5. GEOMETRY & ANALYSIS MEET TOPOLOGY

In this chapter we will meet some startling results which connect the topology of a surface with global aspects of geometry and analysis. For example, the *global Gauss-Bonnet theorem* says that for a closed geometric surface X,

$$\iint_X K \,\mathrm{d}A = 2\pi\chi(X).$$

What is striking about this result is that the term on the RHS is manifestly topological in nature whilst the total curvature on the LHS is ostensibly geometric. It is possible to distort a surface locally to change its Gaussian curvature without changing its topology, but the above theorem shows there will be knock-on effects elsewhere on the surface as the total curvature must remain constant. There are other corollaries to this result such as (a) the sphere is the only orientable closed surface which can have positive curvature everywhere and (b) the torus is the only orientable closed surface which can be everywhere flat (Example 3.26).

5.1 The Gauss-Bonnet theorems

We begin first with a proof of the *local Gauss-Bonnet theorem.* The statement of this theorem is on the syllabus *but its proof is not*; I include the proof here for completeness' sake.

Theorem 5.1 ***(Local Gauss-Bonnet Theorem** – first version) (Proof off syllabus) Let γ be a smooth, simple, closed curve on a patch of surface X, enclosing a region R. Then*

$$\int_\gamma k_g \,\mathrm{d}s + \iint_R K \,\mathrm{d}A = 2\pi.$$

Proof. We will assume that $X = \mathbf{r}(U)$ where $\mathbf{r}$ is an orthogonal parameterization, so that $F = 0$. (The existence of such fields was mentioned in Remark 3.16.) We then set

$$\mathbf{e}_1 = \frac{\mathbf{r}_u}{\sqrt{E}}, \qquad \mathbf{e}_2 = \frac{\mathbf{r}_v}{\sqrt{G}},$$

to be smooth, orthonormal, tangent, vector fields $\mathbf{e}_1, \mathbf{e}_2 \colon V \to \mathbb{R}^3$. Let $\theta(s)$ denote the angle between the unit vector $\dot\gamma(s)$ and $\mathbf{e}_1$ at the point $\gamma(s)$, so that

$$\dot\gamma = \mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta$$

giving

$$\ddot\gamma = \dot\theta\left(-\mathbf{e}_1 \sin\theta + \mathbf{e}_2 \cos\theta\right) + \left(\dot{\mathbf{e}}_1 \cos\theta + \dot{\mathbf{e}}_2 \sin\theta\right).$$

With $\mathbf{n} = \mathbf{e}_1 \wedge \mathbf{e}_2$ then

$$\mathbf{n} \wedge \dot\gamma = -\mathbf{e}_1 \sin\theta + \mathbf{e}_2 \cos\theta,$$

as $\mathbf{n} \wedge \mathbf{e}_1 = \mathbf{e}_2$ and $\mathbf{n} \wedge \mathbf{e}_2 = -\mathbf{e}_1$, so that

$$\ddot{\gamma} = \dot{\theta}\left(\mathbf{n} \wedge \dot{\gamma}\right) + \left(\dot{\mathbf{e}}_1 \cos\theta + \dot{\mathbf{e}}_2 \sin\theta\right).$$

Thus

$$\begin{aligned} k_g &= \ddot{\gamma} \cdot (\mathbf{n} \wedge \dot{\gamma}) \\ &= \dot{\theta} + (\dot{\mathbf{e}}_1 \cos\theta + \dot{\mathbf{e}}_2 \sin\theta) \cdot (-\mathbf{e}_1 \sin\theta + \mathbf{e}_2 \cos\theta) \\ &= \dot{\theta} - \mathbf{e}_1 \cdot \dot{\mathbf{e}}_2 \end{aligned}$$

because $\dot{\mathbf{e}}_1 \cdot \mathbf{e}_1 = 0 = \dot{\mathbf{e}}_2 \cdot \mathbf{e}_2$ and $\dot{\mathbf{e}}_1 \cdot \mathbf{e}_2 = -\mathbf{e}_1 \cdot \dot{\mathbf{e}}_2$ from differentiating $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$ and $\mathbf{e}_1 \cdot \mathbf{e}_1 = 1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = 0$; thus we have

$$2\pi - \int_\gamma k_g \, \mathrm{d}s = \Delta\theta - \int_\gamma k_g \, \mathrm{d}s = \int_\gamma \left(\dot{\theta} - k_g\right) \mathrm{d}s = \int_\gamma \mathbf{e}_1 \cdot \dot{\mathbf{e}}_2 \, \mathrm{d}s.$$

We will then apply Green's theorem to this last integral. Recall Green's theorem states: for a smooth, simple, closed curve β in an open set $V \subseteq \mathbb{R}^2$, bounding a region S, with P, Q being two smooth functions defined on V

$$\int_\beta (P \, \mathrm{d}u + Q \, \mathrm{d}v) = \iint_S \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) \mathrm{d}u \, \mathrm{d}v.$$

Let $\beta = \mathbf{r}^{-1}(\gamma)$. We have

$$\dot{\mathbf{e}}_2 = \frac{\partial \mathbf{e}_2}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}s} + \frac{\partial \mathbf{e}_2}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}s},$$

so that $P = \mathbf{e}_1 \cdot \partial \mathbf{e}_2/\partial u$ and $Q = \mathbf{e}_1 \cdot \partial \mathbf{e}_2/\partial v$. Then

$$\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} = \frac{\partial \mathbf{e}_1}{\partial u} \cdot \frac{\partial \mathbf{e}_2}{\partial v} - \frac{\partial \mathbf{e}_1}{\partial v} \cdot \frac{\partial \mathbf{e}_2}{\partial u}.$$

Differentiating $\mathbf{e}_1 = \mathbf{r}_u/\sqrt{E}$ and $\mathbf{e}_2 = \mathbf{r}_v/\sqrt{G}$ we find

$$\frac{\partial \mathbf{e}_1}{\partial u} = \frac{1}{\sqrt{E}}\mathbf{r}_{uu} - \frac{E_u}{2E^{3/2}}\mathbf{r}_u, \qquad \frac{\partial \mathbf{e}_2}{\partial v} = \frac{1}{\sqrt{G}}\mathbf{r}_{vv} - \frac{G_v}{2G^{3/2}}\mathbf{r}_v.$$

Noting that $\mathbf{r}_u, \mathbf{r}_v$ and $\mathbf{n}$ are mutually orthogonal, we find

$$\begin{aligned} \frac{\partial \mathbf{e}_1}{\partial u} \cdot \frac{\partial \mathbf{e}_2}{\partial v} &= \frac{\mathbf{r}_{uu} \cdot \mathbf{r}_{vv}}{\sqrt{EG}} - \frac{E_u \mathbf{r}_u \cdot \mathbf{r}_{vv}}{2E^{3/2}\sqrt{G}} - \frac{G_v \mathbf{r}_{uu} \cdot \mathbf{r}_v}{2G^{3/2}\sqrt{E}} \\ &= \frac{\Gamma_{11}^1 \Gamma_{22}^1 E + \Gamma_{11}^2 \Gamma_{22}^2 G + LN}{\sqrt{EG}} - \frac{E_u \Gamma_{22}^1}{2\sqrt{EG}} - \frac{G_v \Gamma_{11}^2}{2\sqrt{EG}}. \end{aligned}$$

From Corollary 3.51, when $F = 0$, we have

$$\begin{aligned} \Gamma_{11}^1 &= E_u/2E, & \Gamma_{12}^1 &= E_v/2E, & \Gamma_{22}^1 &= -G_u/2E, \\ \Gamma_{11}^2 &= -E_v/2G, & \Gamma_{12}^2 &= G_u/2G, & \Gamma_{22}^2 &= G_v/2G. \end{aligned}$$

and so the above simplifies to

$$\frac{\partial \mathbf{e}_1}{\partial u} \cdot \frac{\partial \mathbf{e}_2}{\partial v} = \frac{LN}{\sqrt{EG}}$$

Similarly

$$\frac{\partial \mathbf{e}_1}{\partial v} \cdot \frac{\partial \mathbf{e}_2}{\partial u} = \frac{\Gamma^1_{12}\Gamma^1_{21}E + \Gamma^2_{12}\Gamma^2_{21}G - M^2}{\sqrt{EG}} - \frac{E_v\Gamma^1_{21}}{2\sqrt{EG}} - \frac{G_u\Gamma^2_{12}}{2\sqrt{EG}} = \frac{-M^2}{\sqrt{EG}}.$$

Hence

$$I = \iint_S \frac{LN - M^2}{\sqrt{EG}} \, \mathrm{d}u \, \mathrm{d}v = \iint_S K \sqrt{EG} \, \mathrm{d}u \, \mathrm{d}v = \iint_S K \, \mathrm{d}A.$$

■

Theorem 5.2 *(**Local Gauss-Bonnet Theorem** – second version.) (Proof off syllabus)*
Let γ be a piecewise-smooth simple, closed curve on a patch of surface X, enclosing a region R. Then

$$\int_\gamma k_g \, \mathrm{d}s + \iint_R K \, \mathrm{d}A + \sum_{i=1}^n \alpha_i = 2\pi$$

where $\alpha_1, \ldots, \alpha_n$ are the external angles at the points where γ is not smooth.

Proof. The proof is almost identical to the proof of the first version save that at those points where γ is not smooth there is a jump discontinuity in $\theta(s)$ of α_i where α_i is the external angle. The only amendment needed to the proof is that

$$\int_\gamma \dot{\theta} \, \mathrm{d}s = \Delta\theta = 2\pi - \sum_{i=1}^n \alpha_i.$$

■

Example 5.3 *Note that when we use internal angles $\beta_i = \pi - \alpha_i$, and when the curvilinear polygon R is bounded by geodesics, then we obtain*

$$\iint_R K \, \mathrm{d}A = \sum_{i=1}^n \beta_i - (n-2)\pi.$$

Thus the internal angle sum exceeds $(n-2)\pi$ by the total curvature. Focusing on triangles: in the plane, where $K = 0$, we have

$$\beta_1 + \beta_2 + \beta_3 = \pi,$$

whilst in the hyperbolic plane (where $K = -1$) we have Lambert's Theorem

$$A = \pi - \beta_1 - \beta_2 - \beta_3,$$

and on the sphere or elliptic plane (where $K = 1$) then we have Girard's Theorem

$$A = \beta_1 + \beta_2 + \beta_3 - \pi.$$

Theorem 5.4 ***(Global Gauss-Bonnet Theorem)*** *Let X be a smooth, closed, orientable surface. Then*

$$\iint_R K \, \mathrm{d}A = 2\pi\chi(X).$$

Proof. Say that X is subdivided by smooth curves into curvilinear polygons. We apply the local GBT to each of these polygons and sum each of the resulting equations.

The contributions $\int_\gamma k_g \, \mathrm{d}s$ cancel out as each edge bounds two polygons but with different orientations. For one orientation k_g is negative what it equals on the reverse orientation. The sum of the total curvature from each polygon equals the total curvature on the surface. And, using internal angles, we also need to sum the expressions

$$\sum_{i=1}^{n} \beta_i - n\pi + 2\pi.$$

The sum of the internal angles equals $2\pi V$ where V is the number of vertices. This is because at each vertex the internal angles add up to 2π. Now we have F faces so that

$$\sum_{\text{faces}} 2\pi = 2\pi F,$$

and each edge bounds two faces so that

$$\sum_{\text{faces}} n_{\text{face}}\pi = \sum_{\text{edges}} 2\pi = 2\pi E,$$

finally yielding

$$\sum_{\text{faces}} \left(\sum_{i=1}^{n} \beta_i - n_{\text{face}}\pi + 2\pi \right) = 2\pi V - 2\pi E + 2\pi F = 2\pi\chi(X).$$

■

Remark 5.5 *(a) We have assumed, without proof, that every compact, smooth surface has a subdivision. This is true – in fact this is more generally true for any separable smooth surface.*

(b) The above proof is for closed orientable surfaces. It relies on orientability when we refer to the opposite orientations of two curves. However the theorem also holds for non-orientable closed surfaces.

(c) In the next chapter we will discuss closed surfaces of constant curvature. The global Gauss-Bonnet theorem makes plain that only certain surfaces might be endowed with first fundamental forms with constant Gaussian curvature. A closed geometric surface with constant positive/zero/negative Gaussian curvature is necessarily a sphere/torus/torus with more than one hole. That is because their Euler characteristics are $2/0/\leqslant -2$. The theorem only gives necessity. As a sphere has constant positive curvature and as the flat torus has constant zero curvature, such surfaces are clearly possible. When we study quotients of the hyperbolic plane we will construct surfaces of constant curvature -1 of each positive genus.

Example 5.6 *Show that the catenoid $x^2 + y^2 = \cosh^2 z$ has a single, simple, closed geodesic.*

Solution. The 'waist' $z = 0$ is a simple, closed geodesic by Example 4.6(b) – it is a latitude where the radius function is at a minimum.

The catenoid can be parameterized by

$$\mathbf{r}(u, v) = (\cosh v \cos u, \cosh v \sin u, v), \qquad 0 < u < 2\pi,\ v \in \mathbb{R}.$$

The Gaussian curvature at the point $\mathbf{r}(u, v)$ equals $K(u, v) = -\cosh^{-4} v < 0$. Note that there cannot be a simple, closed geodesic that does not wrap once around the catenoid. By the local Gauss-Bonnet theorem we would then have

$$0 > \iint_R K \, \mathrm{d}A = 2\pi,$$

which is a contradiction. Suppose now that there were two simple, closed geodesics wrapping once around the catenoid. If these geodesics do not intersect and enclose a region R between them then we would have

$$0 > \iint_R K \, \mathrm{d}A = 2\pi\chi(R) = 0,$$

as the Euler characteristic of R (which is a cylinder) equals 0. Again we have a contradiction.

Finally suppose that the two geodesics do intersect and let R be the region bounded by them. Should they intersect once we would have

$$\iint_R K \, \mathrm{d}A + (\pi - \beta_1) + (\pi - \beta_2) = 2\pi,$$

where β_1 and β_2 are the two internal angles at the point of intersection. The LHS is less than 2π and so again we have a contradiction. Should the geodesics intersect more than once then we can focus on the geodesics between two points of intersection to get the same contradiction. ■

5.2 The Poincaré-Hopf theorem

Suppose that we are given a tangent vector $\mathbf{v}(x)$ at each point x of a smooth, closed surface X in $\mathbb{R}^3$. We can think of $\mathbf{v}(x)$ as the velocity at x of some fluid flow $\mathbf{v}$ on the surface. A point where $\mathbf{v}(x) = \mathbf{0}$ is called a **stationary (or singular) point** of the flow. It is a well known fact – the *hairy ball theorem* – that a flow on a sphere must have at least one stationary point. This is a consequence of the sphere's topology and we will more generally prove the *Poincaré-Hopf theorem* for surfaces which states that

$$\chi(X) = \sum_{\substack{\text{stationary} \\ \text{points } x}} \operatorname{index}(\mathbf{v}(x))$$

where the *index* (or *multiplicity*) is an integer associated with each stationary point, and assuming there to be finitely many stationary points.

If $x \in X$ is an isolated stationary point of $\mathbf{v}$ then we can find a small neighbourhood U of x such that $\mathbf{v}$ is non-zero on $U \backslash \{x\}$. Now let $\mathbf{e}$ be another smooth, nowhere zero, vector field defined on U; we will use $\mathbf{e}$ as a reference direction with which to compare the behaviour of

$\mathbf{v}(x)$. Let $\gamma(t)$ be a simple, closed, anticlockwise smooth curve in U which encircles x. Then $\mathbf{v}$ and $\mathbf{e}$ are both non-zero on γ and we defined the index as the winding number of $\mathbf{v}$ with respect to $\mathbf{e}$ as γ is traversed once. That is:

$$\text{index } = \frac{\Delta\psi}{2\pi} = \frac{1}{2\pi}\int_\gamma \frac{\mathrm{d}\psi}{\mathrm{d}t}\,\mathrm{d}t$$

where ψ is the angle between $\mathbf{v}$ and $\mathbf{e}$. Whilst ψ is only defined up to multiples of 2π this does not affect the total change $\Delta\psi$ in ψ.

Remark 5.7 *It is not immediately clear that the index of a stationary point is well-defined. It may depend on the choice of vector field* $\mathbf{e}$ *or on the curve* γ. *In Do Carmo, p.280, it is shown that*

$$\Delta\psi = \iint_R K\,\mathrm{d}A,$$

where R is the region bounded by γ. This then is independent of the choice of $\mathbf{e}$.

Say now that γ_0 and γ_1 are two simple, closed, positively oriented curves around x. These curves are then homotopic and it is possible to create a family of simple, closed, positively oriented curves γ_t, where $0 \leqslant t \leqslant 1$, which continuously deform from γ_0 to γ_1. Let $I(t)$ be the index as calculated using γ_t. Then $I(t)$ is a continuous, integer-valued function on $[0,1]$, so by connectedness $I(t)$ is constant and in particular $I(0) = I(1)$. This shows that the index is independent of the choice of γ.

Example 5.8 *Find the index of each of the following stationary points at the origin:*
(a) source: $\mathbf{v}(x,y) = (x,y)$.
(b) sink: $\mathbf{v}(x,y) = (-x,-y)$.
(c) vortex: $\mathbf{v}(x,y) = (-y,x)$.
(d) bifurcation: $\mathbf{v}(x,y) = (x,-y)$.
(e) dipole: $\mathbf{v}(x,y) = (x^2-y^2, 2xy)$.

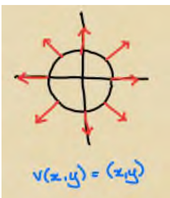

Figure. 5.1
source

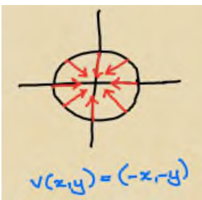

Figure 5.2
sink

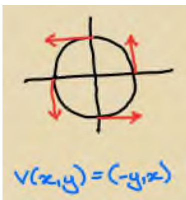

Figure 5.3
vortex

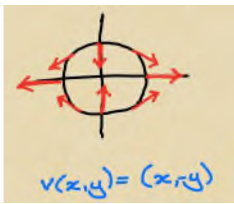

Figure 5.4
bifurcation

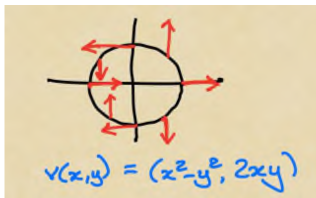

Figure 5.5
dipole

Solution. In each case, we will take γ to be the curve $\gamma(t) = (\cos t, \sin t)$ and $\mathbf{e} = (1,0)$.
(a) $\mathbf{v}(\cos t, \sin t) = (\cos t, \sin t)$ and so we may take $\psi = t$. Thus the index is 1.
(b) $\mathbf{v}(\cos t, \sin t) = (-\cos t, -\sin t) = (\cos(t+\pi), \sin(t+\pi))$ and so we may take $\psi = t+\pi$. Again the index is 1.
(c) $\mathbf{v}(\cos t, \sin t) = (-\sin t, \cos t) = (\cos(t+\pi/2), \sin(t+\pi/2))$ and so we may take $\psi = t+\pi/2$. Once more the index is 1.
(d) $\mathbf{v}(\cos t, \sin t) = (\cos t, -\sin t) = (\cos(-t), \sin(-t))$ and so we may take $\psi = -t$. Thus the index is -1.

(e) $\mathbf{v}(\cos t, \sin t) = (\cos^2 t - \sin^2 t, 2\sin t\cos t) = (\cos(2t), \sin(2t))$ and so we may take $\psi = 2t$. Thus the index is 2. ■

Theorem 5.9 *(Poincaré 1881, Hopf 1926) Let* $\mathbf{v}$ *be a smooth vector field on a smooth closed orientable surface* X *with finitely many stationary points. Then*

$$\chi(X) = \sum_{\substack{\text{stationary} \\ \text{points } x}} \textit{index}(\mathbf{v}(x)).$$

Remark 5.10 *Henri Poincaré proved the above theorem for surfaces in 1881. Heinz Hopf generalized the result to higher-dimenstional manifolds in 1926. Any continuous map of the unit circle can be assigned its degree – an integer describing how many times the circle wraps onto itself in an anticlockwise fashion and the index of a stationary point can be seen in this light. The degree of a map from a higher-dimensional sphere to itself can similarly be defined (Brouwer 1911) and the index of stationary points in higher dimensions can be simiarly understood.*

Proof. Let $x_1, \ldots, x_n$ be the stationary points of the vector field $\mathbf{v}$. Choose a smooth, simple, closed curve γ_i around each x_i enclosing a region R_i. Let

$$Y = X \Big\backslash \bigcup_{i=1}^{n} R_i.$$

At each point $y \in Y$ we may choose an orthonormal basis $\{\mathbf{e}_1(y), \mathbf{e}_2(y)\}$ for the tangent space at y and such that $\mathbf{e}_1(y)$ is in the direction of the non-zero $\mathbf{v}(y)$. Applying the argument of the local Gauss-Bonnet theorem to the region Y we obtain

$$\iint_Y K\,\mathrm{d}A = -\sum_{i=1}^{n} \int_{\gamma_i} \mathbf{e}_1 \cdot \dot{\mathbf{e}}_2\,\mathrm{d}s.$$

The negative sign is because the γ_i are oriented clockwise as part of the boundary of Y.

Choosing a similar orthonormal basis $\{\mathbf{f}_1, \mathbf{f}_2\}$ for the points in R_i we find that

$$\iint_{R_i} K\,\mathrm{d}A = \int_{\gamma_i} \mathbf{f}_1 \cdot \dot{\mathbf{f}}_2\,\mathrm{d}s.$$

Adding each of these equations ($i = 1, \ldots, n$) to the previous equation and applying the global Gauss-Bonnet theorem we obtain

$$2\pi\chi(X) = \iint_X K\,\mathrm{d}A = \sum_{i=1}^{n} \int_{\gamma_i} \left(\mathbf{f}_1 \cdot \dot{\mathbf{f}}_2 - \mathbf{e}_1 \cdot \dot{\mathbf{e}}_2\right)\,\mathrm{d}s.$$

From the proof of the local Gauss-Bonnet theorem we know that

$$\mathbf{e}_1 \cdot \dot{\mathbf{e}}_2 = \dot{\theta} - k_g, \qquad \mathbf{f}_1 \cdot \dot{\mathbf{f}}_2 = \dot{\varphi} - k_g$$

where θ and φ are the angles between $\dot{\gamma}$ and $\mathbf{e}_1$ and $\mathbf{f}_1$ respectively. Setting $\psi = \varphi - \theta$ to be the angle between $\mathbf{f}_1$ and $\mathbf{e}_1$ we obtain

$$\chi(X) = \sum_{i=1}^{n} \frac{1}{2\pi} \int_{\gamma_i} \dot{\psi}\,\mathrm{d}s = \sum_{i=1}^{n} \mathrm{index}(\mathbf{v}(x_i))$$

as required. ■

Corollary 5.11 *(**Hairy ball theorem**) A smooth vector field on a sphere must have at least one stationary point.*

Proof. The follows from the fact that the Euler characteristic of a sphere is two. ■

5.3 Analysis on a closed surface

We now apply the Poincaré-Hopf theorem to demonstrate a first result in *Morse theory.* Morse theory, named after Marston Morse, includes a wide selection of results relating a surface's topology to the behaviour of smooth real functions on the surface.

Proposition 5.12 *(**Gradient vector field**) Let X be a smooth surface in $\mathbb{R}^3$, $p \in X$ and $f\colon X \to \mathbb{R}$ be smooth. Then there is a unique tangent vector, denoted $(\mathrm{grad}_X\, f)(p)$ or $(\nabla_X f)(p)$, such that*

$$(\mathrm{grad}_X\, f)(p) \cdot \mathbf{v} = \mathrm{d}f_p\,(\mathbf{v}) \tag{5.21}$$

for any tangent vector $\mathbf{v} \in T_pX$.

Proof. Parameterize X locally as $\mathbf{r}\,(u,v)$. It follows that

$$(\mathrm{grad}_X\, f)(p) \cdot \mathbf{r}_u = \mathrm{d}f_p\,(\mathbf{r}_u) = f_u(p), \qquad (\mathrm{grad}_X\, f)(p) \cdot \mathbf{r}_v = \mathrm{d}f_p\,(\mathbf{r}_v) = f_v(p).$$

As $\mathbf{r}_u$ and $\mathbf{r}_v$ form a basis for T_pX then this specifies $(\mathrm{grad}_X\, f)(p)$ uniquely. As the scalar product and $\mathrm{d}f_p$ are both linear, then (5.21) holds on the entire tangent space. ■

Exercise 5.13 *In terms of the local co-ordinates u, v, show that*

$$\mathrm{grad}_X\, f = \left(\frac{f_uG - f_vF}{EG - F^2}\right)\mathbf{r}_u + \left(\frac{f_vE - f_uF}{EG - F^2}\right)\mathbf{r}_v.$$

It then follows that $\mathrm{grad}_X\, f = 0$ if and only if $f_u = f_v = 0$. This is left to Sheet 3, Part A, Exercise 1.

Note that when $X = \mathbb{R}^2$, parameterized with Cartesian co-ordinates x, y, then

$$\mathrm{grad}_X\, f = f_x\mathbf{i} + f_y\mathbf{j}$$

concurs with the usual definition of ∇f.

Definition 5.14 *Given a smooth surface X in $\mathbb{R}^3$ and a smooth function $f\colon X \to \mathbb{R}$, we say that $p \in X$ is a **critical point** of f if $(\mathrm{grad}_X\, f)\,(p) = 0$. Equivalently, if $\mathbf{r}(u,v)$ is a local parameterization around p, then p is a critical point if and only if*

$$\frac{\partial f}{\partial u}\,(p) = 0 = \frac{\partial f}{\partial v}\,(p)\,.$$

Example 5.15 *Let $f(x,y) = \cos \pi x + \cos \pi y$ on $\mathbb{R}^2$. Then*

$$\frac{\partial f}{\partial x} = -\pi \sin \pi x, \qquad \frac{\partial f}{\partial y} = -\pi \sin \pi y$$

are zero when x and y are integers.

Definition 5.16 *A critical point p is said to be* ***non-degenerate*** *if the Hessian matrix*

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

is nonsingular. If, further, the Hessian matrix is:

- *positive-definite then p is a local minimum;*
- *negative-definite then p is a local maximum;*
- *indefinite then p is a saddle point.*

A smooth real-valued function with only non-degerate critical points is called a ***Morse function****.*

It will become apparent in the proof of the next proposition that these definitions do indeed correspond to standard notions of a minima, maxima and saddle points.

Example 5.17 *With the above $f(x, y)$,the Hessian equals*

$$-\pi^2 \begin{pmatrix} \cos \pi x & 0 \\ 0 & \cos \pi y \end{pmatrix}$$

At $(0,0)\,, (1,0),$ and $(1,1)$ this respectively equals

$$-\pi^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad -\pi^2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \pi^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

These are respectively negative definite, indefinite and positive definite and so the points are respectively a maximum, a saddle point and a minimum.

Proposition 5.18 *Let f be a Morse function on a smooth patch X which has a critical point at p. Then $\mathrm{grad}_x f$ has:*
(a) index 1 at p if f has a minimum or a maximum at p.
(b) index -1 at p if f has a saddle point at p.

Proof. Take a conformal, local parameterization near $p = \mathbf{r}(0,0)$ and without loss of generality assume that $f(p) = 0$. In terms of these local co-ordinates Taylor's theorem states that

$$f(\mathbf{r}(u,v)) = \frac{1}{2}\left(f_{uu}(p)u^2 + 2f_{uv}(p)uv + f_{vv}(p)v^2\right) + \text{ higher order terms.}$$

Further, by the spectral theorem, we can may rotate the uv-plane so that

$$f(\mathbf{r}(u,v)) = \lambda u^2 + \mu v^2 + \text{ higher order terms,}$$

By assuming u and v to be suitably small we note that f has the same type of critical point as

$$g(\mathbf{r}(u,v)) = \lambda u^2 + \mu v^2.$$

This is (a) a minimum if $\lambda, \mu > 0$, (b) a maximum if $\lambda, \mu < 0$, (c) a saddle point if $\lambda\mu < 0$.

Now

$$\begin{aligned}(\mathrm{grad}_X\, g)(p) \cdot \mathbf{r}_u &= \frac{\mathrm{d}}{\mathrm{d}t} g(\mathbf{r}(t,0)) = 2\lambda t = 2\lambda u;\\ (\mathrm{grad}_X\, g)(p) \cdot \mathbf{r}_v &= \frac{\mathrm{d}}{\mathrm{d}t} g(\mathbf{r}(0,t)) = 2\mu t = 2\mu v.\end{aligned}$$

Taking $\mathbf{r}_u$ as the reference field and recalling that the parameterization is conformal, this means that the angle ψ between $\mathbf{r}_u$ and $\mathrm{grad}_X\, g$ satisfies

$$(\cos\psi, \sin\psi) = \frac{(\lambda u, \mu v)}{\sqrt{\lambda^2 u^2 + \mu^2 v^2}}.$$

A suitably small closed curve $\lambda^2 u^2 + \mu^2 v^2 = r^2$ around the point $p = \mathbf{r}(0,0)$ can be parameterized by

$$u = \frac{r}{|\lambda|} \cos t, \qquad v = \frac{r}{|\mu|} \sin t,$$

giving

$$\cos\psi = \frac{\lambda}{|\lambda|} \cos t, \qquad \sin\psi = \frac{\mu}{|\mu|} \sin t.$$

- Minimum: $\lambda, \mu > 0$ so that $\psi = t$ and the index is 1
- Maximum: $\lambda, \mu < 0$ so that $\psi = \pi + t$ and the index is 1.
- Saddle: $\lambda < 0 < \mu$ so that $\psi = \pi - t$ and the index is -1.
- Saddle: $\lambda > 0 > \mu$ so that $\psi = 2\pi - t$ and the index is -1.

■

Theorem 5.19 *Given a Morse function f on a smooth, orientable surface X then*

$$\chi(X) = \#\,(\mathit{maxima}) - \#\,(\mathit{saddles}) + \#\,(\mathit{minima})\,.$$

Proof. Apply the Poincaré-Hopf theorem to the vector field $\mathrm{grad}_X(f)$, taking note of the previous proposition. ■

Example 5.20 *The function $f(x,y) = \cos\pi x + \cos\pi y$ has period 2 in both the x and y variables. $f(x,y)$ descends to a well-defined smooth function $\tilde{f}(x,y)$ on $\mathbb{R}^2/(2\mathbb{Z})^2$ which is diffeomorphic to the torus $\mathbb{T}$. $\tilde{f}(x,y)$ has a maximum at $(0,0)$, saddle points at $(1,0)$ and $(0,1)$ and a minimum at $(1,1)$. Hence*

$$\chi(\mathbb{T}) = 1 - 2 + 1 = 0.$$

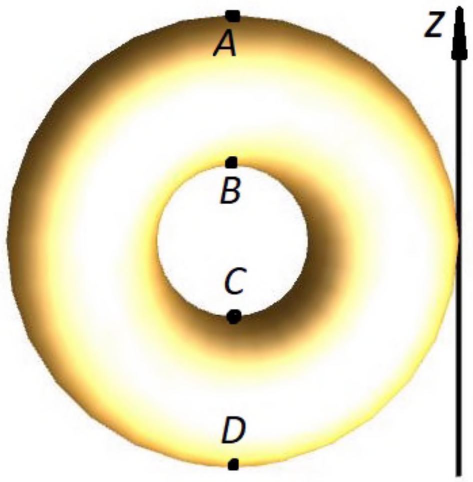

Figure 5.6 – height function on a torus

Example 5.21 *Consider the height function z on the torus $\mathbb{T}$ as depicted in Figure 5.6. There is a maximum at the top of the torus (point A), a minimum at the bottom of the torus (point D) and two saddle points at points B and C. Hence*

$$\chi(\mathbb{T}) = 1 - 2 + 1 = 0.$$

Remark 5.22 *Theorem 5.19 is part of a broader subject called Morse theory, a subject within differential topology which relates differentiable functions on a surface to the surface's topology. It is named after Marston Morse (1892-1977) who first wrote on the subject in 1925.*

Revisiting the example of the height function on a torus, consider the sets

$$X_h = \{(x, y, z) \in \mathbb{T} \mid z \leqslant h\}.$$

Note that the topology of these sets only changes as h achieves the value of one of the critical points' heights. In fact Morse showed that two such sets X_h and X_k would have the same 'homotopy type' if no critical height lay between h and k. This notion of homotopy equivalence is a type of topological equivalence, though weaker than that of being homeomorphic. Further Morse showed how the topology of X_h changes as h passes through a critical height. When h passes through a maximum (at A) or a minimum (at D) a 2-cell (a disc) is attached to the set, but when h passes through a saddle point (at B and C) a 0-cell (a point) is adjoined to the set.