyhedron of the same *shape* (such as any pyramid or a cuboid). That is $V - E + F$ will equal 2 for any polyhedron that is homeomorphic to a sphere. So this number 2 is known as the Euler characteristic of the sphere.

Remark 2.3 *Euler arrived at his $V - E + F = 2$ formula for convex polyhedra in 1750 (in a letter to Goldbach) and this is arguably one of the first topological results. It is, in fact, equivalent to a result of Descartes' from 1639 but Euler's formulation of the result was more obviously topological in nature. The formula had been noted as early as 1537 by Francesco Maurolico. In 1811 Cauchy gave a semi-rigorous proof of the formula, though it would not be considered watertight by modern standards.*

We need to be a little careful in how we assign vertices, edges and faces to the surface. For example, were we to assign no vertices and no edges to a sphere and treat the entire surface as a face then we would arrive at an Euler characteristic of $0 - 0 + 1 = 1 \neq 2$, so presumably this should not be permitted. Likewise a single edge as an equator, no vertices and two hemispherical faces gives $0 - 1 + 2 = 1 \neq 2$ and should again not be admissable. The important point is that our vertices, edges and faces make a *subdivision* of the surface.

Definition 2.4 *Let X be a closed topological surface.*

(a) An ***edge*** *on X is the image of a continuous map $f\colon [0,1] \to X$ which is 1-1 except possibly that $f(0) = f(1)$.*

(b) A ***subdivision*** *of X is a finite set of edges, together with a finite set of points of X, called* ***vertices*** *(singular: vertex), such that*

(i) each edge begins and ends in a vertex and passes through no other vertices;

(ii) two edges intersect, at most, at their ends;

(iii) if Γ is the union of the edges then each connected component of $X \backslash \Gamma$ is homeomorphic to $\mathbb{R}^2$.

(c) The closure of a connected component of $X \backslash \Gamma$ is known as a ***face.***

With the earlier inadmissible examples: we cannot use the entire surface of the sphere as a face as it is not homeomorphic to $\mathbb{R}^2$ invalidating (iii) – if we included a single solitary vertex on the sphere we would then have a valid subdivision; for the second example the edge does not begin and end in a vertex invalidating (i) – if we included a vertex on the edge then we would have a valid subdivision.

For those that did A5 we note the following:

Example 2.5 *If a topological surface is the realisation $|K|$ of a simplicial complex K then the simplicial complex is a valid subdivision of $|K|$ with the 0-simplices as vertices, the 1-simplices as edges and the 2-simplices as faces.*

The important result – which we shall not prove in this course – is the following:

Theorem 2.6 *Let X be a topological surface. Then the number*

$$\chi(X) = V - E + F$$

is the same for any subdivision, where V, E, F are respectively the number of vertices, edges and faces in the subdivision. The number $\chi(X)$ is known as the ***Euler characteristic*** *of X, and also sometimes as its* ***Euler number*** *or its* ***Euler-Poincaré characteristic****.*

Consequently the Euler characteristic is a topological invariant of the surface – that is, it is preserved by homeomorphisms.

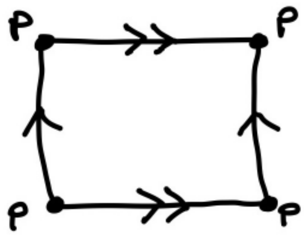


Figure 2.7 – torus

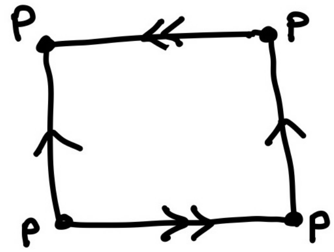


Figure 2.8 – Klein bottle

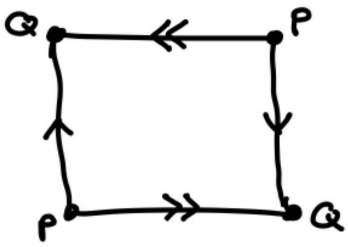


Figure 2.9 – projective plane

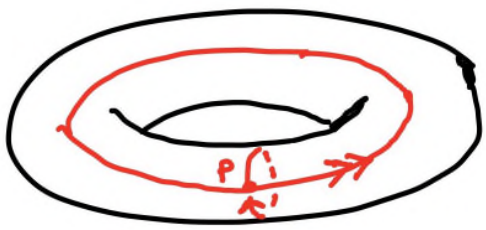


Figure 2.10 – torus with subdivision

Example 2.7 *Find the Euler characteristics of (a) the torus, (b) the Klein bottle, (c) the projective plane.*

Solution. Each of these surfaces begin with a square face, bounded by four edges and four vertices. The important point is how many vertices and edges remain after the identifications are made. In each case there is just one face, the square itself.

(a) For the torus $\mathbb{T}$ (Figure 2.7) the four edges are pairwise identified to leave two edges – the single arrows and the double arrows. Following the identifications around the four vertices are all identified to become a single vertex P. How these edges and vertices would look on a torus is drawn in Figure 2.10. This means that the Euler characteristic of the torus is

$$\chi(\mathbb{T}) = 1 - 2 + 1 = 0.$$

(b) For the Klein bottle $\mathbb{K}$ (Figure 2.8) the four edges are pairwise identified to leave two edges – the single arrows and the double arrows. Following the identifications around the four vertices are all identified to become a single vertex P. This means that the Euler characteristic of the Klein bottle is

$$\chi(\mathbb{K}) = 1 - 2 + 1 = 0.$$

So $\mathbb{T}$ and $\mathbb{K}$ have the same Euler characteristic despite not being homeomorphic – $\mathbb{T}$ is orientable, whilst $\mathbb{K}$ is not.

(c) For the projective plane $\mathbb{P}$ (Figure 2.8) the four edges are pairwise identified to leave two edges – the single arrows and the double arrows. Following the identifications around the four vertices become identified a two vertices, P and Q. This means that the Euler characteristic of the Klein bottle is

$$\chi(\mathbb{P}) = 2 - 2 + 1 = 1.$$

It's apparent from the identification that $\mathbb{P}$ is the sphere with 1 cross-cap. Just treat the single and double arrows as one edge and we see that $\mathbb{P}$ is the surface $a_1 a_1$. ■

Example 2.8 *Find the Euler characteristic of the surface created from the three polygons below. Is the surface orientable?*

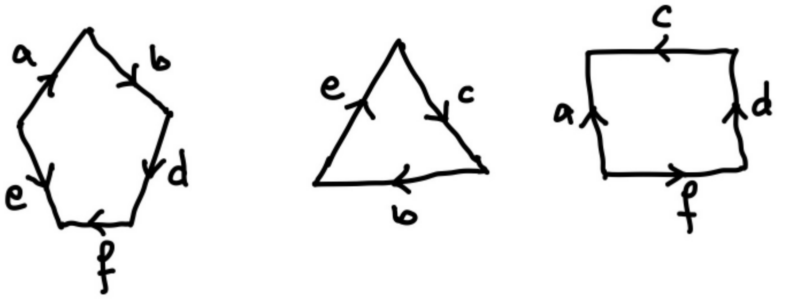


Figure 2.11 – a more complicated example

Solution. The surface, as drawn, comes with a natural subdivision. There are 3 faces – the pentagon, triangle and square – and 6 edges, namely a, b, c, d, e, f. It's not immediately clear how the original 12 vertices identify though.

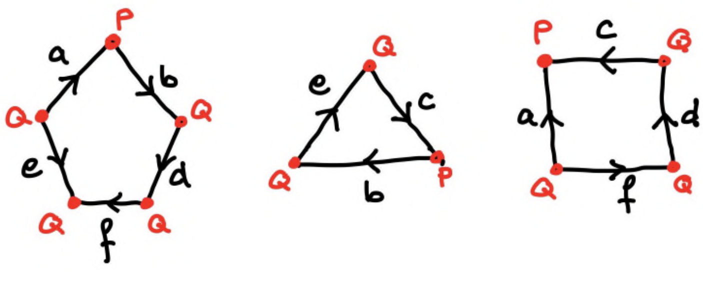


Figure 2.12 – counting the vertices

If we label the vertex at the top of the pentagon as P then, by following around the identifications, we can see what other vertices it is identified with. P is at the front end of a and so we follow around the identifications

$$\text{front of } a \;\rightarrow\; \text{front of } c \rightarrow \text{back of } b \rightarrow \text{front of } a$$

and we are back where we started. So the three vertices labelled P in Figure 2.12 are identified together. Labelling another vertex Q we can follow around the identifications and see in this case that the remaining 9 vertices are identified with Q. Thus there are 2 vertices once identified and we find

$$\chi = 2 - 6 + 3 = -1.$$

We cannot immediately see whether there is a Möbius strip within the surface as each edge is identified with an edge on a different face. However if we bring the pentagon and triangle together as in Figure 2.13

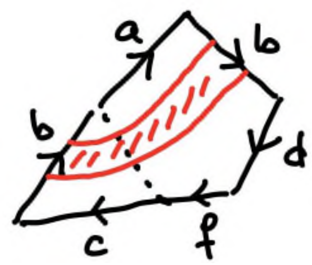


Figure 2.13 – non-orientability

we now see that the senses of the two b-edges are the same or equivalently the shaded region is a Möbius strip. Thus the surface is non-orientable. ■

Proposition 2.9 *(a) Adding a handle to a surface reduces the Euler characteristic by* 2.
(b) Adding a cross-kap to a surface reduces the Euler characteristic by 1.

Proof. (a) As shown in Figure 2.14 a handle can be added to a surface and subdivided with two further edges. The vertex shown is already part of the original surface's subdivision. As E increases by 2 then $V - E + F$ reduces by 2.

(b) As shown in Figure 2.15 a cross-cap can be added to a surface and subdivided using two new edges and a new vertex. The unlabelled vertex shown is already part of the original surface's subdivision. As E increases by 2 and V by 1 then $V - E + F$ reduces by 1 overall. ■

Corollary 2.10 *(a) The Euler characteristic of the torus with $g \geqslant 0$ holes equals $2 - 2g$.*
(b) The Euler characteristic of the sphere with $k \geqslant 1$ cross-caps equals $2 - k$.

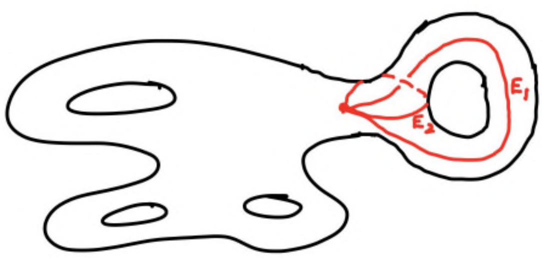


Figure 2.14 – adding a handle

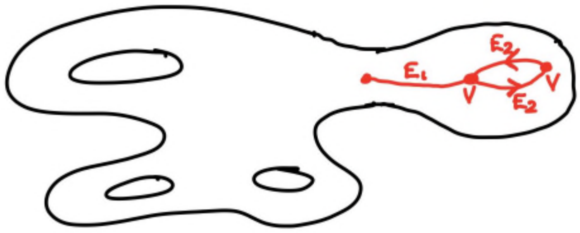


Figure 2.15 – adding a cross-cap

We are now in a position to state the classification theorem as we see that, between them, orientability and the Euler characteristic are enough to distinguish the homeomorphism classes.

Theorem 2.11 ***(Classification Theorem for Closed Surfaces)*** *Let X be a closed topological surface. Then X is homeomorphic to precisely one of the following.*

(a) If X is orientable, then X is homeomorphic to a torus with $g \geqslant 0$ holes. g is called the ***genus*** *of X.*

(b) If X is non-orientable, then X is homeomorphic to a sphere with $k \geqslant 1$ cross-caps.

Proof. From the A5 result we know that X is homeomorphic to one of these surfaces. None of the surfaces in list (a) is homeomorphic to a surface in list (b) by orientability. Further the Euler characteristics of the surfaces in list (a) are distinct, thus separating them topologically. And the same can be said of the surfaces in list (b). ■

Remark 2.12 *It is worth noting that the early topologists who 'proved' the classification theorem did not have available in their time the rigorous definitions necessary to prove their results to modern standards. In 1861 Möbius gave an early sketch proof of the classification for orientable surfaces, and Von Dyck gave a sketch proof for all closed surfaces in 1888. But without any formal definition of what a surface is, these proofs can at best be considered incomplete. Somewhat differently expressed rigorous versions of the classification theorem would be proved by Dehn and Heegaard in 1907 and by Brahana in 1921.*

Remark 2.13 *The above classification theorem relates to closed topological surfaces up to homeomorphism; we could easily consider instead closed differentiable surfaces up to diffeomorphism and the classification theorem would essentially read the same. The situation is similar in 3 dimensions but there are topological 4-manifolds which admit no differentiable structure and others which admit many; indeed there are 'exotic' versions of $\mathbb{R}^4$ which are homeomorphic to the standard $\mathbb{R}^4$ but not diffeomorphic to it.*

When it comes to 'complex structures' on surfaces the situation is very different and considerably more subtle. Riemann surfaces are necessarily orientable so no complex structure can be given to a sphere with k cross-caps. Only one structure, up to biholomorphism, can be put on the sphere but uncountably many can be put on a torus.

Remark 2.14 *We now see that the surface created in Example 2.8 is a sphere with 3 cross-caps. Indeed having worked out that the Euler characteristic equals -1 we did not need to determine the orientability as this is the only surface, up to homeomorphism, with this Euler characteristic.*

Remark 2.15 *Euler noted his formula for polyhedra that are topologically a sphere around 1750. The French-Swiss mathematician, Simon Lhuilier, noted in 1812 that $V-E+F = 2(1-g)$ when a polyhedron has g holes – this number g is called the polyhedron's genus.*

A modern demonstration of the topological invariance of the Euler characteristic usually appears in an algebraic topology course – see the Part C course of that name. In fact, the Euler characteristic is a homotopy invariant – homotopy equivalence is a more general notion than that of being homeomorphic. The Euler characteristic of a surface is the alternating sum of its Betti numbers.

$$\chi = b_0 - b_1 + b_2.$$

For an n-dimensional manifold, Betti numbers $b_0, b_1, \ldots, b_n$ can be defined which are the ranks of the manifold's homology groups which are topological invariants by definition. For the torus with g holes we have

$$b_0 = 1, \qquad b_1 = 2g, \qquad b_2 = 1,$$

giving $\chi = 2 - 2g$. That $b_0 = 1$ signifies the surface to be connected and that $b_2 = 1$ signifies that it has an 'inside' or is orientable. b_1 equalling $2g$ represents the loops that go through or go around each of the g holes. For the sphere with k cross-caps,

$$b_0 = 1, \qquad b_1 = k - 1, \qquad b_2 = 0,$$

giving $\chi = 2 - k$. That $b_0 = 1$ signifies connectedness and $b_2 = 0$ signifies non-orientability. b_1 equalling $k - 1$ represents $k - 1$ that loops are (in some technical sense) independent. Much of this early work was due to Poincaré around the end of the nineteenth century and the start of the twentieth and consequently the Euler characteristic is commonly referred to as the Euler-Poincaré characteristic.

More complicated surfaces can be created from simpler ones using the *connected sum*.

Definition 2.16 *Given two closed topological surfaces X_1 and X_2, their connected sum $X_1 \# X_2$ is created by removing two small discs, one from each surface, and identifying the circumferences of the two discs.*

Note that $X_1 \# X_2$ is orientable if and only if X_1 and X_2 are both orientable. The Euler characteristic of the connected sum can be quickly determined – as below – and we can then see that the torus $\mathbb{T}$ and projective plane $\mathbb{P}$ can be used as the building blocks for general closed topological surfaces.

Theorem 2.17 *Let X_1 and X_2 be closed topological surfaces. Then*

$$\chi(X_1 \# X_2) = \chi(X_1) + \chi(X_2) - 2.$$

Corollary 2.18 *(a) For $g \geqslant 0$, $\chi(\mathbb{T}^{\#g}) = 2 - 2g$.*
(b) For $k \geqslant 1$, $\chi(\mathbb{P}^{\#k}) = 2 - k$.

Proof. Say that X_i has subdivisions with V_i, E_i, F_i vertices, edges and faces and suppose that one of the faces in each subdivision is a triangle. When those two triangles are removed, and their boundaries identified, then 6 vertices become 3, 6 edges become 3 and 2 faces are lost. Thus

$$V_\# = V_1 + V_2 - 3, \qquad E_\# = E_1 + E_2 - 3, \qquad F_\# = F_1 + F_2 - 2$$

so that

$$\begin{aligned} \chi(X_1 \# X_2) &= (V_1 + V_2 - 3) - (E_1 + E_2 - 3) + (F_1 + F_2 - 2) \\ &= (V_1 - E_1 + F_1) + (V_2 - E_2 + F_2) - 2 \\ &= \chi(X_1) + \chi(X_2) - 2. \end{aligned}$$

The corollaries then follow by induction noting

$$\begin{aligned}\chi(\mathbb{T}^{\#g}) &= \chi(\mathbb{T}^{\#g-1}\#\mathbb{T}) = \chi(\mathbb{T}^{\#g-1}) + 0 - 2 = \chi(\mathbb{T}^{\#g-1}) - 2;\\ \chi(\mathbb{P}^{\#k}) &= \chi(\mathbb{P}^{\#k-1}\#\mathbb{P}) = \chi(\mathbb{P}^{\#k-1}) + 1 - 2 = \chi(\mathbb{P}^{\#k-1}) - 1,\end{aligned}$$

with the initial steps verified by

$$\begin{aligned}\chi(\mathbb{T}^{\#0}) &= \chi\left(S^2\right) = 2 = 2 - 2 \times 0;\\ \chi(\mathbb{P}^{\#1}) &= \chi(\mathbb{P}) = 1 = 2 - 1.\end{aligned}$$

The corollaries are essentially alternative proofs of Proposition 2.9. ■

We shall see later,in Chapter 5, with the Gauss-Bonnet theorem, the Poincaré-Hopf theorem and in elements of Morse theory, that the Euler characteristic is a topological obstruction to the global analysis and geometry of a surface.