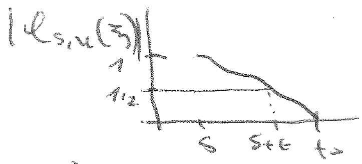
1. a) ✓

b) ✓

c) Set $t_0 = \inf\{t \geq s : \ell_{s,t}(\xi) = 0\}$.



Want to prove $t_0 = \infty$.

$\ell_{s,s}(\xi)=1$ and $u \mapsto |\ell_{s,u}(\xi)|$ cont., so

ex. $\epsilon > 0$ w/ $|\ell_{s,u}(\xi)| \geq \frac{1}{2}$ for $u \in [s, s+\epsilon)$

$\Rightarrow$ $t_0 > s$ and by b)

$$0 = \ell_{s,t_0}(\xi) = \ell_{s,t}(\xi)\,\ell_{t,t_0}(\xi).$$

If $t \in [s, s+\epsilon)$, then $\ell_{s,t}(\xi) \neq 0$, so $\ell_{t,t_0}(\xi) = 0 \ \forall t \in [s, s+\epsilon)$.

Thus,

$$0 = \lim_{t \downarrow s} \ell_{t,t_0}(\xi) = \ell_{t_0,t_0}(\xi) = 1 \quad ↯$$

$\Rightarrow$ $t_0 = \infty$.

d) ✓

2. a) Y_n and K_n are clear.

Need surely cont. or complete p.s.

$[0,1)$ is uncountable and "$\exists s \in [0,1)$" is equivalent to union over uncountable set $[0,1)$, so measurability not immediately obvious.

But BM B has continuous trajectories (a.s.), so can typically replace union over $[0,1)$ by union over $[0,1) \cap \mathbb{Q}$.

Absolute condition

$$\cancel{\exists s \in [0,} \quad |B_t - B_s| \leq C|t-s| \text{ if } |t-s| \leq \frac{3}{n}$$

$$\cancel{\Leftrightarrow}\ \zeta_s^n := \sup_{\substack{t \in (0,1) \\ |t-s| \leq \frac{3}{n}}} |B_t - B_s| - C|t-s| \leq 1,$$

so

$$A_n = \{\cancel{\zeta_s \leq 0} : \exists s \in [0,1) : \zeta_s^n \leq 0\}.$$

Now $s \mapsto \zeta_s^n$ is cont., so

$$A_n = \{\exists s \in [0,1) : \zeta_s^n \leq 0\} = \bigcap_{i \geq 1} \{\exists s \in \mathbb{Q} \cap [0,1) : \zeta_s^n \leq \tfrac{1}{i}\}.$$

b) $n \to A_n$ increasing clear.

$\omega \in A_n \Rightarrow \exists s$ w/ $|B_t - B_s| \le C|t-s|$ $\forall t$ $|t-s| \le \frac{3}{n}$.

Take $k \in \{1,\dots,n-2\}$ w/ $\frac{k}{n} \le s < \frac{k+1}{n}$. (if $s \in [0,\frac{1}{n}]$ or $s \in [\frac{n-1}{n},1]$ need to slightly modify)

Then for $j = 0,1,2$,

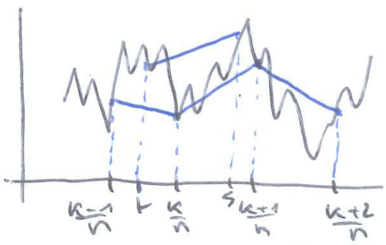
$\cancel{|B_{\frac{k}{n}} - B_{\frac{k-1}{n}}|}$

$$|B_{\frac{k+j}{n}} - B_{\frac{k+j-1}{n}}| \le |B_{\frac{k+j}{n}} - B_s| + |B_s - B_{\frac{k+j-1}{n}}|, \quad (1)$$

but $\left|\frac{k+j}{n} - s\right| \le \frac{j+1}{n} \le \frac{2}{n}$ and

$\left|s - \frac{k+j-1}{n}\right| \le \frac{j+1}{n} \le \frac{2}{n}$ so

$$(1) \le C\left|\frac{2}{n}\right| + C\left(\frac{2}{n}\right) = \frac{4C}{n}.$$



(In edge cases $s \in [0,\frac{1}{n}]$ or $\in [\frac{n-1}{n},1]$ get factor $\frac{5C}{n}$).

c) ✓

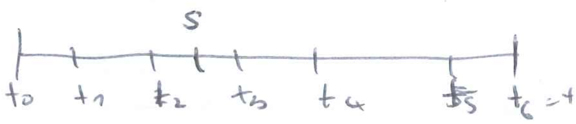
d) ✓

e) ✓

3. a) ✓

b) need to show

$\mathbb{E}[X_{n-1} | \mathcal{G}_n] = X_n$.



~~Here~~ wlog $D_{n-1} = \{0 = t_0 < t_1 < \dots < t_{n-1} = t\}$ and A_1 insert $s \in (t_{j-1}, t_j)$.

$\cancel{\mathbb{E}[X_{n-1}}$

$s \in \cancel{(t_i, t_{i+1})}$ (t_{j-1}, t_j), so

$\cancel{X_n = \sum_{j=1, j \ne t_i, t_{i+1}}^{n-1} |X_{t_j} - X_{t_{j-1}}|^2 + |X}$

$$\text{Then } X_n = \underbrace{\sum_{\substack{i=1\\ i\ne j}}^{n-1} |B_{t_i} - B_{t_{i-1}}|^2}_{} + |B_{t_j} - B_s|^2 + |B_s - B_{t_{j-1}}|^2 \underbrace{+ |B_{t_j} - B_{t_{j-1}}|^2}_{} - |B_{t_j} - B_{t_{j-1}}|^2$$

$$= \underbrace{X_{n-1}}_{=X_{n-1}} - |B_{t_j} - B_{t_{j-1}}|^2 + |B_{t_j} - B_s|^2 + |B_s - B_{t_{j-1}}|^2.$$

so

$$\mathbb{E}[X_{n-1} | \mathcal{G}_n] = X_n + \mathbb{E}[|B_{t_j} - B_{t_{j-1}}|^2 - |B_{t_j} - B_s|^2 - |B_s - B_{t_{j-1}}|^2 | \mathcal{G}_n]$$

Idea: decompose $\mathcal{G}_n$ into stuff below t_{j-1}, after t_j, and in between

Def $\hat{\mathcal{G}}_n = \mathcal{F}_{t_{j-1}} \vee \sigma(|B_r - B_u|^2 : t_j \le u \le r \le 1)$ « I added this square to make things more simple — all independent

$$\vee\ \sigma\Big(\cancel{B_{t_{j-1}}} \ldots + \sum_{i=1}^{m_-} |B_{s_i^-} - B_{s_{i-1}^-}|^2 + \sum_{i=1}^{m_+} |B_{s_i^+} - B_{s_{i-1}^+}|^2 : t_{j-1} = s_0^- < \ldots < s_{m_-}^- = t = s_0^+ < \ldots < s_{m_+}^+ = t_j \Big)$$

$$\supset \mathcal{G}_n,$$

so $$\mathbb{E}[X_{n-1} \mid \mathcal{G}_n] = \mathbb{E}\big[\mathbb{E}[X_{n-1} \mid \hat{\mathcal{G}}_n] \mid \mathcal{G}_n\big]$$

Hence, need to show

$$\mathbb{E}\big[|B_{t_{j+1}} - B_{t_{j-1}}|^2 - |B_{t_j} - B_s|^2 - |B_s - B_{t_{j-1}}|^2 \mid \hat{\mathcal{G}}_n\big] = 0.$$

Have

$$\mathbb{E}\big[|\underbrace{(B_{t_j} - B_s)}_{Y} + \underbrace{(B_s - B_{t_{j-1}})}_{Z}|^2 \mid \hat{\mathcal{G}}_n\big]$$

symmetry (* see page 5)

$$= \mathbb{E}\big[|-(B_{t_j} - B_s) + (B_s - B_{t_{j-1}})|^2 \mid \hat{\mathcal{G}}_n\big].$$

new "−" here

taking difference implies

$$\mathbb{E}\big[\underbrace{(B_{t_j} - B_s)}_{Y}\underbrace{(B_s - B_{t_{j-1}})}_{Z} \mid \hat{\mathcal{G}}_n\big] = 0,$$

so that

$$\mathbb{E}[(Y+Z)^2 \mid \hat{\mathcal{G}}_n] = \mathbb{E}[Y^2 + 2YZ + Z^2 \mid \hat{\mathcal{G}}_n]$$
$$= \mathbb{E}[Y^2 + Z^2 \mid \hat{\mathcal{G}}_n] = Y^2 + Z^2$$

Then by Levy's downward thm w/ $\mathcal{G}_\infty = \bigcap_{n \ge 1} \mathcal{G}_n$,

$$\lim_{n\to\infty} X_n = \mathbb{E}[X_1 \mid \mathcal{G}_\infty].$$

But $\mathbb{E}X_1 = t$ and $\operatorname{var} X_n \to 0$, so that $\lim_{n\to\infty} X_n$ is a.s. constant.
Thus must have $\lim X_n = t$.

c) i) ✓
ii) ✓
iii) ✓

4 a) ✓

b) By $\mathbb{P}(M_t \geq y, W_t \in dx)$ mean the measure
$$\mu_{t,y}(A) := \mathbb{P}(M_t \geq y, W_t \in A) \quad \text{(not a prob).}$$

Idea: apply Girsanov to remove drift. Def $Z_t = \cancel{\exp(\mu W_t - \frac{1}{2}\mu^2 t)}$, $\mathcal{E}(-\mu W)_t = \exp(-\mu W_t - \frac{1}{2}\mu^2 t)$
then under $\mathbb{Q}$ def via $\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = Z_t$,

$\cancel{W^{\mathbb{Q}}} = W_t - \langle W, -\mu W \rangle_t = W_t + \mu t = W_t^{\mu}$ is BM. So by a),

$$\frac{1}{h} \int_x^{x+h} \underbrace{\frac{1}{\sqrt{2\pi}} \exp(-(2y-z)^2/2t)}_{=: f(z)} dz \to \frac{1}{\sqrt{2\pi}} \exp(-(2y-x)^2/2t)$$

$$= \frac{1}{h} \mathbb{Q}(M_t^{\mu} \geq y, W_t^{\mu} \in [x, x+h]) \quad (= \mathbb{E}^{\mathbb{Q}} \mathbb{1}_{\{M_t^{\mu} \geq y, W_t^{\mu} \in [x, x+h]\}})$$

$$= \frac{1}{h} \mathbb{E}^{\mathbb{P}}[Z_t \mathbb{1}_{\{M_t^{\mu} \geq y, W_t^{\mu} \in [x, x+h]\}}]$$

$$Z_t = \exp(-\mu W_t - \tfrac{1}{2}\mu^2 t) = \cancel{\exp(-\mu} \exp(-\mu W_t - \mu^2 t)\exp(\tfrac{1}{2}\mu^2 t)$$
$$= \exp(-\mu W_t^{\mu}) \exp(\tfrac{1}{2}\mu^2 t)$$
$$= \exp(-\mu x) \exp(-\mu(W_t^{\mu} - x)) \exp(\tfrac{1}{2}\mu^2 t)$$
$$= \underbrace{\exp(-\mu x + \tfrac{1}{2}\mu^2 t)}_{=: a} \text{ -- '' --}$$
$$= a + a(e^{-\mu(W_t^{\mu} - x)} - 1)$$

c) ✓

$$= \frac{1}{h} \exp(-\mu x + \tfrac{1}{2}\mu^2 t) \, \mathbb{P}(M_t^{\mu} \geq y, W_t^{\mu} \in [x, x+h])$$

$$+ \frac{1}{h} a \, \mathbb{E}\Big[\underbrace{(e^{-\mu(W_t^{\mu} - x)} - 1)}_{|\cdot| \leq (1 - e^{-\mu h})} \mathbb{1}_{\{M_t^{\mu} \geq y, W_t^{\mu} \in [x, x+h]\}}\Big] \quad \in (x, x+h)$$

$$|\cdot| \leq (1 - e^{-\mu h}) \, \mathbb{P}(W_t^{\mu} \in [x, x+h]) = o(h)$$

$$\to \exp(-\mu x + \tfrac{1}{2}\mu^2 t) \, \mathbb{P}(M_t^{\mu} \geq y, W_t^{\mu} \in dx)$$

Now rearrange

c) ✓

To show that

$$\mathbb{E}\Big[\big(\underbrace{(B_{t_j} - B_s)}_{=:Y} + \underbrace{(B_s - B_{t_{j-1}})}_{=:Z}\big)^2 \,\Big|\, \mathcal{G}_n\Big]$$

new "–" here

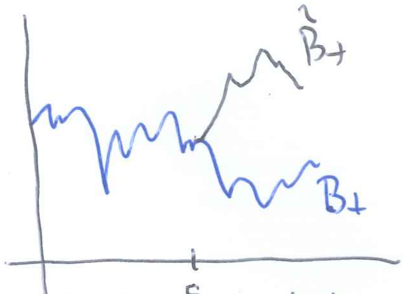
$$= \mathbb{E}\big[(-(B_{t_j} - B_s) + (B_s - B_{t_{j-1}}))^2 \,\big|\, \mathcal{G}_n\big],$$

we must show that $\forall A \in \mathcal{G}_n$

$$\mathbb{E}[(Y+Z)^2 \mathbb{1}_A] = \mathbb{E}[(-Y+Z)^2 \mathbb{1}_A]. \quad (1)$$

Let us define $\hat{B} = (\hat{B}_t)_{0 \le t \le 1}$ by

$$\hat{B}_t = \begin{cases} B_t, & s \le t \\ B_s - (B_t - B_s), & s > t, \end{cases}$$



The process $\hat{B}$ is a BM and for all $A \in \mathcal{G}_n$ we have $(B, \mathbb{1}_A) \sim (\hat{B}, \mathbb{1}_A)$. Indeed, note that in the definition of $\mathcal{G}_n$, ~~that~~ all dependences on B ~~are~~ after time s are through increments $|B_r - B_u|^2$. But we have

$$|B_r - B_u|^2 = |\hat{B}_r - \hat{B}_u|^2.$$

Hence, $(B, \mathbb{1}_A) \sim (\hat{B}, \mathbb{1}_A)$. Now, this gives

$$\mathbb{E}[((B_{t_j} - B_s) + (B_s - B_{t_{j-1}}))^2 \mathbb{1}_A]$$

$$= \mathbb{E}[(\underbrace{(\hat{B}_{t_j} - \hat{B}_s)}_{= -(B_{t_j} - B_s)} + \underbrace{(\hat{B}_s - \hat{B}_{t_{j-1}})}_{= B_s - B_{t_{j-1}}})^2 \mathbb{1}_A]$$

$$= \mathbb{E}[(-(B_{t_j} - B_s) + (B_s - B_{t_{j-1}}))\mathbb{1}_A],$$

which is (1).