C3.10 Additive and Combinatorial Number Theory, HT23 Sheet 0

This example sheet will not be marked, and nor will solutions be provided. I am happy to discuss any issues with students after lectures or classes. The aim is to get you a little practice with some basic techniques of the kind that we will see in the course.

Question 1. This question is about basic practice with analytic quantities and asymptotic notation.

(i) Show that $x^{10} \ll e^x$ for real numbers $x \geqslant 1$. (Hint: series expansion.)

(ii) Show that $\log^2 n = O(\sqrt{n})$ for positive integers n. (Hint: previous part. Also, note that $\log^2 n$ is standard notation meaning the same thing as $(\log n)^2$.)

(iii) Let $x \geqslant 1$. Show that $f(x) \ll e^{\sqrt{\log x}}$ when $f(x)$ is any fixed power of $\log x$, but that $e^{\sqrt{\log x}} \ll f(x)$ when $f(x)$ is any fixed power of x.

(iv) Let $f(x) = x^{1/\log\log x}$ and $g(x) = e^{(\log x)^{2/3}}$, for $x \geqslant 10$. Is either $f(x) \ll g(x)$ or $g(x) \ll f(x)$ true, and if so which?

(v) Show that $\sqrt{n} \ll \sum_{i=1}^n \frac{1}{\sqrt{i}} \ll \sqrt{n}$ for positive integer n.

Question 2. The aim of this question is to think about what are typically called 'averaging arguments'. Let X be a finite set, and let $f : X \to \mathbb{R}$ be a function with $|f(x)| \leqslant 1$ for all x. Let $0 < \delta < 1$.

(i) Suppose that $|X|^{-1}|\sum_{x\in X} f(x)| \geqslant \delta$. Show that, for at least $\delta|X|/2$ values of x, we have $|f(x)| \geqslant \delta/2$.

(ii) Suppose that $|X|^{-1}|\sum_{x\in X} f(x)| \geqslant 1-\delta$. Show that, for at least $(1-\sqrt{\delta})|X|$ values of x, we have $|f(x)| \geqslant 1-\sqrt{\delta}$.

Question 3. The aim of this question is to think a little bit about the behaviour of the divisor function $\tau(n)$, that is to say the number of divisors of n.

(i) Show that $\sum_{n=1}^N \tau(n) \ll N\log N$, uniformly for $N \geqslant 2$. (Hint: write $\tau(n) = \sum_{d|n} 1$ and then interchange the order of summation.)

(ii) Show that there are infinitely many n for which $\tau(n) > \log^{10} n$.

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