3. SMOOTH AND GEOMETRIC SURFACES

In the introductory lecture we recalled the definition of a parameterized surface and introduced the notion of a differentiable structure on a surface. Here we will mainly be discussing the *local* geometric structure of surfaces, so it will be sufficient to focus on parameterized surfaces, though we will wish to make sure our definitions are not dependent on the choice of parameterization.

Definition 3.1 *Let $\mathbf{r}(U)$ be a smooth parameterized surface in $\mathbb{R}^3$ and let $p = \mathbf{r}(u_0, v_0)$. The **tangent plane** to $\mathbf{r}(U)$ at p is the plane through p that is parallel to*

$$\mathbf{r}_u(u_0, v_0) \quad \text{and} \quad \mathbf{r}_v(u_0, v_0).$$

*The **tangent space** $T_p(\mathbf{r}(U))$ is the vector space spanned by the above two vectors and any element of $T_p(\mathbf{r}(U))$ is called a **tangent vector.** It is easy to check that the tangent space at p consists of all the tangent vectors to all curves in $\mathbf{r}(U)$ which pass through p.*

Note that a parameterized surface is a surface in $\mathbb{R}^3$ with a preferred choice of co-ordinates from a particular chart $\mathbf{r}^{-1}$. But $\mathbf{r}(U)$ can also be associated with other charts, technically giving a different parameterized surface but we would hope that any questions asked of X (simply as a subspace of $\mathbb{R}^3$) such as, 'what is the area of X?' and 'what is the length of a curve in X?', will yield the same answers, irrespective of what chart we use. This will be an important consideration in all future definitions, namely that any new definitions are chart independent.

Proposition 3.2 *The tangent space is independent of the choice of parameterization.*

Proof. Let $\mathbf{r}(U) = \mathbf{s}(X)$ be two parameterizations

$$(u, v) \mapsto \mathbf{r}(u, v), \qquad (x, y) \mapsto \mathbf{s}(x, y).$$

If we have $\mathbf{r}(u, v) = \mathbf{s}(x, y)$ then by the chain rule

$$\mathbf{r}_u = x_u \mathbf{s}_x + y_u \mathbf{s}_y, \qquad \mathbf{r}_v = x_v \mathbf{s}_x + y_v \mathbf{s}_y.$$

Applying the vector product, we find

$$\mathbf{r}_u \wedge \mathbf{r}_v = (x_u y_v - x_v y_u)\, \mathbf{s}_x \wedge \mathbf{s}_y = \frac{\partial(x, y)}{\partial(u, v)} \mathbf{s}_x \wedge \mathbf{s}_y$$

are parallel. This is the normal direction to the tangent space which we see is also independent of the choice of parameterization. ■

Definition 3.3 *Let $\mathbf{r}(U)$ be a smooth parameterized surface in $\mathbb{R}^3$. A **normal vector** to $\mathbf{r}(U)$ at the point p is any (non-zero) vector orthogonal to $T_p(\mathbf{r}(U))$.*

The normal vectors are non-zero scalar multiples of $\mathbf{r}_u \wedge \mathbf{r}_v$ where $\wedge$ denotes the vector product in $\mathbb{R}^3$. The two unit vectors

$$\pm \frac{\mathbf{r}_u \wedge \mathbf{r}_v}{|\mathbf{r}_u \wedge \mathbf{r}_v|}$$

are the choices of **unit normal** to $\mathbf{r}(U)$ at p.

Definition 3.4 *The map* $\mathbf{n}$ *from* $\mathbf{r}(U)$ *to* S^2*, the unit sphere, which continuously sends* $\mathbf{r}(u, v)$ *to a unit normal* $\mathbf{n}(u, v)$ *is called the **Gauss map**.*

The definition of the differential of a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ then extends to maps between parameterized surfaces in an obvious way.

Definition 3.5 *Let* X *and* Y *be smooth parameterized surfaces in* $\mathbb{R}^3$ *and let* $p \in X$*. For a smooth map* $f : X \to Y$ *(see Definition 1.10). Then the **differential** of* f *at* p *is the linear map*

$$\mathrm{d}f_p : T_pX \to T_{f(p)}Y$$

defined as follows. Let $\mathbf{v} \in T_pX$ *and let* $\gamma : (-\epsilon, \epsilon) \to X$ *be a smooth curve such that*

$$\gamma(0) = p \quad \text{and} \quad \gamma'(0) = \mathbf{v}.$$

Then $f \circ \gamma$ *is a smooth curve in* Y *and as before we define*

$$\mathrm{d}f_p(v) = \mathrm{d}f_p(\gamma'(0)) = (f \circ \gamma)'(0).$$

A quick check shows that this definition is independent of the choice of curve γ*.*

Before we discuss any of the theory of surfaces, we should introduce some standard examples. We have already introduced differentiable atlases for the sphere and real projective plane, but we introduce two other parameterizations for (most of) the sphere here.

Example 3.6 *(**Parameterizing the sphere**) Consider the map* $\mathbf{r}_1 : (-\pi, \pi) \times (0, \pi) \to \mathbb{R}^3$ *(see Figure 3.1) given by*

$$\mathbf{r}_1 : (u, v) \mapsto (\cos u \sin v, \sin u \sin v, \cos v).$$

It is easy to check that the image of this map is contained in S^2*, the unit sphere centred at the origin. In fact the image is the whole sphere save for half a great circle. The parameter* u *is the angle between the projection of* $\mathbf{r}_1(u, v)$ *onto the xy-plane and the x-axis and* v *is the angle between* $\mathbf{r}_1(u, v)$ *and the z-axis.*

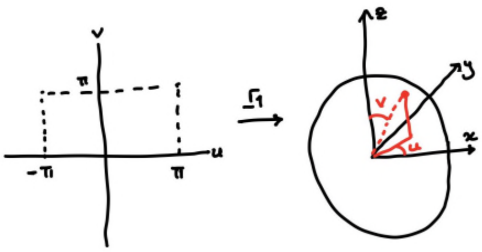

Figure 3.1 – spherical polars

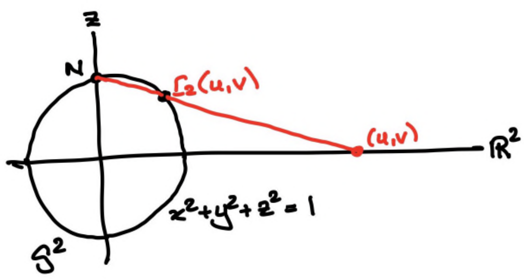

Figure 3.2 – stereographic projection

Consider also the map $\mathbf{r}_2 : \mathbb{R}^2 \to \mathbb{R}^3$ *defined by*

$$\mathbf{r}_2 : (u, v) \mapsto \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right).$$

This again is a chart of the unit sphere. The map $\mathbf{r}_2$ *is in fact stereographic projection (see Figure 3.2) from the 'north pole'* $N = (0,0,1)$*; that is a point of* $(u,v) \in \mathbb{R}^2$ *is mapped to the intersection of the sphere with the line joining* $(u,v,0)$ *and* N*. In this case the image of the sphere is the whole sphere minus* N*. This map is particularly relevant for setting up the extended complex plane with the* ***Riemann sphere****, a first example of a compact Riemann surface,*

Example 3.7 *(**Graphs**) Amongst the simplest examples of parameterized surfaces are graphs. Let* $f(x,y)$ *be a smooth function defined on an open set* $U \subseteq \mathbb{R}^2$*. Then the* ***graph*** *of* f *is the surface* $z = f(x,y)$ *and may be parameterized by*

$$\mathbf{r}(u,v) = (u, v, f(u,v)), \qquad (u,v) \in U.$$

These graphs seem almost too simple a family of surfaces to be of interest. One point of importance though is that any smooth surface in $\mathbb{R}^3$ *is, locally at least, a graph. That is:*

- *About any point of a smooth surface in* $\mathbb{R}^3$ *there is an open neighbourhood* U *such that* U *is a graph of the form* $z = f(x,y)$ *or* $y = f(x,z)$ *or* $x = f(y,z)$ *for some smooth function* f*. (Do Carmo, p.63).*

Indeed, for a general smooth surface $(x(u,v), y(u,v), z(u,v))$, *provided the normal is not horizontal or equivalently*

$$\frac{\partial(x,y)}{\partial(u,v)} = x_u y_v - x_v y_u \neq 0,$$

then the surface can be locally parameterized as $z = f(x,y)$ *for some* f*.*

Example 3.8 *(**The cone**) The punctured cone* $x^2 + y^2 = z^2$, $(z > 0)$ *in* $\mathbb{R}^3$ *may be smoothly parameterized by*

$$\mathbf{r}(u,v) = (u, v, \sqrt{u^2 + v^2}), \qquad u, v \in \mathbb{R},\ u^2 + v^2 \neq 0.$$

Note that the two sheeted cone $x^2 + y^2 = z^2$ *is not the image of any parameterization as no neighbourhood of the cone about* $(0,0,0)$ *is homeomorphic to an open subset of* $\mathbb{R}^2$*. (To see this consider the topological effect of removing the origin.)*

Consider now the one sheeted cone C *given by* $x^2 + y^2 = z^2$, $(z \geqslant 0)$*. This certainly is the image of a parameterization* $\mathbf{s}\colon \mathbb{R}^2 \to C$*, but for no such map is* C *smooth at the point* $(0,0,0)$*. To prove this we assume that the cone may be locally parameterized about* $(0,0,0)$ *as the graph of a smooth function. The only possibility (from* $z = f(x,y)$ *or* $y = f(x,z)$ *or* $x = f(y,z)$*) is a graph of the form* $z = f(x,y)$ *and by the definition of* C *we see that*

$$f(x,y) = \sqrt{x^2 + y^2}.$$

As f *is not differentiable at* $(0,0)$ *then* $(0,0,0)$ *is not a smooth point of* C *for any parameterization. Such points on a surface are called* ***singular points****.*

Example 3.9 *(**Surfaces of revolution**) Surfaces may also be formed by taking a curve in $\mathbb{R}^3$ and using this curve to generate a surface. One such family are the surfaces of revolution. A **surface of revolution** is formed by rotating a smooth curve in, say, the xz-plane about the z-axis. For example, the cylinder in the above exercise is a surface of revolution.*

Assume the curve has equation $x = f(z) > 0$. Then the surface of revolution generated has equation $x^2 + y^2 = f(z)^2$. The surface cannot entirely be parameterized with one co-ordinate system but the map

$$\mathbf{r}(\theta, z) = (f(z)\cos\theta, f(z)\sin\theta, z), \qquad \theta \in (0, 2\pi), z \in \mathbb{R}$$

*parameterizes all of the surface except for the original generating curve. The curves of the form $\theta = const.$ are called **meridians;** this includes the original generating curve (where $\theta = 0$). Those curves with equations $z = const.$ are called **parallels**.*

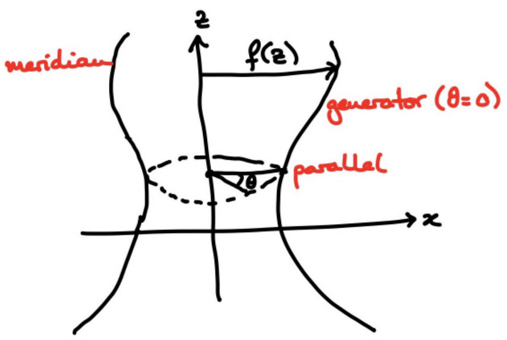

Figure 3.3 – surface of revolution

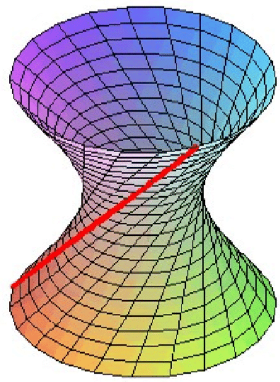

Figure 3.4 – hyperboloid of one sheet

Example 3.10 *(**Ruled surfaces**) Let $\gamma\colon I \to \mathbb{R}^3$ be a smooth curve in $\mathbb{R}^3$ and let $\mathbf{w}\colon I \to \mathbb{R}^3\backslash\{\mathbf{0}\}$ be a second non-vanishing vector function on I. Then the parameterized surface given by*

$$\mathbf{r}(u, v) = \gamma(u) + v\mathbf{w}(u) \quad u \in I, v \in \mathbb{R}$$

*is an example of a **ruled surface**. The curve γ is known as the directrix and the lines in the surface given by $u =$ constant are known as rulings.*

Note that the parameterization $\mathbf{r}$ above need not be a homeomorphism onto its image and so such a ruled surface may have self-intersections, although these may be avoided by limiting the domain of the co-ordinate v. For example, the image of the map

$$\mathbf{r}(u, v) = (v\cos u, v\sin u, v), \qquad u \in (0, 2\pi), v \in \mathbb{R},$$

is all of the two sheeted cone except for two rays (two halves of the line $x = z$). The map $\mathbf{r}$ is not a parameterization as $(0, 0, 0)$ is a self-intersection. However the restriction of $\mathbf{r}$ to $(0, 2\pi) \times (0, \infty)$ is a valid parameterization for the one sheeted cone with the omission of a single ruling.

Exercise 3.11 *Show that the hyperbolic paraboloid $z = xy$ and the hyperboloid of one sheet $x^2 + y^2 = z^2 + 1$ in $\mathbb{R}^3$ are ruled surfaces.*

3.1 The First Fundamental Form

Let $U \subseteq \mathbb{R}^2$ be an open subset of the plane and $\mathbf{r}\colon U \to \mathbb{R}^3$ be a parameterization of a smooth surface X. Let

$$\gamma\colon I \to X \quad \text{be given} \quad \gamma(t) = \mathbf{r}(u(t), v(t))$$

be a smooth curve lying in X.

Definition 3.12 *We define the **length** of γ to be*

$$\mathcal{L}(\gamma) = \int_I \left| \frac{\mathrm{d}\gamma}{\mathrm{d}t} \right| \mathrm{d}t. \tag{3.1}$$

Using the chain rule it is easy to see that the length of γ does not depend on the choice of parameter t. Now

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}t}\frac{\partial \mathbf{r}}{\partial u} + \frac{\mathrm{d}v}{\mathrm{d}t}\frac{\partial \mathbf{r}}{\partial v}$$

or written more concisely

$$\dot{\gamma} = \dot{u}\mathbf{r}_u + \dot{v}\mathbf{r}_v.$$

So the length of γ equals

$$\int_I \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\, \mathrm{d}t \tag{3.2}$$

where

$$E = \mathbf{r}_u \cdot \mathbf{r}_u, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v, \quad G = \mathbf{r}_v \cdot \mathbf{r}_v.$$

Definition 3.13 *The quadratic form $I_p\colon T_pX \to \mathbb{R}$ on the tangent space T_pX, defined by*

$$I\left({}_p\alpha\mathbf{r}_u + \beta\mathbf{r}_v\right) = E\alpha^2 + 2F\alpha\beta + G\beta^2$$

*is called the **first fundamental form** of X.*

Remark 3.14 *What does this actually mean? The first fundamental form is the restriction to T_pX of the quadratic form*

$$\mathbf{x} \mapsto |\mathbf{x}|^2.$$

Now $\{\mathbf{r}_u, \mathbf{r}_v\}$ is a basis for the tangent space and with respect to this basis the first fundamental form has coefficients $E, 2F$ and G. Geometrically it can be thought of as the square of the element of arc length, often conveyed as

$$\mathrm{d}s^2 = E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2.$$

For $X = \mathbf{r}(U)$, a smooth parameterized surface, let

$$u\colon \mathbf{r}(u,v) \mapsto u \quad \text{and} \quad v\colon \mathbf{r}(u,v) \mapsto v$$

denote the co-ordinate maps. For $p = \mathbf{r}(u_0, v_0)$, consider the differentials $\mathrm{d}u_p, \mathrm{d}v_p\colon T_pX \to \mathbb{R}$. We define two curves along the co-ordinate curves through p. Set

$$\begin{aligned} \gamma(t) &= \mathbf{r}(u_0 + t, v_0), \qquad t \in (-\epsilon, \epsilon), \\ \Gamma(t) &= \mathbf{r}(u_0, v_0 + t), \qquad t \in (-\epsilon, \epsilon). \end{aligned}$$

Note that $\gamma'(0) = \mathbf{r}_u(p)$ *and* $\Gamma'(0) = \mathbf{r}_v(p)$. *So*

$$\begin{aligned} \mathrm{d}u_p(\mathbf{r}_u) &= \mathrm{d}u_p(\gamma'(0)) = (u \circ \gamma)'(0) = (t \mapsto u_0 + t)'(0) = 1, \\ \mathrm{d}u_p(\mathbf{r}_v) &= \mathrm{d}u_p(\Gamma'(0)) = (u \circ \Gamma)'(0) = (t \mapsto u_0)'(0) = 0. \end{aligned}$$

Similarly $\mathrm{d}v_p(\mathbf{r}_u) = 0$ *and* $\mathrm{d}v_p(\mathbf{r}_v) = 1$. *So* $\mathrm{d}u_p$ *and* $\mathrm{d}v_p$ *are elements of the dual tangent space* T_pX; *in fact they are the dual basis of* $\{\mathbf{r}_u(p), \mathbf{r}_v(p)\}$. *So* $E\mathrm{d}u_p^2 + 2F\mathrm{d}u_p\mathrm{d}v_p + G\mathrm{d}v_p^2$ *is the quadratic form on* T_pX *given by*

$$I_p \colon \alpha\mathbf{r}_u + \beta\mathbf{r}_v \mapsto E\alpha^2 + 2F\alpha\beta + G\beta^2.$$

However one thinks about the first fundamental form, remember that the form is associated with the surface. When we change co-ordinates the quadratic form does not change, but its expression will generally look different in terms of the new co-ordinates.

Example 3.15 *Find the first fundamental form of the plane using (a) Cartesian co-ordinates and (b) polar co-ordinates.*

Solution. Using Cartesian co-ordinates we find

$$\mathbf{r}(u, v) = (u, v), \qquad u, v \in \mathbb{R}$$

and with polar co-ordinates

$$\mathbf{R}(r, \theta) = (r\cos\theta, r\sin\theta), \qquad r > 0, \theta \in (0, 2\pi).$$

So

$$\begin{aligned} \mathbf{r}_u &= (1, 0) & \mathbf{r}_v &= (0, 1), \\ \mathbf{R}_r &= (\cos\theta, \sin\theta), & \mathbf{R}_\theta &= (-r\sin\theta, r\cos\theta). \end{aligned}$$

With respect to the two co-ordinate systems the first fundamental form is:

$$\mathrm{d}u^2 + \mathrm{d}v^2 \qquad \text{and} \qquad \mathrm{d}r^2 + r^2\mathrm{d}\theta^2.$$

■

Remark 3.16 *It is always possible to introduce local co-ordinates such that the first fundamental form has certain preferential forms.*

- *[Do Carmo, p.183] There exists a local parameterization around any point of a surface such that* $F = 0$. *Such a parameterization is called* ***orthogonal***.
- *[Do Carmo, p.227] There exists a local parameterization around any point of a surface such that* $F = 0$ *and* $E = G$. *Such a parameterization is called* ***isothermal***. *This is equivalent to the parameterization being conformal from the plane; the existence of isothermal co-ordinates implies all smooth surfaces are locally conformal.*
- *[Do Carmo, p.287] Using geodesic polar co-ordinates, it is possible to parameterize a surface locally such that* $E = 1$ *and* $F = 0$.

The following argument was previously given in Prelims Geometry as a definition for area. Let $V \subseteq U$ be an open subset of U; we wish to calculate the area of $\mathbf{r}(V)$. Consider a small parallelogram with vertices

$$\mathbf{r}(u,v), \quad \mathbf{r}(u+\delta u, v), \quad \mathbf{r}(u, v+\delta v), \quad \mathbf{r}(u+\delta u, v+\delta v).$$

Now

$$\mathbf{r}(u+\delta u, v) - \mathbf{r}(u,v) = \mathbf{r}_u(u,v)\delta u + O(\delta u^2)$$

and there is a similar expression for varying v. So the area of the parallelogram is, ignoring higher order terms,

$$|\mathbf{r}_u \wedge \mathbf{r}_v|\, \delta u\, \delta v.$$

It thus seems reasonable to define:

Definition 3.17 *The **area** of* $\mathbf{r}(V)$ *equals*

$$\iint_V |\mathbf{r}_u \wedge \mathbf{r}_v|\, \mathrm{d}u\, \mathrm{d}v. \tag{3.3}$$

Now

$$\begin{aligned} |\mathbf{r}_u \wedge \mathbf{r}_v|^2 &= (\mathbf{r}_u \wedge \mathbf{r}_v) \cdot (\mathbf{r}_u \wedge \mathbf{r}_v) \\ &= (\mathbf{r}_u \cdot \mathbf{r}_u)(\mathbf{r}_v \cdot \mathbf{r}_v) - (\mathbf{r}_u \cdot \mathbf{r}_v)(\mathbf{r}_v \cdot \mathbf{r}_u) \\ &= EG - F^2. \end{aligned}$$

Thus the expression (3.3) for the area of $\mathbf{r}(V)$ can be rewritten as

$$\iint_V \sqrt{EG - F^2}\, \mathrm{d}u\, \mathrm{d}v. \tag{3.4}$$

See Proposition 0.11 for a proof that this definition is independent of the choice of co-ordinates.

Example 3.18 *Show that the area of a sphere of radius a equals $4\pi a^2$.*

Solution. We may parameterize the sphere using spherical polar co-ordinates

$$\mathbf{r}(u,v) = (a\cos u \sin v, a \sin u \sin v, a\cos v), \qquad u \in (-\pi, \pi),\ v \in (0, \pi),$$

omitting only half a great circle. Then

$$\begin{aligned} \mathbf{r}_u &= (-a\sin u \sin v, a\cos u \sin v, 0), \\ \mathbf{r}_v &= (a\cos u \cos v, a \sin u \cos v, -a\sin v). \end{aligned}$$

Thus (with respect to the co-ordinates u and v) the first fundamental form is given by

$$E = a^2 \sin^2 v, \quad F = 0, \quad G = a^2$$

and the area is given by

$$\int_0^\pi \int_{-\pi}^{\pi} a^2 |\sin v|\, \mathrm{d}u\, \mathrm{d}v = 2\pi a^2 \int_0^\pi \sin v\, \mathrm{d}v = 4\pi a^2$$

as required. ■

Example 3.19 *The* ***tractoid*** *(see Figure 3.5) is the surface of revolution formed by rotating the curve*

$$x(t) = -(\cos t + \log \tan \frac{t}{2}), \qquad y(t) = \sin t, \qquad t \in (0, \pi/2)$$

(known as the ***tractrix****) about the x-axis.*
(a) Show that, when the tractrix is parameterized by arc-length s, the first fundamental form of the tractoid is

$$\mathrm{d}s^2 + e^{-2s}\mathrm{d}\theta^2. \tag{3.5}$$

(b) Show that the area of the tractoid equals 2π.

Solution. (a) We may parameterize the tractoid by writing

$$\mathbf{r}(t, \theta) = (x(t), y(t)\cos\theta, y(t)\sin\theta), \qquad t \in (0, \infty),\ \theta \in (0, 2\pi),$$

omitting only the original tractrix. Differentiating with respect to t and θ we find that

$$\begin{aligned} \mathbf{r}_t &= (-\cos t \cot t, \cos t \cos\theta, \cos t \sin\theta), \\ \mathbf{r}_\theta &= (0, -\sin t \sin\theta, \sin t \cos\theta). \end{aligned}$$

Thus the first fundamental form is given by

$$\cot^2 t \,\mathrm{d}t^2 + \sin^2 t \,\mathrm{d}\theta^2. \tag{3.6}$$

Now

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \left(\frac{\cos^2 t}{\sin t}\right)^2 + \cos^2 t = \cot^2 t\left(\cos^2 t + \sin^2 t\right) = \cot^2 t.$$

As s is decreasing with respect to t then $\mathrm{d}s/\mathrm{d}t = -\cot t$ and hence $s = -\log\sin t$. Substituting these expressions into (3.6) we obtain $E = 1, F = 0, G = e^{-2s}$ as in (3.5).

(b) The area of the tractoid is then given by the integral

$$\int_0^\infty \int_0^{2\pi} e^{-s}\,\mathrm{d}\theta\,\mathrm{d}s = 2\pi.$$

■

Exercise 3.20 *Show that the area of the torus in $\mathbb{R}^3$, given by*

$$\mathbf{r}(u, v) = ((a + b\cos v)\cos u, (a + b\cos v)\sin u, b\sin v)$$

for $u, v \in (0, 2\pi)$ and $a > b > 0$, equals $4\pi^2 ab$.

Properties of surfaces which depend solely on the first fundamental such as length and area (and geodesics and Gaussian curvature – see later) are called **intrinsic**. Maps between surfaces which preserve the intrinsic geometry are called *isometries.*

Definition 3.21 *An* ***isometry*** *between two surfaces X and Y is a diffeomorphism $f\colon X \to Y$ which maps curves in X to curves in Y of the same length. X and Y are then said to be isometric.*

As the first fundamental form represents an element of arc length then the following theorem should be intuitively clear.

Theorem 3.22 *Two smooth, parameterized surfaces X and Y are isometric if and only if there exists an open subset $U \subset \mathbb{R}^2$ and parameterizations*

$$\mathbf{r}\colon U \to X, \qquad \mathbf{s}\colon U \to Y,$$

such that the first fundamental forms of X and Y are the same.

Proof. Sufficiency is straightforward. Suppose two such parameterizations $\mathbf{r}$ and $\mathbf{s}$ exist with the same fundamental forms – I claim $f = \mathbf{sr}^{-1}\colon X \to Y$ is the required isometry. Let C be a smooth curve in U. The lengths of $\mathbf{r}(C)$ and $\mathbf{s}(C) = f(\mathbf{r}(C))$ are identical as they are given by the same integral (3.2).

Conversely, suppose now that $f\colon X \to Y$ is an isometry of two smooth, parameterized surfaces and suppose that $\mathbf{r}\colon U \to X$ is a parameterization of X. Let $\mathbf{s} = f\mathbf{r}\colon U \to Y$. We shall write $E, 2F, G$ and $\tilde{E}, 2\tilde{F}, \tilde{G}$ for the coefficients of the first fundamental forms of X and Y with respect to $\mathbf{r}$ and $\mathbf{s}$. As f is an isometry we have that

$$\int_a^b \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\,\mathrm{d}t = \int_a^b \sqrt{\tilde{E}\dot{u}^2 + 2\tilde{F}\dot{u}\dot{v} + \tilde{G}\dot{v}^2}\,\mathrm{d}t \tag{3.7}$$

for **all** smooth curves $(u(t), v(t))$, $a \leqslant t \leqslant b$, in U.

As the above is an identity for all b then

$$\sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2} = \sqrt{\tilde{E}\dot{u}^2 + 2\tilde{F}\dot{u}\dot{v} + \tilde{G}\dot{v}^2}$$

And choosing part of a co-ordinate curve, namely: $u(t) = u_0 + t$ and $v(t) = v_0$ in U, it follows that $E = \tilde{E}$. By a similar argument using $u =$const. curves we may conclude that $G = \tilde{G}$. Finally then $F = \tilde{F}$. ■

Example 3.23 *The* ***catenoid*** *(with a meridian removed) and* ***helicoid*** *are respectively parameterized by*

$$\begin{aligned}
\mathbf{r}(u, v) &= (u, \cosh u, \cos v, \cosh u \sin v), \qquad u \in \mathbb{R},\ v \in (0, 2\pi)\,, \\
\mathbf{s}(\tilde{u}, \tilde{v}) &= (\tilde{u}, \tilde{v}\cos\tilde{u}, \tilde{v}\sin\tilde{u}), \qquad \tilde{u} \in \mathbb{R},\ \tilde{v} \in \mathbb{R}.
\end{aligned}$$

Show that the catenoid is isometric to part of the helicoid, in such a way that meridians of the catenoid map to rulings of the helicoid.

Solution. The first fundamental form of the catenoid equals

$$\cosh^2 u\,\mathrm{d}u^2 + \cosh^2 u\,\mathrm{d}v^2$$

and the first fundamental form of the helicoid equals

$$(1 + \tilde{v}^2)\,\mathrm{d}\tilde{u}^2 + \mathrm{d}\tilde{v}^2. \tag{3.8}$$

Now consider the map

$$\mathbf{r}(u,v) \mapsto \mathbf{s}(v, \text{sinh}u), \quad \text{for} \quad u \in \mathbb{R}, v \in (0, 2\pi) \tag{3.9}$$

between the catenoid and the helicoid. Under the substitution $\tilde{u} = v$ and $\tilde{v} = \text{sinh}u$ then the form (3.8) becomes

$$(1 + \text{sinh}^2 u)\, \mathrm{d}v^2 + \mathrm{d}(\text{sinh}u)^2 = \text{cosh}^2 u\, \mathrm{d}u^2 + \text{cosh}^2 u\, \mathrm{d}v^2$$

which is the first fundamental form of the catenoid. Thus the map (3.9) is indeed an isometry.

The meridians of the catenoid are given by the equations $v = \textit{constant}$. Under the above isometry the meridians map to the curves on the helicoid given by $\tilde{u} = \textit{constant}$ – i.e. the rulings. ■

Exercise 3.24 *(First part is Sheet 2, Part A, Exercise 1) Two curves on the same smooth parameterized surface are given parameterically by* $t \mapsto (u(t), v(t))$ *and* $t \mapsto (\tilde{u}(t), \tilde{v}(t))$*. Suppose that the curves intersect at* $t = 0$*. (i.e.* $u(0) = \tilde{u}(0)$ *and* $v(0) = \tilde{v}(0)$*.) Prove that the angle of intersection* θ *is given by*

$$\cos\theta = \frac{E\dot{u}\dot{\tilde{u}} + F(\dot{u}\dot{\tilde{v}} + \dot{\tilde{u}}\dot{v}) + G\dot{v}\dot{\tilde{v}}}{\sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\sqrt{E\dot{\tilde{u}}^2 + 2F\dot{\tilde{u}}\dot{\tilde{v}} + G\dot{\tilde{v}}^2}}$$

Deduce that a parameterization is conformal if and only if the first fundamental form satisfies $E = G$ *and* $F = 0$ *everywhere.*

Exercise 3.25 *A diffeomorphism between surfaces* X *and* Y *is said to be* ***conformal*** *if the angle between any two intersecting curves on* X *equals the angle between their images on* Y *and is said to be* ***area-preserving*** *if each subset of* X *is mapped to a subset of* Y *of equal area. Show that a diffeomorphism is an isometry if and only if it is area-preserving and conformal.*

Thus far we have not made any calculations of lengths and areas which couldn't have been done as easily with the old expressions (3.1), (3.3) as with the new expressions (3.2), (3.4) which are in terms of coefficients of the first fundamental form. The calculations in the following examples however can only be done using the new definitions of length and area.

Example 3.26 *The* ***flat torus*** $\mathbb{T}$ *is the surface in* $\mathbb{R}^4$ *given by*

$$\mathbb{T} = \{(x, y, z, t) \in \mathbb{R}^4 \mid x^2 + y^2 = z^2 + t^2 = 1\}.$$

Show that $\mathbb{T}$ *is locally isometric to* $\mathbb{R}^2$ *and calculate the area of* $\mathbb{T}$*.*

Solution. We may parameterize (a dense open subset of) $\mathbb{T}$ by

$$\mathbf{r}(u,v) = (\cos u, \sin u, \cos v, \sin v), \qquad u, v \in (0, 2\pi).$$

Then the first fundamental form of $\mathbb{T}$ is $\mathrm{d}u^2 + \mathrm{d}v^2$ and so $\mathbb{T}$ is locally isometric to the plane. $\mathbb{T}$ is certainly not globally isometric to $\mathbb{R}^2$ since $\mathbb{T}$ is compact and $\mathbb{R}^2$ is non-compact. (In fact the flat torus is isometric to no surface in $\mathbb{R}^3$ – see Sheet 2, Part B, Exercise 4.) The area of $\mathbb{T}$

is easily seen using (3.4) to equal $4\pi^2$ but as the vector product is not defined in $\mathbb{R}^4$ then our original definition (3.3) is not applicable. ■

So far we have only considered examples where the metric structure of the surface is precisely that induced on the surface by the Euclidean space (usually $\mathbb{R}^3$) in which the surface lies. There is no reason why we should limit ourselves to these cases – in fact there are good reasons not to.

From Example 3.19 the tractoid (with the original tractrix removed) has first fundamental form

$$\mathrm{d}s^2 + e^{-2s}\,\mathrm{d}\theta^2, \qquad s > 0,\ \theta \in (0, 2\pi),$$

when the tractrix is parameterized by arc-length s. The map f from the tractoid to $(0, 2\pi) \times (1, \infty)$ which sends the point on the tractoid with co-ordinates (s, θ) to (θ, e^s) is a diffeomorphism but is not an isometry. We could however ask:

Example 3.27 *In terms of the co-ordinates x and y, find the first fundamental form on $(0, 2\pi) \times (1, \infty)$ for which f is an isometry.*

Solution. The co-ordinates x and y are related to s and θ by

$$x = \theta, \quad \text{and} \quad y = e^s.$$

For f to be an isometry we need to endow $(0, 2\pi) \times (1, \infty)$ with the first fundamental form

$$\mathrm{d}s^2 + e^{-2s}\,\mathrm{d}\theta^2 = \mathrm{d}(\log y)^2 + \frac{1}{y^2}\mathrm{d}x^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}.$$

■

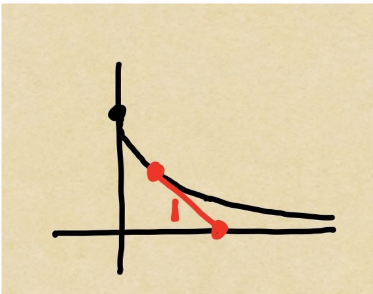

Figure 3.5 – tractrix

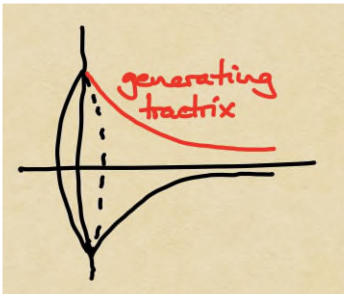

Figure 3.6 – tractoid

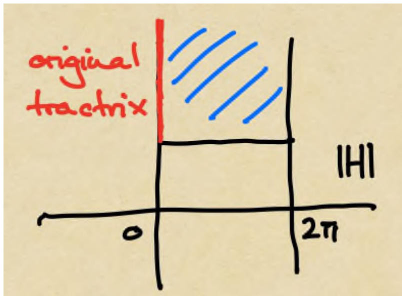

Figure 3.7 – tractoid as a subset of $\mathbb{H}$

What we have shown above is that the tractoid (without a meridian) is isometric to part of $\mathbb{H}$, the *hyperbolic plane* (Figure 3.7).

Example 3.28 ***(Poincaré's half plane model of the hyperbolic plane)*** *$\mathbb{H}$ is the surface created by endowing the upper half plane $\{(x, y) \mid y > 0\}$ with the first fundamental form*

$$\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}. \tag{3.10}$$

$\mathbb{H}$ is of interest because it was the first model for a non-Euclidean geometry.

Whilst the infinite rectangle $(0, 2\pi) \times (1, \infty)$ with the first fundamental form (3.10) is isometric to a surface in $\mathbb{R}^3$, the hyperbolic plane is not. This is a consequence of *Hilbert's Theorem* (Do Carmo, p. 446). We could isometrically embed $\mathbb{H}$ in a higher dimensional Euclidean space, although the isometry may be a little complicated, but there is no need. From our formulas (3.2),(3.4) we may find the length and area of curves and regions in $\mathbb{H}$ without having to be working in a particular Euclidean space. Indeed we could create a *geometric* surface by endowing any open subset of $\mathbb{R}^2$ with any first fundamental form $E\mathrm{d}x^2 + 2F\mathrm{d}x\mathrm{d}y + G\mathrm{d}y^2$ provided that E, F, G are smooth functions and

$$E > 0, \quad G > 0, \quad EG - F^2 > 0.$$

Conversely any parameterized surface which is diffeomorphic to an open subset of $\mathbb{R}^2$ would be isometric to one of these surfaces.

Example 3.29 *Find the length of the curve $\gamma(t) = (0, t)$ for $1 \leqslant t \leqslant 2$ in $\mathbb{H}$.*

Solution. We have $E = G = y^{-2}$ and $F = 0$. Substituting these into (3.2) we find

$$\mathcal{L}(\gamma) = \int_1^2 \sqrt{\frac{1}{t^2}}\,\mathrm{d}t = [\log t]_1^2 = \log 2.$$

■

Exercise 3.30 *Show that the surfaces created by endowing $(0, \alpha) \times (0, \infty)$ with the first fundamental form (3.10) are isometric for any $\alpha > 0$.*

Definition 3.31 *A* ***smooth geometric surface*** *or* ***smooth Riemannian 2-manifold*** *is a Hausdorff topological space X together with*

(a) homeomorphisms $\phi_\alpha \colon U_\alpha \to V_\alpha$ between open sets $U_\alpha \subseteq X$ and open sets $V_\alpha \subseteq \mathbb{R}^2$,

(b) first fundamental forms $E_\alpha \mathrm{d}x^2 + 2F_\alpha \mathrm{d}x\mathrm{d}y + G_\alpha \mathrm{d}y^2$ on U_α where $E_\alpha, F_\alpha, G_\alpha$ are smooth functions satisfying

$$E_\alpha > 0, \quad G_\alpha > 0, \quad E_\alpha G_\alpha - (F_\alpha)^2 > 0,$$

such that

(a) $\bigcup_\alpha U_\alpha = X$,

(b) when $U_\alpha \cap U_\beta \neq \emptyset$ then

$$(\phi_\alpha) \circ \phi_\beta^{-1} \colon (\phi_\beta)(U_\alpha \cap U_\beta) \to (\phi_\alpha)(U_\alpha \cap U_\beta)$$

is an isometry.

Example 3.32 ***(The elliptic plane)*** *Topologically the elliptic plane is the real projective plane. Geometrically it is the surface endowed with the first fundamental form from the unit sphere.*

Let D denote the unit disc $\{(u, v) \mid u^2 + v^2 < 1\}$. Then $\mathbf{r}_1 \colon D \to S^2$, defined by

$$\mathbf{r}_1(u, v) = (u, v, \sqrt{1 - u^2 - v^2}),$$

is a parameterization of a unit hemisphere, so that $\mathbf{s}_1 = \pi \circ \mathbf{r}_1 \colon D \to S^2/\{\pm 1\} = \mathbb{P}$ *is a parameterization of (a dense open subset of) the real projective plane* $\mathbb{P}$*. The first fundamental form on* $\mathbf{r}_1(D)$ *is*

$$\frac{(1-v^2)\mathrm{d}u^2 + 2uv\mathrm{d}u\mathrm{d}v + (1-u^2)\mathrm{d}v^2}{1-u^2-v^2},$$

and we can endow $\mathbf{s}_1(D)$ *with this first fundamental form to form a geometric surface.*

A second parameterization $\mathbf{s}_2 = \pi \circ \mathbf{r}_2 \colon D \to \mathbb{P}$ *arises from the parameterization* $\mathbf{r}_2 \colon D \to S^2$ *given by*

$$\mathbf{r}_2(U,V) = (U, \sqrt{1-U^2-V^2}, V),$$

which is endowed with the same first fundamental form once u *is replaced with* U *and* v *with* V*. Now the transition map* $\mathbf{s}_2^{-1} \circ \mathbf{s}_1$ *is given by*

$$U(u,v) = u \qquad \text{and} \qquad V(u,v) = \sqrt{1-u^2-v^2}.$$

Substituting these values into the first fundamental form on $\mathbf{s}_2(D)$ *we note* $\mathrm{d}U = \mathrm{d}u$ *and*

$$\mathrm{d}V = \frac{-u\mathrm{d}u - v\mathrm{d}v}{\sqrt{1-u^2-v^2}}$$

and then

$$\begin{aligned}
&\frac{(1-V^2)\mathrm{d}U^2 + 2UV\mathrm{d}U\mathrm{d}V + (1-U^2)\mathrm{d}V^2}{1-U^2-V^2} \\
=\ & \frac{(u^2+v^2)\mathrm{d}u^2 - 2u\sqrt{1-u^2-v^2}\mathrm{d}u\left(\frac{u\mathrm{d}u+v\mathrm{d}v}{\sqrt{1-u^2-v^2}}\right) + (1-u^2)\left(\frac{u\mathrm{d}u+v\mathrm{d}v}{\sqrt{1-u^2-v^2}}\right)^2}{v^2} \\
=\ & \frac{(u^2+v^2)\mathrm{d}u^2 - 2u\mathrm{d}u\,(u\mathrm{d}u+v\mathrm{d}v) + \frac{(1-u^2)}{(1-u^2-v^2)}\left(u^2\mathrm{d}u^2 + 2uv\mathrm{d}u\mathrm{d}v + v^2\mathrm{d}v^2\right)}{v^2} \\
=\ & \frac{\left[(v^2-u^2)(1-u^2-v^2) + (1-u^2)u^2\right]\mathrm{d}u^2 + 2\left[-uv(1-u^2-v^2) + uv(1-u^2)\right]\mathrm{d}u\mathrm{d}v + (1-u^2)v^2\mathrm{d}v^2}{v^2(1-u^2-v^2)} \\
=\ & \frac{\left[v^2-v^4\right]\mathrm{d}u^2 + 2\left[uv^3\right]\mathrm{d}u\mathrm{d}v + \left[(1-u^2)v^2\right]\mathrm{d}v^2}{v^2(1-u^2-v^2)} \\
=\ & \frac{(1-v^2)\mathrm{d}u^2 + 2uv\mathrm{d}u\mathrm{d}v + (1-u^2)\mathrm{d}v^2}{1-u^2-v^2}.
\end{aligned}$$

Hence the transition map is an isometry as required, because one first fundamental form transforms in the other.

We can similarly extend the notion of orientability to abstract surfaces by requiring that transition maps are orientation-preserving as well as diffeomorphisms. It is even possible to define the tangent space for a smooth abstract surface, even though the surface is not embedded in any ambient Euclidean space. On an abstract surface we still have local co-ordinates, so it still possible to differentiate smooth functions with respect to those co-ordinates.

Definition 3.33 *Let* X *be a smooth abstract surface and* $p \in X$*. Let* V_p *denote the vector space (algebra, in fact) of all functions* $\varphi \colon X \to \mathbb{R}$ *which are smooth at* p*. Then the* ***tangent space*** *at* p*, written* T_pX*, is the set of all linear maps* $D \colon V_p \to \mathbb{R}$ *which satisfy the product rule*

$$D(\varphi\psi) = \varphi(p)\,D\psi + \psi(p)\,D\varphi \qquad \text{for all } \varphi, \psi \in V_p.$$

Such a D is called a ***derivation.*** *Note T_pX is a vector space with addition and scalar multiplication defined by*

$$(D_1 + D_2)\,\varphi = D_1\varphi + D_2\varphi, \qquad (\lambda D)\,\varphi = \lambda(D\varphi).$$

Given a smooth map $f\colon X \to Y$ between two smooth abstract surfaces X, Y with $p \in X$ the differential $\mathrm{d}f_p\colon T_pX \to T_{f(p)}Y$ is defined by

$$\left(\mathrm{d}f_p\,(D)\right)(\alpha) = D(\alpha \circ f)$$

where $D \in T_pX$ and α is a real map $\alpha\colon Y \to \mathbb{R}$ which is smooth at $f(p)$.

Exercise 3.34 *T_pX is two dimensional and a basis is*

$$\left\{ \left.\frac{\partial}{\partial u}\right|_p, \left.\frac{\partial}{\partial v}\right|_p \right\}$$

where u and v are co-ordinates local to p.

3.2 Curvature and the Weingarten map

Let X be a smooth parameterized surface in $\mathbb{R}^\mathbf{3}$ described by $\mathbf{r}\colon U \to X$ and let

$$\mathbf{n} = \frac{\mathbf{r}_u \wedge \mathbf{r}_v}{|\mathbf{r}_u \wedge \mathbf{r}_v|}$$

denote a choice of unit normal. When $\gamma(s)$ is a curve in X, parameterized by arc length then the curvature $\kappa(s)$ of γ at the point $\gamma(s)$ is simply the magnitude of $\ddot{\gamma}(s)$.

When looking at such a curve, the vector $\ddot{\gamma}(s)$ has two natural components, a tangential component and a normal component. As $\dot{\gamma}(s)$ is a unit vector for all s, its derivative $\ddot{\gamma}(s)$ is perpendicular to $\dot{\gamma}(s)$. So we may decompose $\ddot{\gamma}(s)$ in the form:

$$\ddot{\gamma} = k_n\mathbf{n} + k_g(\mathbf{n} \wedge \dot{\gamma}). \tag{3.11}$$

Definition 3.35 *We define:*

(a) $k_n(s)$ is the ***normal curvature*** *of γ at $\gamma(s)$.*

(b) $k_g(s)$ is the ***geodesic curvature*** *of γ at $\gamma(s)$.*

It follows that $\kappa^2 = \left|\dot{\mathbf{t}}\right|^2 = |\ddot{\gamma}|^2 = k_n^2 + k_g^2$.

(c) A curve in X whose geodesic curvature is everywhere zero is called a ***geodesic.***

We shall consider, for the moment, the normal curvature of curves and we shall use this to define a second quadratic form on the tangent space of a point of X. We shall see later (Theorem 4.2 and Sheet 2, Part B, Exercise 3) that the geodesics of a surface and the geodesic curvature of a curve are intrinsic; that is they depend only on the first fundamental form of the surface and the direction of the curve. This is very much not the case with normal curvature, which gives information on how a geometric surface has been embedded in $\mathbb{R}^3$.

The normal curvature k_n of γ equals $\ddot{\gamma} \cdot \mathbf{n}$. By the chain rule we have

$$\dot{\gamma} = \dot{u}\mathbf{r}_u + \dot{v}\mathbf{r}_v,$$

and applying the chain rule again we find

$$\ddot{\gamma} = \ddot{u}\mathbf{r}_u + \ddot{v}\mathbf{r}_v + \dot{u}^2\mathbf{r}_{uu} + 2\dot{u}\dot{v}\mathbf{r}_{uv} + \dot{v}^2\mathbf{r}_{vv}.$$

Hence the normal curvature $k_n = \ddot{\gamma} \cdot \mathbf{n}$ equals

$$k_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2,$$

where

$$\begin{aligned} L &= \mathbf{r}_{uu} \cdot \mathbf{n} = -\mathbf{r}_u \cdot \mathbf{n}_u, \\ M &= \mathbf{r}_{uv} \cdot \mathbf{n} = -\mathbf{r}_u \cdot \mathbf{n}_v = -\mathbf{r}_v \cdot \mathbf{n}_u, \\ N &= \mathbf{r}_{vv} \cdot \mathbf{n} = -\mathbf{r}_v \cdot \mathbf{n}_v. \end{aligned} \tag{3.12}$$

Note that the alternative expressions for L, M, N come from the differentiating the equations

$$\mathbf{r}_u \cdot \mathbf{n} = 0 = \mathbf{r}_v \cdot \mathbf{n}.$$

Definition 3.36 *The quadratic form $II_p \colon T_pX \to \mathbb{R}$ given by*

$$\alpha\mathbf{r}_u + \beta\mathbf{r}_v \mapsto L\alpha^2 + 2M\alpha\beta + N\beta^2$$

is called the ***second fundamental form*** *of X. (Note that some authors, including Do Carmo, use e, f, g instead of L, M, N for the coefficients of the second fundamental form.)*

The first fundamental form describes the intrinsic properties of the surface, whereas the second fundamental form relates to the surface's embedding in $\mathbb{R}^3$. Although the proof of the following theorem is far beyond the scope of this course, I include an abridged statement of:

Theorem 3.37 *(Off-syllabus)* ***The Fundamental Theorem of the Local Theory of Surfaces.***

Let E, F, G, L, M, N be differentiable functions on an open set $U \subset \mathbb{R}^2$ which satisfy
(a) $E > 0, G > 0, EG - F^2 > 0$,
(b) certain compatibility equations (Remark 3.56, Do Carmo p.235).

Then for each $p \in U$ there is an open set $V \subset U$ containing p and a smooth parameterization $\mathbf{r}(V)$ of a surface in $\mathbb{R}^3$ with $E, 2F, G$ and $L, 2M, N$ as the coefficients of the first and second fundamental forms. Further a second surface $\tilde{\mathbf{r}}(V)$ in $\mathbb{R}^3$ with the same first and second fundamental forms differs from $\mathbf{r}(V)$ only by a rigid motion of $\mathbb{R}^3$.

One equation of compatability is the Gauss formula, which we will meet shortly, and which expresses the Gaussian curvature – ostensibly defined in terms of both fundamental forms – solely in terms of the first fundamental form.

Example 3.38 *Show that the xy-plane and cylinder $x^2 + y^2 = a^2$ are locally isometric but have different second fundamental forms.*

Solution. A parameterization of the xy-plane is $\mathbf{r}(u, v) = (u, v, 0)$ which leads to

$$E = 1, \quad F = 0, \quad G = 1, \quad L = 0, \quad M = 0, \quad N = 0.$$

The cylinder, except for one meridian, can be parameterized by $\mathbf{s}(u,v) = (a\cos(u/a), a\sin(u/a), v)$ where $0 < u < 2\pi a$ and $v \in \mathbb{R}$. This leads to

$$E = 1, \quad F = 0, \quad G = 1, \quad L = -a^{-1}, \quad M = 0, \quad N = 0.$$

Thus the cylinder and plane are locally isometric. They are not globally isometric as they are not homeomorphic – the cylinder is not simply connected whereas the plane is. ■

In order to define the curvature of the surface at a point we need to introduce the *Weingarten map* or *shape operator*. The Weingarten map is the differential of the Gauss (normal) map $\mathbf{n}$ and consequently is written as $\mathrm{d}\mathbf{n}_p$ in some texts. Curvature, for a curve, is a measure of how quickly the tangent is varying. Similarly for a surface we need to investigate how quickly the tangent plane, or equivalently the normal to the surface is varying. Note that as $\mathbf{n} \cdot \mathbf{n} = 1$ then

$$\mathbf{n} \cdot \mathbf{n}_u = 0 = \mathbf{n} \cdot \mathbf{n}_v.$$

Thus $\mathbf{n}_u$ and $\mathbf{n}_v$ are tangents vectors to the surface.

Definition 3.39 *The* ***Weingarten map*** *(or* ***shape operator****) at the point p is the linear map $W_p \colon T_pX \to T_pX$ defined by*

$$W_p\mathbf{r}_u = \mathbf{n}_u, \qquad W_p\mathbf{r}_v = \mathbf{n}_v. \tag{3.13}$$

More generally note that $W_p(\gamma'(s)) = (\mathbf{n}\circ\gamma)'(s)$ and so $W_p = \mathrm{d}\mathbf{n}_p$ is the differential of the Gauss map.

Proposition 3.40 *The Weingarten map $W_p \colon T_pX \to T_pX$ is a self-adjoint linear map independent of the choice of parameters u and v. In particular, as W_p is self-adjoint, it is diagonalisable.*

Proof. Let $\mathbf{s}(\tilde{u},\tilde{v})$ be a second parameterization for X with $\mathbf{s}(\tilde{u},\tilde{v}) = \mathbf{r}(u,v)$. Then by the chain rule we have

$$\mathbf{s}_{\tilde{u}} = \frac{\partial u}{\partial \tilde{u}}\mathbf{r}_u + \frac{\partial v}{\partial \tilde{u}}\mathbf{r}_v, \qquad \mathbf{s}_{\tilde{v}} = \frac{\partial u}{\partial \tilde{v}}\mathbf{r}_u + \frac{\partial v}{\partial \tilde{v}}\mathbf{r}_v.$$

Hence by the above definition of the Weingarten map and the chain rule we have

$$W_p\mathbf{s}_{\tilde{u}} = \frac{\partial u}{\partial \tilde{u}}\mathbf{n}_u + \frac{\partial v}{\partial \tilde{u}}\mathbf{n}_v = \mathbf{n}_{\tilde{u}}, \qquad W_p\mathbf{s}_{\tilde{v}} = \frac{\partial u}{\partial \tilde{v}}\mathbf{n}_u + \frac{\partial v}{\partial \tilde{v}}\mathbf{n}_v = \mathbf{n}_{\tilde{v}}.$$

It is also easy to check that W_p is a self-adjoint linear map – that is

$$(W_p\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (W_p\mathbf{y}) \tag{3.14}$$

for any two tangent vectors $\mathbf{x}, \mathbf{y} \in T_pX$. We note from equation (3.12) that

$$W_p\mathbf{r}_u \cdot \mathbf{r}_v = \mathbf{n}_u \cdot \mathbf{r}_v = \mathbf{n}_v \cdot \mathbf{r}_u = W_p\mathbf{r}_v \cdot \mathbf{r}_u.$$

Equation (3.14) then follows for all tangent vectors $\mathbf{x}, \mathbf{y}$ by linearity. ■

As W_p is self-adjoint it is diagonalizable and has real eigenvalues. Let γ be a curve in X with $\gamma(0) = p$. Then

$$\begin{aligned} W_p(\gamma'(0)) \cdot \gamma'(0) &= \mathbf{n}'(\gamma(0)) \cdot \gamma'(0) \\ &= -\mathbf{n} \cdot \gamma''(0) = -k_n. \end{aligned}$$

Thus the eigenvalues of W_p are $-k_1$ and $-k_2$ where k_1 and k_2 are the extreme values of the normal curvature, called the **principal curvatures** of X at p and the eigenvectors of W_p are the **principal directions**. The **lines of curvature** are curves whose tangents are the principal directions.

We make the following definitions:

Definition 3.41 *The **Gaussian curvature** $K(p)$ at the point p is the product of the principal curvatures or equivalently the determinant* $\det W_p$ *of the Weingarten map.*

Definition 3.42 *(Off-syllabus) The average of the principal curvatures is known as the **mean curvature** at p. It is given by the formula*

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)}.$$

The mean curvature is important in the study of minimal surfaces. A minimal surface is a surface with an area that is a local minimum, such as with soap films. A soap film – in order to reduce the surface tension – has minimal area compared with all perturbations of the surface. This is equivalent to the mean curvature of the surface being zero (Segal, Theorem 9.1).

The tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ form a basis for the tangent plane T_pX and $W_p \colon T_pX \to T_pX$ is a linear map. We now work out the matrix for W_p with respect to this basis.

Let us suppose that the matrix for W_p with respect to the basis $\{\mathbf{r}_u, \mathbf{r}_v\}$ is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then

$$W_p\mathbf{r}_u = \mathbf{n}_u = A\mathbf{r}_u + C\mathbf{r}_v, \tag{3.15}$$
$$W_p\mathbf{r}_v = \mathbf{n}_v = B\mathbf{r}_u + D\mathbf{r}_v. \tag{3.16}$$

Dotting equation (3.15) with $\mathbf{r}_u$ and with $\mathbf{r}_v$ we find

$$-L = AE + CF, \qquad -M = AF + CG.$$

Doing the same for equation (3.16) we obtain

$$-M = BE + DF, \qquad -N = BF + DG.$$

Putting these equations into matrix form gives

$$-\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and hence with respect to the basis $\{\mathbf{r}_u, \mathbf{r}_v\}$

$$W_p = \frac{1}{EG - F^2} \begin{pmatrix} -G & F \\ F & -E \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix}. \tag{3.17}$$

Corollary 3.43 *The Gaussian curvature $K(p)$ at p, which equals* $\det W_p$*, is given by the formula*

$$K(p) = \frac{LN - M^2}{EG - F^2}.$$

Despite the above expression for K, which is in terms of the coefficients of the first *and* second fundamental forms, the Gaussian curvature may be written solely in terms of the coefficients of the first fundamental form and is invariant under isometries. This is a theorem due to Gauss and known as the *Theorema Egregium* (Theorem 3.48).

Gauss originally did not define K by the above formula but rather as the following more intuitive limit. Let U be a small open subset of X about the point p. Then if we let the area of U tend to zero (see Sheet 3, Part B, Exercise 3)

$$|K| = \lim_{\text{Area}(U)\to 0} \frac{\text{Area}(\mathbf{n}(U))}{\text{Area}(U)}.$$

The more 'curved' the surface at a point, the greater the variety in the normal vectors about the point.

We end this section with two worked examples – we continue with the earlier examples – the sphere and the tractoid – where we calculated the first fundamental form.

Example 3.44 *Find the Gaussian curvature of a sphere of radius a.*

Solution. In Example 3.18 we parameterized the sphere with

$$\mathbf{r}(u,v) = (a\cos u \sin v, a \sin u \sin v, a\cos v), \qquad u \in (-\pi, \pi),\ v \in (0, \pi),$$

omitting only half a great circle and found

$$E = a^2 \sin^2 v, \quad F = 0, \quad G = a^2.$$

The outward-pointing unit normal equals

$$\mathbf{n}(u,v) = (\cos u \sin v, \sin u \sin v, \cos v) = \frac{1}{a}\mathbf{r}(u,v).$$

So we can avoid further calculation by noting

$$\begin{aligned} L &= \mathbf{r}_{uu}\cdot\mathbf{n} = -\mathbf{r}_u\cdot\mathbf{n}_u = -\frac{1}{a}\mathbf{r}_u\cdot\mathbf{r}_u = -\frac{E}{a} = -a\sin^2 v; \\ M &= \mathbf{r}_{uv}\cdot\mathbf{n} = -\mathbf{r}_u\cdot\mathbf{n}_v = -\frac{1}{a}\mathbf{r}_u\cdot\mathbf{r}_v = -\frac{F}{a} = 0; \\ N &= \mathbf{r}_{vv}\cdot\mathbf{n} = -\mathbf{r}_v\cdot\mathbf{n}_v = -\frac{1}{a}\mathbf{r}_v\cdot\mathbf{r}_v = -\frac{G}{a} = -a. \end{aligned}$$

Hence

$$K = \frac{LN - M^2}{EG - F^2} = \frac{a^2\sin^2 v}{a^4 \sin^2 v} = \frac{1}{a^2}.$$

■

Example 3.45 *A torus of revolution is formed by rotating the circle with equation*

$$(x-b)^2 + y^2 = a^2, \qquad (b > a),$$

about the y-axis. Parameterize the torus and find its Gaussian curvature.

Solution. We can parametrize (an open dense subset of) the torus as

$$\mathbf{r}(u,v) = ((b + a\sin u)\cos v,\ (b + a\sin u)\sin v, a\cos u) \qquad 0 < u, v < 2\pi.$$

We have

$$\mathbf{r}_u = (a\cos u\cos v, a\cos u\sin v, -a\sin u), \qquad \mathbf{r}_v = (-(b + a\sin u)\sin v, (b + a\sin u)\cos v, 0)$$

giving

$$\begin{aligned}
E &= a^2\cos^2 u\left(\cos^2 v + \sin^2 v\right) + a^2\sin^2 u = a^2, \\
F &= a(b + a\sin u)(-\cos u\sin v\cos v + \cos v\cos u\sin v) = 0, \\
G &= (b + a\sin u)\left(\sin^2 v + \cos^2 v\right) = b + a\sin u.
\end{aligned}$$

Further

$$\begin{aligned}
\mathbf{r}_u \wedge \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos u\cos v & a\cos u\sin v & -a\sin u \\ -(b + a\sin u)\sin v & (b + a\sin u)\cos v & 0 \end{vmatrix} \\
&= a(b + a\sin u)\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos u\cos v & \cos u\sin v & -\sin u \\ -\sin v & \cos v & 0 \end{vmatrix} \\
&= a(b + a\sin u)(\sin u\cos v, \sin y\sin v, \cos u),
\end{aligned}$$

giving

$$\mathbf{n} = (\sin u\cos v, \sin u\sin v, \cos u).$$

We then have

$$\begin{aligned}
\mathbf{r}_{uu} &= (-a\sin u\cos v, -a\sin u\sin v, -a\cos u), \\
\mathbf{r}_{uv} &= (-a\cos u\sin v, a\cos u\cos v, 0), \\
\mathbf{r}_{vv} &= (-(b + a\sin u)\cos v, -(b + a\sin u)\sin v, 0),
\end{aligned}$$

and so

$$\begin{aligned}
L &= -a\sin^2 u\left(\cos^2 v + \sin^2 v\right) - a\cos^2 u = -a, \\
M &= -a\cos u\sin v\sin u\cos v + a\cos u\sin v\sin u\cos v = 0, \\
N &= -(b + a\sin u)\sin u\left(\cos^2 v + \sin^2 v\right) = -(b + a\sin u)\sin u.
\end{aligned}$$

Hence

$$K = \frac{LN - M^2}{EG - F^2} = \frac{a(b + a\sin u)\sin u}{a^2(b + a\sin u)} = \frac{1}{a}\sin u.$$

Note that $K > 0$ on the outside of the torus when $0 < u < \pi$ and $K < 0$ when $\pi < u < 2\pi$. ■

Remark 3.46 *(**Parity of Gaussian curvature**) The sign of Gaussian curvature can be readily appreciated. If we choose an outward pointing normal in the example of the torus, on the outside of the outside of the torus the lines of curvature are both bending away from the normal, the principal curvatures are negative and their product K is positive. If we had instead had an inward pointing normal then the principal curvatures would still have had the same sign and $K > 0$ would still be true. On the inside of the torus, one line of curvature is around the hole of the torus and one through the hole of the torus. The principal curvatures have different signs and then $K < 0$.*

Exercise 3.47 *Find the lines of curvature and the principal curvatures on a surface of revolution in terms of the distance ρ of the generating curve from the axis. Show that the Gaussian curvature K equals $\kappa \cos\phi/\rho$ where κ is the curvature of the generating curve and ϕ is the angle between the axis and the tangent line to the curve.*

3.3 Theorema Egregium

Theorem 3.48 *(**Theorema Egregium**, Gauss, 1827) Gaussian curvature is intrinsic, and so preserved by isometries.*

Remark 3.49 *Recall that we define Gaussian curvature as*

$$K = \frac{LN - M^2}{EG - F^2}.$$

The first fundamental form is intrinsic but the second fundamental form is not (as we saw earlier with Example 3.38). Hence there is no reason to expect that K is intrinsic. The Latin title 'Theorema Egregium' translates as 'remarkable theorem'.

Proof. Let $(u, v) \mapsto \mathbf{r}(u, v)$ be a parameterization for a patch of surface X in $\mathbb{R}^3$. Let $\mathbf{n}$ be a unit normal vector field on X and let the first and second fundamental forms respectively be

$$E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2 \quad \text{and} \quad L\mathrm{d}u^2 + 2M\mathrm{d}u\mathrm{d}v + N\mathrm{d}v^2.$$

And recall the Weingarten map $W = \mathrm{d}\mathbf{n}$ (equation (3.17)) is represented by the matrix

$$\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} = -\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

with respect to the basis $\{\mathbf{r}_u, \mathbf{r}_v\}$ for the tangent space.

We introduce now the **Christoffel symbols** Γ^k_{ij}, defined by writing

$$\begin{aligned} \mathbf{r}_{uu} &= \Gamma^1_{11}\mathbf{r}_u + \Gamma^2_{11}\mathbf{r}_v + L\mathbf{n}, \\ \mathbf{r}_{uv} &= \Gamma^1_{12}\mathbf{r}_u + \Gamma^2_{12}\mathbf{r}_v + M\mathbf{n}, \\ \mathbf{r}_{vv} &= \Gamma^1_{22}\mathbf{r}_u + \Gamma^2_{22}\mathbf{r}_v + N\mathbf{n}. \end{aligned}$$

Our aim will be first to show that the Christoffel symbols are intrinsic – that is they depend only on E, F and G and their derivatives – and then show that the Gaussian curvature can be written in terms of the Christoffel symbols. ■

Lemma 3.50 *The Christoffel symbols depend only on E, F and G and their derivatives.*

Proof. Dotting the equations above with $\mathbf{r}_u$ and $\mathbf{r}_v$ we find

$$\begin{cases} \Gamma_{11}^1 E + \Gamma_{11}^2 F &= \mathbf{r}_{uu} \cdot \mathbf{r}_u = \frac{1}{2}(\mathbf{r}_u \cdot \mathbf{r}_u)_u &= \frac{1}{2}E_u, \\ \Gamma_{11}^1 F + \Gamma_{11}^2 G &= \mathbf{r}_{uu} \cdot \mathbf{r}_v = (\mathbf{r}_u \cdot \mathbf{r}_v)_u - \frac{1}{2}(\mathbf{r}_u \cdot \mathbf{r}_u)_v &= F_u - \frac{1}{2}E_v, \end{cases}$$
$$\begin{cases} \Gamma_{12}^1 E + \Gamma_{12}^2 F &= \mathbf{r}_{uv} \cdot \mathbf{r}_u = \frac{1}{2}(\mathbf{r}_u \cdot \mathbf{r}_u)_v &= \frac{1}{2}E_v, \\ \Gamma_{12}^1 F + \Gamma_{12}^2 G &= \mathbf{r}_{uv} \cdot \mathbf{r}_v = \frac{1}{2}(\mathbf{r}_v \cdot \mathbf{r}_v)_u &= \frac{1}{2}G_u, \end{cases}$$
$$\begin{cases} \Gamma_{22}^1 E + \Gamma_{22}^2 F &= \mathbf{r}_{vv} \cdot \mathbf{r}_u = (\mathbf{r}_u \cdot \mathbf{r}_v)_v - \frac{1}{2}(\mathbf{r}_v \cdot \mathbf{r}_v)_u &= F_v - \frac{1}{2}G_u, \\ \Gamma_{22}^1 F + \Gamma_{22}^2 G &= \mathbf{r}_{vv} \cdot \mathbf{r}_v = \frac{1}{2}(\mathbf{r}_v \cdot \mathbf{r}_v)_v &= \frac{1}{2}G_v. \end{cases}$$

Each of the braced equations are invertible as the determinant $EG - F^2$ is non-zero. Thus each Christoffel symbol may be written in terms of E, F, G and their derivatives. ■

Corollary 3.51 *Suppose that the parameterization $\mathbf{r}$ is orthogonal, that is $F = 0$. Then:*

$$\Gamma_{11}^1 = E_u/2E, \quad \Gamma_{12}^1 = E_v/2E, \quad \Gamma_{22}^1 = -G_u/2E,$$
$$\Gamma_{11}^2 = -E_v/2G, \quad \Gamma_{12}^2 = G_u/2G, \quad \Gamma_{22}^2 = G_v/2G.$$

Lemma 3.52 ***(The Gauss formula)***

$$(\Gamma_{12}^2)_u - (\Gamma_{11}^2)_v + \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{11}^1\Gamma_{12}^2 = -EK,$$

where K denotes the Gaussian curvature.

Proof. Note by the product rule that

$$(\mathbf{r}_{uu})_v = \Gamma_{11}^1\mathbf{r}_{uv} + (\Gamma_{11}^1)_v\mathbf{r}_u + \Gamma_{11}^2\mathbf{r}_{vv} + (\Gamma_{11}^2)_v\mathbf{r}_v + L\mathbf{n}_v + L_v\mathbf{n},$$

and that

$$(\mathbf{r}_{uv})_u = \Gamma_{12}^1\mathbf{r}_{uu} + (\Gamma_{12}^1)_u\mathbf{r}_u + \Gamma_{12}^2\mathbf{r}_{uv} + (\Gamma_{12}^2)_u\mathbf{r}_v + M\mathbf{n}_u + M_u\mathbf{n}.$$

We may write $(\mathbf{r}_{uu})_v$ and $(\mathbf{r}_{uv})_u$ in terms of the basis $\{\mathbf{r}_u, \mathbf{r}_v, \mathbf{n}\}$. By comparing the coefficients of $\mathbf{r}_v$ in these expressions we obtain

$$\Gamma_{11}^1\Gamma_{12}^2 + \Gamma_{11}^2\Gamma_{22}^2 + (\Gamma_{11}^2)_v + Lw_{22} = \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 + (\Gamma_{12}^2)_u + Mw_{21}.$$

Hence

$$\begin{aligned} & (\Gamma_{12}^2)_u - (\Gamma_{11}^2)_v + \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{11}^1\Gamma_{12}^2 \\ = \; & Lw_{22} - Mw_{21} \\ = \; & \frac{L(FM - EN) - M(LF - EM)}{EG - F^2} \\ = \; & -EK. \end{aligned}$$

These two lemmas prove our claims. The Christoffel symbols are intrinsic, so by the Gauss formula K is also intrinsic. ■

Corollary 3.53 *When $F = 0$ the Gaussian curvature K equals*

$$K = \frac{-1}{2\sqrt{EG}} \left\{ \left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right\}.$$

Solution. From the Gauss formula, and the above formulae for the Christoffel symbols when $F = 0$, we have that $-EK$ equals

$$\begin{aligned}
& \left(\frac{G_u}{2G}\right)_u + \left(\frac{E_v}{2G}\right)_v - \frac{E_v^2}{4EG} + \frac{G_u^2}{4G^2} + \frac{E_v G_v}{4G^2} - \frac{E_u G_u}{4EG} \\
= & \frac{GG_{uu} - G_u^2}{2G^2} + \frac{GE_{vv} - E_v G_v}{2G^2} - \frac{E_v^2}{4EG} + \frac{G_u^2}{4G^2} + \frac{E_v G_v}{4G^2} - \frac{E_u G_u}{4EG} \\
= & \frac{G_{uu}}{2G} + \frac{E_{vv}}{2G} - \frac{E_v^2}{4EG} - \frac{G_u^2}{4G^2} - \frac{E_v G_v}{4G^2} - \frac{E_u G_u}{4EG}.
\end{aligned}$$

Hence $-2\sqrt{EG}K$ equals

$$\left(\frac{E_{vv}}{\sqrt{EG}} - \frac{E_v}{2EG\sqrt{EG}} (E_v G + G_v E) \right) + \left(\frac{G_{uu}}{\sqrt{EG}} - \frac{G_u}{2EG\sqrt{EG}} (G_u E + E_u G) \right).$$

to give the required result. ■

Example 3.54 *The (Poincaré half-plane model for the)* ***hyperbolic plane*** *is the half-plane $\mathbb{H} = \{(u, v) \in \mathbb{R}^2 : v > 0\}$ with the first fundamental form $E = G = v^{-2}$ and $F = 0$. Find the Gaussian curvature of $\mathbb{H}$.*

Solution.

$$K = \frac{-v^2}{2} \frac{\mathrm{d}}{\mathrm{d}v} \left(\frac{-2}{v} \right) = -1.$$

■

Example 3.55 *Show that there exists no surface $\mathbf{r}(u, v)$ with first and second fundamental forms respectively*

$$\mathrm{d}u^2 + \mathrm{d}v^2 \quad \textit{and} \quad \mathrm{d}u^2 - \mathrm{d}v^2.$$

Solution. On the one hand the surface has Gaussian curvature everywhere -1 as $E = G = L = 1, F = M = 0, N = -1$. On the other the surface is isometric to a subset of the plane and hence has Gaussian curvature 0. ■

Remark 3.56 ***(Equations of compatability)*** *The Gauss formula is a necessary condition connecting the coefficients of the first and second fundamental forms – it is one of the equations of compatability that form part of the fundamental theorem (Theorem 3.37). The other equations are called the Mainardi-Codazzi equations and they require*

$$\begin{aligned}
L_v - M_u &= L\Gamma_{12}^1 + M(\Gamma_{12}^2 - \Gamma_{11}^1) - N\Gamma_{11}^2, \\
M_v - N_u &= L\Gamma_{22}^1 + M(\Gamma_{22}^2 - \Gamma_{12}^1) - N\Gamma_{11}^2.
\end{aligned}$$