1 Disclaimer

These are **draft** lecture notes in expanded form. *If you would like to have a coherent, checked and correct account of Analytic Topology, you are better served by reading the relevant part of various textbooks.* These notes are **not** carefully checked (so will contain errors), may contain material which is not part of the course and may not contain all material I will cover in the lectures. In particular, motivation and order of concepts may differ and I may choose other (equivalent) definitions.

2 Basics

I expect this to have been covered in a first Topology course.

Definition 2.1. A topology on a set X is a collection τ of subsets of X such that τ is closed under finite intersections and arbitrary unions.

A topological space is a pair (X, τ) where X is a set and τ is a topology on X.

Note that I use the convention that $\bigcup \emptyset = \emptyset$ and $\bigcap \emptyset$ is everything (where everything depends on the context).

Definition 2.2. Suppose (X, τ) is a topological space and $A \subseteq X$.

A is X-*open* or *an open subset of* X if and only if $A \in \tau$.

A is X-*closed* if and only if $X \setminus A \in \tau$.

The *closure* of A, $\overline{A}$, is the intersection of all closed subsets of X containing A. If the space/topology with respect to which the closure is taken is unclear we will use $\overline{A}^X$.

The *interior* of A, $\operatorname{int}(A)$, is the union of all open subsets of X contained in A.

Lemma 2.3. *Suppose X is a topological space and $A \subseteq X$.*

1. *Finite unions and arbitrary intersections of closed sets are closed.*
2. *$\overline{A}$ is closed and*
$$\overline{A} = \{x \in X : \forall \text{ open } U \subseteq X \ U \cap A \neq \emptyset\}.$$
Hence $\overline{A}$ is the smallest closed set containing A.
3. $\operatorname{int}(A)$ *is open and*
$$\operatorname{int}(A) = \{x \in X : \exists \text{ open } U \subseteq X \ x \in U \subseteq A\}.$$
Hence $\operatorname{int}(A)$ *is the largest open set contained in A.*

Definition 2.4. Suppose X, Y are topological spaces and $f: X \to Y$ a function.

f is *continuous* if and only if the preimage of every Y-open set under f is X-open.

f is a homeomorphism if and only if f is a continuous bijection with continuous inverse.

Lemma 2.5. *Suppose X, Y are topological spaces and $f: X \to Y$ a function. TFAE:*

1. *f is continuous.*
2. *the preimage of every Y-closed set under f is X-closed.*
3. *for every $A \subseteq X$, $f\left(\overline{A}\right) \subseteq \overline{f(A)}$.*

Definition 2.6. Suppose X is a topological space with topology τ and $A \subseteq X$.

The *subspace topology* on A is $\{U \cap A : U \in \tau\}$.

Lemma 2.7. *Suppose X is a topological space, $Y \subseteq X$ and $A \subseteq Y$.*

Then $\overline{A}^Y = Y \cap \overline{A}^X$.

Lemma 2.8. *Suppose X, Y are topological spaces, $f: X \to Y$ a function, $A \subseteq X$ and $B \subseteq Y$ (equipped with the subspace topology).*

If $f(X) \subseteq B$ then f is continuous if and only if $f': X \to B; x \mapsto f(x)$ is continuous.

If f is continuous then $f|_A$ is continuous.

Lemma 2.9 (Pasting Lemma)**.** *Suppose X, Y are topological spaces, A, B closed $\subseteq X$ such that $A \cup B = X$, $f: A \to Y, g: B \to Y$ functions.*

If f, g are continuous and agree on $A \cap B$ (i.e. $f|_A = g|_A$ then the function $h = f \cup g: X \to Y$ given by $x \mapsto \begin{cases} f(x); & x \in A \\ g(x); & x \in B \end{cases}$ is well-defined and continuous.

3 Bases, Subbases

Definition 3.1. Suppose (X, τ) is a topological space. A collection $\mathcal{B}$ of subsets of X is a *basis* (of open sets) for τ if and only if

(B0) $\mathcal{B} \subseteq \tau$;

(B1) $\forall U \in \tau\ \exists \mathcal{B}' \subseteq \mathcal{B}: \ U = \bigcup \mathcal{B}'$.

We say that $\mathcal{B}$ generates τ or that τ is generated by (the basis)

If a basis for a topological space has been fixed, we call its elements *basic open* sets.

Note that (B1) can be restated as $\forall U \in \tau\ \forall x \in U\ \exists B \in \mathcal{B}\ x \in B \subseteq U$.

Lemma 3.2. *Suppose X is a set and $\mathcal{B}$ is a collection of subsets of X such that*

(B0') $X = \bigcup \mathcal{B}$*;*

(B1') $\forall A, B \in \mathcal{B}\ \exists \mathcal{B}' \subseteq \mathcal{B}\ A \cap B = \bigcup \mathcal{B}'$*.*

Then $\mathcal{B}$ *is a basis for a unique topology* τ *on* X*, namely*

$$\left\{ \bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B} \right\}.$$

Proof. It is straightforward to check that the given τ is a topology (note that we use the convention that $\bigcup \emptyset = \emptyset$), that $\mathcal{B}$ is a basis for it and that if σ is a topology on X with basis $\mathcal{B}$ then $\tau = \sigma$. □

Note that (B1') is satisfied if $\mathcal{B}$ is closed under binary (finite) intersections. Also (B1') is of course equivalent to

$$\forall A, B \in \mathcal{B}\ \forall x \in A \cap B\ \exists C \in \mathcal{B}\ x \in C \subseteq A \cap B.$$

Definition 3.3. Suppose (X, τ) is a topological space. A collection $\mathcal{S}$ of subsets of X is a *subbasis* (of open sets) for τ if and only if

$$\left\{ \bigcap \mathcal{F} : \mathcal{F} \text{ finite } \subseteq \mathcal{S} \right\}$$

is a basis for (X, τ).

We say that $\mathcal{S}$ generates τ or that τ is generated by (the subbasis) $\mathcal{S}$.

If a subbasis for a topological space has been fixed, we call its elements *subbasic open* sets.

Lemma 3.4. *Suppose* X *is a set and* $\mathcal{S}$ *a collection of subsets of* X*.*

Then

$$\mathcal{B}_{\mathcal{S}} = \left\{ \bigcap \mathcal{F} : \mathcal{F} \textit{ finite } \subseteq \mathcal{S} \right\}$$

satisfies [B0'] and [B1'] and hence $\mathcal{S}$ *is a subbasis for a topology* τ *on* X*.*

Moreover, this topology is uniquely determined by $\mathcal{S}$*.*

Proof. Note that $\bigcap \emptyset = X$ (by convention in this course) so that $\mathcal{B}$ satisfies [B0']. By construction $\mathcal{B}_{\mathcal{S}}$ is closed under binary intersections so that [B1'] holds as well.

Uniqueness follows from 3.2. □

Lemma 3.5. *Suppose* X, Y *are topological space,* $\mathcal{S}$ *is a subbasic for* Y *and* $f\colon X \to Y$ *is a function. Then* f *is continuous if and only if for each subbasic open set (of* Y*) has open preimage.*

Proof. Preimages commute with unions and intersections. □

3.1 Examples

1. If (X,d) is a metric space then $\{B_\epsilon(x) : x \in X, \epsilon > 0\}$ is a basis for X.
2. $\{(a,b) \colon a,b \in \mathbb{Q}\}$ is a basis for $\mathbb{R}$
3. $\{[a,b) \colon a,b \in \mathbb{R}\}$ is a basis for the Sorgenfrey line.
4. If X is a linearly ordered space, $\{(-\infty,a) \colon a \in X\} \cup \{(a,\infty) \colon a \in X\}$ is a subbasis for the order topology on X.
5. $\{U \times \mathbb{R} \colon U \text{ open } \subseteq \mathbb{R}\} \cup \{\mathbb{R} \times U \colon U \text{ open } \subseteq \mathbb{R}\}$ is a subbasis for $\mathbb{R}^2$.

4 The Lattice of Topologies

Definition 4.1. Suppose X is a set.

Write $\mathcal{T}$ for the set of all topologies on X and define a relation $\leq$ on $\mathcal{T}$ by $\tau \leq \sigma \iff \tau \subseteq \sigma$, in which case we say that τ is *coarser/weaker/smaller* than σ and σ is *finer/stronger/greater* than τ.

Lemma 4.2. *If $\tau_i, i \in I$ are topologies on X then so is $\tau = \bigcap_{i\in I} \tau_i$. Moreover, τ is the greatest topology contained in each of the τ_i.*

Proof. It is straightforward to verify the conditions and check the last sentence. □

Lemma 4.3. *$(\mathcal{T}, \leq)$ is a complete lattice, i.e. a partially ordered set such that any (non-empty) subset $\mathcal{T}'$ of $\mathcal{T}$ has a* sup *and* inf*.*

Proof. That $\leq$ is a partial order (reflexive, transitive, anti-symmetric) is immediate. That infs exist follows from the previous lemma (i.e. $\inf \mathcal{T}' = \bigcap \mathcal{T}'$) and finally $\sup \mathcal{T}' = \inf\{\tau \in \mathcal{T} \colon \forall \sigma \in \mathcal{T}' : \sigma \leq \tau\}$ (noting that $\mathcal{P}(X)$ is the greatest topology on X so the inf is well-defined). □

Note that $\bigcup \tau_i$ is, in general, not a topology but only a subbasis for the topology $\sup \tau_i$.

Lemma 4.4. *The topology generated by a subbasis is the smallest topology containing this subbasis.*

Proof. Let τ be the topology generated by a subbasis $\mathcal{S}$. Clearly $\mathcal{S} \subseteq \tau$ since $\bigcap\{A\} = A$ for each A.

If σ is a topology containing $\mathcal{S}$ then σ must contain all finite intersections of elements of $\mathcal{S}$, i.e. $\mathcal{B}_{\mathcal{S}}$. But every element of τ is the union of elements of $\mathcal{B}_{\mathcal{S}}$ and hence must be in σ. Thus $\tau \subseteq \sigma$ as required. □

5 Initial Topology and Products

Theorem 5.1. *Suppose X is a set, $(Y_i, \tau_i), i \in I$ is a family of topological spaces and $f_i \colon X \to Y_i, i \in I$ is a family of functions. There is a unique topology τ on X, called the* initial topology *on X with respect to the family $\{f_i \colon i \in I\}$, such that:*

(Test Condition) for every topological space Z and for every function $f \colon Z \to X$, f is continuous with respect to τ if and only if for every $i \in I$ $f_i \circ f$ is continuous.

This unique topology τ is generated by the subbasis $\left\{ {f_i}^{-1}(U) : i \in I, U \in \tau_i \right\}$. Moreover, τ is the coarsest topology on X such that all the f_i are continous.

Proof. We first show uniqueness: suppose σ and τ are two topologies satisfying the Test Condition and consider the map $\mathrm{id}_{\tau,\tau} \colon (X, \tau) \to (X, \tau)$. Since $\mathrm{id}_{\tau,\tau}$ is continuous and τ satisfies the Test Condition, each $f_i = f_i \circ \mathrm{id}_{\tau,\tau}$ is continuous (from τ to τ_i). Now consider $\mathrm{id}_{\tau,\sigma} \colon (X, \tau) \to (X, \sigma)$. Since $f_i \circ \mathrm{id}_{\tau,\sigma} = f_i$ is continous (from τ to τ_i) and σ satisfies the Test Condition, $\mathrm{id}_{\tau,\sigma}$ is continuous so that $\sigma \subseteq \tau$. By symmetry $\tau \subseteq \sigma$ and hence uniqueness follows.

Clearly each f_i is continuous from the given τ.

For existence, we only need to verify that the given τ has satisfies the Test Condition: so suppose that Z is a topological space and $f \colon Z \to X$ is a function. If f is continuous (into (X, τ)), then, since each f_i is continuous, $f_i \circ f$ is a composition of continuous function so continuous. Conversely, if each $f_i \circ f$ is continuous and S is subbasic open then $S = {f_i}^{-1}(U)$ for some open $U \subseteq Y_i$ so that $f^{-1}(S) = (f_i \circ f)^{-1}(U)$ is Z-open.

The final sentence follows from elementary set algebra. □

Definition 5.2. Suppose $X_i, i \in I$ are topological spaces. The *(Tychonoff) product topology* on $P = \prod_{i \in I} X_i = \left\{ f \colon I \to \bigcup_{i \in I} X_i \colon \forall i \in I \ f(i) \in X_i \right\}$ is the initial topology with respect to the natural projections $\pi_i \colon P \to X_i$ given by $\pi_i(f) = f(i)$.

Lemma 5.3. *Suppose $X_i, i \in I$ and Y are topological spaces.*

The product topology has basis consisting of all sets of the form $\prod_{i \in I} U_i$ where each U_i is an open subset of X_i and for all but finitely many $i \in I$ we have $U_i = X_i$. When no explicit basis for the product topology is given, we assume that this basis has been selected.

A function $f \colon Y \to \prod_i X_i$ is continuous if and only if all of the compositions $\pi_i \circ f \colon Y \to X_i$ are continuous and the product topology is the unique topology on $\prod_i X_i$ satisfying this result.

Proof. Straightforward verification: we note that

$$\prod_{i \in I} U_i = \bigcap_{i \in I} {\pi_i}^{-1}(U_i) = \bigcap_{i \in I : U_i \neq X_i} {\pi_i}^{-1}(U_i)$$

so that the given collection of sets is indeed a basis for the product topology.

The sentence about functions follows immediately from the fact that the product topology is the initial topology with respect to the π_i. □

Definition 5.4. Suppose X and $Y_i, i \in I$ are sets and $f_i \colon X \to Y_i, i \in I$ are functions. The *diagonal map* of the f_i is the map

$$\Delta_{i \in I} f_i \colon X \to \prod Y_i ; x \mapsto (f_i(x))_{i \in I}.$$

Lemma 5.5. *If all the f_i are continuous then so is the diagonal map.*

Proof. By the Test Condition for initial topologies (recall that the product topology is the initial topology with respect to the π_i) it is enough to observe that for each $i \in I$

$$\pi_i \circ \Delta = f_i.$$

□

6 Low Separation Properties

Definition 6.1. Suppose X is a topological space.

X is T_0 if and only if for any distinct $x, y \in X$ there is open U containing exactly one of x and y.

X is T_1 if and only if for any ordered pair $(x, y) \in X^2 \setminus \Delta$ of distinct points there is open $U \subseteq X$ such that $x \in U \not\ni y$ (which is equivalent to $x \in U \subseteq X \setminus \{y\}$).

X is T_2 (*Hausdorff*) if and only if for any distinct $x, y \in X$ there are disjoint open $U \ni x$, $V \ni y$ if and only if for any distinct $x, y \in X$ there is open U such that

$$x \in U \subseteq \overline{U} \subseteq X \setminus \{y\}.$$

X is T_3 (*regular*) if and only if X is T_1 and for any $x \notin C$ closed $\subseteq X$ there are disjoint open $U \ni x, V \supseteq C$ if and only if X is T_1 and for any $x \notin C$ closed $\subseteq X$ there is open U such that

$$x \in U \subseteq \overline{U} \subseteq X \setminus C.$$

X is $T_{3.5}$ (*Tychonoff*) if and only if X is T_1 and for any $x \notin C$ closed $\subseteq X$ there is a continuous $f \colon X \to [0,1]$ such that $f(x) = 0$, $f(C) \subseteq \{1\}$ if and only if X is T_1 and for any $x \in U$ open $\subseteq X$ there is a continous $f \colon X \to [0,1]$ such that $f(x) = 0$ and $\subseteq (X \setminus U)\{1\}$.

Lemma 6.2. *Suppose X is a topological space and $\mathcal{B}$ a basis for X.*

Then X is regular (resp. Tychonoff) if and only the condition holds for basic open sets.

Also in the Tychonoff condition, we can swap the roles of 0 *and* 1.

Proof. The forward directions are clear.

For the backwards directions: if $x \in U$ open X find basic open B with $x \in B \subseteq U$, apply the condition to the pair x, B and observe that the open set (resp. continuous function) also works for the pair x, U.

For the last sentence, note that $x \mapsto 1 - x$ is a homeomorphism of $[0,1]$ to itself swapping 0 and 1. □

Lemma 6.3. *X is T_1 if and only if every singleton $\{x\}$ of X is closed.*
Thus $T_{3.5} \implies T_3 \implies T_2 \implies T_1 \implies T_0$.

Proof. Suppose X is T_1 and $x \in X$. For each $y \in X \setminus \{x\}$ find $U_y \ni y$ such that $x \in X \setminus U_y$. Then

$$\{x\} = \bigcap_{y \neq x} X \setminus U_y$$

is an intersection of closed sets, so closed.

For the converse, let $x, y \in X$ be distinct. By assumption $U = X \setminus \{y\}$ is open and clearly $x \in U \not\ni y$.

For the (non-trivial) implications: If X is $T_{3.5}$, $x \in X$ and C closed $\subseteq X$ with $x \notin C$, let $f\colon X \to [0,1]$ be a continuous function with $f(x) = 0$ and $f(C) \subseteq \{1\}$. Then $U = f^{-1}([0,1/3))$ and $V = f^{-1}((2/3,1])$ are the required open sets.

If X is T_3 then it is T_1 so singletons are closed. If $x, y \in X$ are distinct then apply regularity to $x \notin \{y\}$ closed $\subseteq X$. □

Lemma 6.4. *For $i \leq 3.5$ all the T_i are preserved by subspaces and products.*

Proof. Suppose X and $X_k, k \in K$ are topological spaces satisfying T_i (for some $i \leq 3.5$) and $A \subseteq X$. We write $P = \prod_k X_k$ for the Tychonoff product.

T_0, subspace: If $x, y \in A$ are distinct, let U be X-open containing exactly one of x and y. Then $A \cap U$ is as required.

T_1, subspace: As for T_0.

T_2, subspace: As for T_0 (intersect both the open sets).

T_3, subspace: Suppose $x \in A$ and $C \subseteq A$ is A-closed. Then $x \notin \overline{C}^X = D$ since $C = \overline{C}^A = A \cap \overline{C}^X$ and $x \in A$. Thus we can apply regularity to $x \notin D$ closed $\subseteq X$ and intersect the open sets we obtain with A.

$T_{3.5}$, subspace: As for T_3, except we of course restrict the continuous map to A.

T_0, products: If $x, y \in P$ are distinct, then there is some $k \in K$ such that $x_k = \pi_k(x), y_k = \pi_k(y)$ are distinct. Thus in X_k we can find and open U containing exactly one of x_k, y_k. Then ${\pi_k}^{-1}(U)$ is as required.

T_1, products: As for T_0.

T_2, products: As for T_0.

T_3**, products:** It is enough to check the condition for standard basic open sets. So let $x \in P$ and $U = \bigcap_{k \in F} \pi_k{}^{-1}(U_k)$ be basic open (i.e. each U_k is an open subset of X_k and F is a finite subset of K). If $x \in U$ then for $k \in F$, $x_k = \pi_k(x) \in U_k$ so that there is X_k-open V_k and X_k-closed $D_k = \overline{V_k}$ with $x_k \in V_k \subseteq D_k \subseteq U_k$. Then

$$x \in V = \bigcap_{k \in F} \pi_k{}^{-1}(V_k) \subseteq D = \bigcap_{k \in F} \pi_k{}^{-1}(D_k) \subseteq U$$

and V is open and D closed so that $\overline{V} \subseteq D$ as well.

$T_{3.5}$**, products:** It is enough to check the condition for standard basic open sets. So let $x \in P$ and $U = \bigcap_{k \in F} \pi_k{}^{-1}(U_k)$ be basic open (i.e. each U_k is an open subset of X_k and F is a finite subset of K). If $x \in U$ then for $k \in F$, $x_k = \pi_k(x) \in U_k$ so that there is a continuous function $f_k \colon X_k \to [0,1]$ such that $f_k(x) = 1$ and $f_k(X_k \setminus U_k) \subseteq \{0\}$ (note that 0 and 1 have interchanged roles).

Now consider the functions

$$\begin{aligned} \pi_F \colon P &\to \prod_{k \in F} X_k \\ g &\mapsto g|_F, \end{aligned}$$

$$\begin{aligned} \prod_{k \in F} f_k \colon \prod_{k \in F} X_k &\to [0,1]^F \\ (x_k)_{k \in F} &\mapsto (f_k(x_k))_{k \in F} \end{aligned}$$

and

$$\begin{aligned} m \colon [0,1]^F &\to [0,1] \\ (r_k) &\mapsto \prod_k r_k. \end{aligned}$$

Each of them is continuous (the first two by the Test Condition, the last by elementary Analysis) and hence so is there composition

$$h = m \circ \prod_k f_k \circ \pi_F \colon P \to [0,1].$$

It is straightforward to check that $h(x) = 1$ and $h(P \setminus U) \subseteq \{0\}$. $\square$

Theorem 6.5 (The Embedding Lemma)**.** *Suppose X is a topological space, $Y_i, i \in I$ are topological spaces and $f_i \colon X \to Y_i, i \in I$ are continuous such that*

- *the f_i separate points, i.e. for distinct $x, y \in X$ there is $i \in I$ with $f_i(x) \neq f_i(y)$;*

- $\left\{ {f_i}^{-1}(U) : i \in I, U \text{ open } \subseteq Y_i \right\}$ *is a basis for* X.

Then the diagonal $\Delta = \Delta_{i \in I} f_i$ *is an embedding of* X *into* $\prod_{i \in I} Y_i$*, i.e. a continuous map which is a homeomorphism onto its image.*

Proof. The diagonal is continuous and as the f_i separate points, it is injective. It remains to check that it is open into $f(X)$: so let $U \subseteq X$ be basic open, i.e of the form ${f_i}^{-1}(V)$ for some $i \in I$, V open $\subseteq Y_i$. Then by construction $\Delta(U) \subseteq {\pi_i}^{-1}(V) \cap \Delta(X)$. On the other hand, if $y \in {\pi_i}^{-1}(V) \cap \Delta(X)$ then we can find $x \in X$ with $\Delta(x) = y$ and $f_i(x) = \pi_i(\Delta(x)) = \pi_i(y) \in V$ so that $x \in {f_i}^{-1}(V)$ giving $\supseteq$. Thus basic open sets are mapped to $\Delta(X)$-open sets. Finally, images and unions commute so the result follows. □

Corollary 6.6. *If* X *is* T_1 *then the two conditions in the previous theorem are equivalent to*

- *the* f_i *separate points from closed sets, i.e. for* $x \notin C$ *open* $\subseteq X$ *there is* $i \in I$ *and disjoint open* $U, V \subseteq Y_i$ *with* $f(x) \in U$, $f(C) \subseteq V$.

Definition 6.7. The *Sierpinsky space* $\mathbb{S}$ is $(\{0,1\}, \{\emptyset, \{1\}, \{0,1\}\})$.

Theorem 6.8. *A topological space is* T_0 *if and only if it is homeomorphic to a subspace of some* $\mathbb{S}^\kappa$.

A topological space is $T_{3.5}$ *if and only if it is homeomorphic to a subspace of some* $[0,1]^\kappa$.

Proof. For the first result, apply the Embedding Lemma to the family χ_U, U open $\subseteq X$.

For the second result, apply the Embedding Lemma to the family of all $[0,1]$-valued continuous functions. □

Lemma 6.9. *A topological space* X *is Hausdorff if and only if the diagonal* $\Delta = \{(x,x) \colon x \in X\}$ *is closed in* $X \times X$.

Proof. Suppose X is Hausdorff. If $(x,y) \in X^2 \backslash \Delta$ then $x \neq y$ so by Hausdorffness there are disjoint X-open $U \ni x, V \ni y$. As U, V are disjoint $U \times V \cap \Delta = \emptyset$ so that $(x,y) \in U \times V \subseteq X^2 \setminus \Delta$. As (x,y) was arbitrary this shows that Δ is closed.

Suppose now that Δ is closed and $x, y \in X$ are distinct. Then $(x,y) \notin \Delta$ so there is a basic open $W = U \times V$ (in X^2) with $(x,y) \in W \subseteq X \setminus \Delta$ and U, V open in X. Then $U \cap V = \emptyset$ so that $U \ni x, V \ni y$ are as required. □

Lemma 6.10. *Suppose* $f, g \colon X \to Y$ *are continuous functions and* Y *a Hausdorff space. Then* $\{x \in X \colon f(x) = g(x)\}$ *is closed in* X.

Proof.

$$\{x \in X \colon f(x) = g(x)\} = (f \Delta g)^{-1}(\Delta).$$

□

7 Normality

Definition 7.1. Suppose X is a topological space.

X is *normal* if and only if X is T_1 and for every pair C, D of disjoint closed subsets of X there are disjoint open subsets U, V of X such that $C \subseteq U$ and $D \subseteq V$.

Equivalently (by duality) X is normal if and only if X is T_1 and for every closed $C \subseteq U$ open $\subseteq X$ there is open $V \subseteq X$ with $C \subseteq V \subseteq \overline{V} \subseteq U$.

Definition 7.2. Suppose X is a topological space.

X is *functionally normal* if and only if X is T_1 and for every pair C, D of disjoint closed subsets of X there is a continuous function $f: X \to [0,1]$ such that $f(C) \subseteq \{0\}$ and $f(D) \subseteq \{1\}$. We call f a Urysohn function for the pair C, D.

Equivalently (by duality) X is functionally normal if and only if X is T_1 and for every closed $C \subseteq U$ open $\subseteq X$ there is a continuous function $f: X \to [0,1]$ with $f(C) \subseteq \{0\}$ and $f(U) \subseteq [0,1)$.

Lemma 7.3. *Suppose X is a functionally normal topological space. Then X is normal.*

Proof. For disjoinct closed C, D with Urysohn function f consider $U = f^{-1}([0,1/2))$, $V = f^{-1}((1/2,1])$. □

Theorem 7.4 (Urysohn's Lemma)**.** *Suppose X is a topological space.*

If X is normal then X is functionally normal.

Proof. Suppose we have a closed $C \subseteq U$ open $\subseteq X$.

Constructing the Onion Slices Let $D = \{d_n : n \in \omega\}$ be a countable dense subset of $(0,1)$ (e.g. $D = \mathbb{Q} \cap (0,1)$ or D as the dyadic rationals) and write $D_{-1} = \{0,1\}$ and $D_n = \{d_0, \ldots, d_n\} \cup \{0,1\} = \{d_n\} \cup D_{n-1}$ for $n \in \omega$. Let $\overline{U_0} = C$ and $U_1 = U$.

By (strong) induction on n we will construct open sets $U_{d_n}, n \in \omega$ such that

Induction Hypothesis for $r, s \in D_n$ with $r < s$ we have $\overline{U_r} \subseteq U_s$

(note that we never talk abaout U_0 or $\overline{U_1}$ here!).

Suppose we have defined U_{d_k} for $k < n$.

Let

$$L = \{d \in D_{n-1} : d < d_n\} \quad G = \{d \in D_{n-1} : d_n < d\}$$

and note that these are both finite non-empty sets. Let $l = \max L$ and $g = \min G$ and apply normality to $\overline{U_l} \subseteq U_g$ to obtain open U_{d_n} with $\overline{U_l} \subseteq U_{d_n} \subseteq \overline{U_{d_n}} \subseteq U_g$. By choice of l and g (and transitivity of $\subseteq$) the inductive hypothesis has been preserved.

Defining the Urysohn function We define $f, g\colon X \to [0,1]$ by

$$f(x) = \inf\{d \in D\colon x \in U_d\}\, g(x) = \inf\left\{d \in D\colon x \in \overline{U_d}\right\}$$

(treating $\inf \emptyset = 1$).

Next we claim that $f = g$: clearly $g(x) \leq f(x)$ (as if $x \in U_d$ then $x \in \overline{U_d}$). Next, if $g(x) < r \in [0,1]$ then there are $e, d \in D$ with $g(x) < e < d < r$ such that $x \in \overline{U_e} \subseteq U_d$, giving $f(x) \leq d < r$. Hence $f(x) \leq g(x)$, as required.

Continuity For $r \in (0,1)$ we see: if $f(x) \in [0,r)$ then there is $d \in D$ with $f(x) < d < r$ so that $x \in U_d$ and $f(U_d) \subseteq [0,r)$; if on the other hand $f(x) = g(x) \in (r,1]$ then there is $d \in D$ with $r < d < g(x)$ so that $x \notin \overline{U_d}$ and $g\left(X \setminus \overline{U_d}\right) \subseteq (r,1]$. Hence f is continuous.

Finally observe that if $x \in C = \overline{U_0}$ then $x \in U_d$ for all $d \in D$ so that $f(x) = 0$ and if $x \notin U = U_1$ then $x \notin \overline{U_d}$ for all $d \in D$ so that $g(x) = 1$. $\square$

The following example shows that subspaces of normal spaces need not be normal. Note that you can replace $\aleph_1$ with any uncountable set.

Example 7.5. Let $T = \aleph_1 \cup \{r\}$ with topology $\mathcal{P}(\aleph_1) \cup \{T \setminus F\colon F \text{ finite } \subseteq T\}$ and let $R = \aleph_0 \cup \{t\}$ with topology $\mathcal{P}(\aleph_0) \cup \{R \setminus F\colon F \text{ finite } \subseteq R\}$.

Note that T and R are compact Hausdorff so that $T \times R$ is compact Hausdorff and hence normal.

Let $X = T \times R \setminus \{(r,t)\}$.

X is not normal, specifically the disjoint X-closed subsets $C = \aleph_1 \times \{t\}$ and $D = \{r\} \times \aleph_0$ cannot be separated by open sets. For suppose $U \supseteq D$ is open. For each $n \in \aleph_0$ we can then choose some finite $F_n \subseteq \aleph_1$ such that $(r,n) \in (T \setminus F_n) \times \{n\} \subseteq U$. Letting $\alpha \in \aleph_1 \setminus \bigcup_{n \in \aleph_0} F_n$ (the latter is non-empty as $\aleph_1$ is uncountable but $\bigcup_n F_n$ is a countable union of finite sets so countable), we see that $\{\alpha\} \times \aleph_0 \subseteq U$ and hence $(\alpha, t) \in \overline{U} \cap C$.

7.1 Urysohn's Metrization Theorem

Definition 7.6. A topological space is metrizable if and only if there is a metric on that induces the topology of the space.

Theorem 7.7. *A compact Hausdorff space is metrizable if and only if it is second countable.*

Proof. For the forward direction note that compact implies Lindeloef and Lindeloef metric spaces are second countable (Sheet 1).

We now prove the reverse implication. By Sheet 0, compact Hausdorff spaces are normal, hence functionally normal and Tychonoff.

Thus

$$\mathcal{B} = \left\{f^{-1}(U) : f\colon X \to [0,1] \text{ continuous }, U \text{ open } \subseteq [0,1]\right\}$$

is a basis for X.

By Sheet 2, Q1 and the fact that X has a countable basis, there is a countable $\mathcal{B}' \subseteq \mathcal{B}$ which is still a basis for X. Hence there are countably many continuous functions $f_i \colon X \to [0,1], i \in \mathbb{N}$ such that

$$\mathcal{B}' \subseteq \left\{ {f_i}^{-1}(U) : i \in \mathbb{N}, U \text{ open } \subseteq [0,1] \right\} =: \mathcal{B}''.$$

Observe that $\mathcal{B}''$ is thus still a basis for X and X is T_1 so that $f_i, i \in I$ satisfies the conditions for the Embedding Lemma.

Thus $\Delta_{i \in \mathbb{N}} f_i \colon X \to [0,1]^{\mathbb{N}}$ is a homeomorphic embedding. But $[0,1]^{\mathbb{N}}$ is metrizable (Sheet 1) so that X is homeomorphic to a subspace of a metrizable space and hence metrizable. □

Note that we can replace 'compact Hausdorff' with 'Lindelöf regular' or 'separable normal' in the above theorem.

Alternative proof. Instead of appealing to Sheet 2, Q1, fix a countable basis $\mathcal{B}$ of X. Note that there are only countably many pairs $(U, V) \in \mathcal{B}^2$. For each such pair which satisfies $\overline{U} \subseteq V$, use Urysohn's Lemma (i.e. functional normality) to find a continuous $f = f_{U,V} \colon X \to [0,1]$ such that $f\left(\overline{U}\right) \subseteq \{0\}$ and $f(X \setminus V) \subseteq \{1\}$. Note that then $\overline{U} \subseteq f^{-1}([0,1/2)) \subseteq V$.

We then claim that

$$\left\{ {f_{U,V}}^{-1}([0,1/2)) : (U,V) \in \mathcal{B}^2, \overline{U} \subseteq V \right\}$$

is a basis for X and hence the family $f_{U,V}, (U,V) \in \mathcal{B}^2$ with $\overline{U} \subseteq V$ satisfies the conditions of the Embedding Lemma.

To prove the claim, let $x \in W$ open $\subseteq X$. Find $V \in \mathcal{B}$ with $x \in V \subseteq W$. Use regularity to find open W' such that $x \in W' \subseteq \overline{W'} \subseteq V$ and then find $U \in \mathcal{B}$ such that $x \in U \subseteq W'$. We thus have $(U,V) \in \mathcal{B}^2$ such that $x \in U \subseteq \overline{U} \subseteq {f_{U,V}}^{-1}([0,1/2)) \subseteq V \subseteq W$ as required.

□

8 Paracompactness

8.1 General Theory of Paracompactness

Definition 8.1. Suppose X is a set and $\mathcal{A}$ is a collection of subsets of X.

A collection $\mathcal{A}'$ is a *refinement* of $\mathcal{A}$ if and only if $\forall A' \in \mathcal{A} \exists A' \in \mathcal{A}'\ A' \subseteq A$.

Definition 8.2. Suppose X is a topological space.

A collection $\mathcal{A}$ of subsets of X is *locally finite* if and only if $\forall x \in X \exists$ open $U \ni x\ |\{A \in \mathcal{A} \colon A \cap U \neq \emptyset\}| < \aleph_0$.

A collection $\mathcal{A}$ of subsets of X is *discrete* if and only if $\forall x \in X \exists$ open $U \ni x\ |\{A \in \mathcal{A} \colon A \cap U \neq \emptyset\}| \leq 1$.

A collection $\mathcal{A}$ of subsets of X is *closure preserving* if and only if for every $\mathcal{A}' \subseteq A$, $\overline{\bigcup_{A \in \mathcal{A}'} A} = \bigcup_{A \in \mathcal{A}'} \overline{A}$.

Lemma 8.3. *Suppose X is a topological space and $\mathcal{A}$ a collection of subsets of X.*

If $\mathcal{A}$ is discrete then it is locally finite.

If $\mathcal{A}$ is locally finite then it is closure preserving.

Proof. The first statement is trivial ($1 < \aleph_0$).

For the second, suppose $\mathcal{A}' \subseteq \mathcal{A}$. As $\overline{\bigcup \mathcal{A}'}$ is a closed set containing each $A \in \mathcal{A}'$ it contains each $\overline{A}$ for $A \in \mathcal{A}'$ yielding $\overline{\bigcup \mathcal{A}'} \supseteq \bigcup_{A \in \mathcal{A}'} \overline{A}$.

Thus all we need to show that $\bigcup_{A \in \mathcal{A}'} \overline{A}$ is closed in X. So let $x \in X \setminus \bigcup_{A \in \mathcal{A}'} \overline{A}$ and find open $U \ni x$ such that $\mathcal{A}_U = \{A \in \mathcal{A}' \colon U \cap A \neq \emptyset\}$ is finite. Note that since U is open, $U \cap A \neq \emptyset \iff U \cap \overline{A} \neq \emptyset$ so that $V = U \setminus \bigcup_{A \in \mathcal{A}_U} \overline{A}$ is an open set containing x which is contained in $X \setminus \bigcup_{A \in \mathcal{A}'} \overline{A}$. □

Definition 8.4. Suppose X is a topological space.

X is paracompact if and only if every open cover of X has a locally finite open refinement covering X.

Lemma 8.5. *A compact space is paracompact.*

Lemma 8.6. *Suppose X is a paracompact topological space and C a closed subset of X.*

Then every X-open cover of C has an X-locally finite X-open refinement covering C.

Proof. Suppose $\mathcal{U}$ is an X-open cover of C. Then $\mathcal{U} \cup \{X \setminus C\}$ is an X-open cover of X, so has a locally finite open refinement $\mathcal{V}$ covering X. We claim that $\mathcal{W} = \{V \in \mathcal{V} \colon V \cap C \neq \emptyset\}$ is the required X-locally finite X-open refinement of $\mathcal{U}$ covering C.

The only non-trivial claim is that $\mathcal{W}$ is a refinement of $\mathcal{U}$. For this, note that if $V \cap C \neq \emptyset$ then $V \nsubseteq X \setminus C$ so there is $U \in \mathcal{U}$ with $V \subseteq U$. □

Lemma 8.7. *A closed subspace of a paracompact space is paracompact.*

Proof. Suppose C is a closed subspace of a paracompact space X. Let $\mathcal{U}$ be a C-open cover of C and for each $U \in \mathcal{U}$ choose X-open $V_U \subseteq X$ such that $V_U \cap C = U$. Then $\{V_U \colon U \in \mathcal{U}\}$ is an X-open cover of C and by the previous lemma has a X-locally finite open refinement $\mathcal{V}$ covering C. Let $\mathcal{W} = \{V \cap C \colon V \in \mathcal{V}\}$ and check that this is the required C-locally finite C-open refinement of $\mathcal{U}$ covering C. □

Theorem 8.8. *A paracompact regular space is normal.*

Proof. Suppose X is paracompact regular and that C, D are disjoint closed subsets of X. For $c \in C$ use regularity to find open U_c such that $c \in U_c \subseteq \overline{U_c} \subseteq X \setminus D$. Then $\{U_c \colon c \in C\}$ is an X-open cover of C and thus has a locally finite open refinement $\mathcal{V}$ covering C. We then note that $\mathcal{V}$ is closure preserving and since $\mathcal{V}$ refines $\{U_c \colon c \in C\}$ we have

$$C \subseteq \bigcup \mathcal{V} \subseteq \overline{\bigcup \mathcal{V}} = \bigcup_{V \in \mathcal{V}} \overline{V} \subseteq \bigcup_{c \in C} \overline{U_c} \subseteq X \setminus D$$

as required. ∎

Definition 8.9. Suppose X is a topological space and $\mathcal{P}$ is a property of families of subsets of X (e.g. 'locally finite').

A family $\mathcal{A}$ of subsets of X is σ-$\mathcal{P}$ if and only if there are families $\mathcal{A}_n, n \in \mathbb{N}$ of subsets of X such that each $\mathcal{A}_n$ is $\mathcal{P}$ and $\mathcal{A} = \bigcup_n \mathcal{A}_n$.

For example a family $\mathcal{A}$ is σ-locally finite if and only if it can be written as a countable union of locally finite families.

Lemma 8.10. *Suppose X is a regular space.*

TFAE:

1. *X is paracompact.*
2. *X has a σ-locally finite open refinement covering X.*
3. *every open cover of X has a locally finite refinement covering X.*
4. *every open cover of X has a locally finite closed refinement covering X.*

Proof. Exercise (Probably). ∎

8.2 Paracompactness and Metrizability

Theorem 8.11 (Stone's Theorem)**.** *Suppose X is a metric space.*

Then X is paracompact and in fact every open cover of X has a σ-discrete open refinement covering X.

Proof. Suppose $\mathcal{U}$ is an open cover of X. Well order $\mathcal{U}$ by $\leq$ (by the well-ordering principle).

For each $U \in \mathcal{U}$ and $n \in \omega$ we define

$$U' = \{x \in U : \forall V < U \ x \notin V\},$$

$$V_{U,n} = \left\{x \in U' : B_{3/2^n}(x) \subseteq U\right\}$$

and

$$W_{U,n} = \bigcup_{x \in V_{U,n}} B_{1/2^n}(x).$$

We claim that for each $n \in \omega$, $\mathcal{W}_n = \{W_{U,n} : U \in \mathcal{U}\}$ is a discrete family of open subsets of X refining $\mathcal{U}$. The only non-trivial claim is discreteness. So assume $y_1 \in W_{U_1,n}, y_2 \in W_{U_2,n}$ with $U_1 \neq U_2$ and without loss of generality $U_1 \leq U_2$. For each $i = 1, 2$ pick $x_i \in V_{U_i,n}$ such that $y_i \in B_{1/2^n}(x)$. Since $x_2 \in U_2'$ we have $x_2 \notin U_1$ and hence $d(x_1, x_2) \geq 3/2^n$. But then by the triangle law

$$d(y_1, y_2) \geq d(x_1, x_2) - d(y_1, x_1) - d(y_2, x_2) \geq 1/2^n.$$

Hence the $B_{1/2^{n+1}}(x), x \in X$ witness discreteness of $\mathcal{W}_n$.

Next observe that $\bigcup_n \mathcal{W}_n$ is a cover of X: if $x \in U' \subseteq U$ for some $U \in \mathcal{U}$ then for sufficiently large n we have $B_{3/2^n}(x) \subseteq U$.

Hence $\bigcup_n \mathcal{W}_n$ is a σ-discrete open refinement of X covering X as required.

As σ-discrete certainly implies σ-locally finite, the previous lemma shows that X must be paracompact.

□

It is possibly to define $W'_{U,n}$ such that they form a locally finite family: we define by recursion on n

$$W'_{U,n} = \bigcup \left\{ B_{1/2^n}(x) \colon x \in V_{U,n} \setminus \bigcup \left\{ W'_{U_1,k} \colon k < n, U_1 \in \mathcal{U} \right\} \right\}.$$

We can then manually check that this is locally finite - see Engelking Theorem 4.4.1 for details.

Lemma 8.12. *Every metric space has a σ-discrete basis.*

Proof. For every $n \in \mathbb{N}$, consider the open cover $\mathcal{U}_n = \{B_{2^{-n}}(x) \colon x \in X\}$ and use the previous theorem to obtain a σ-discrete open refinement $\mathcal{W}_n$ of $\mathcal{U}_n$.

Then observe that $\bigcup_n \mathcal{W}_n$ is still a basis for X (easy) and is σ-discrete. □

Lemma 8.13. *Suppose X is a normal space with a σ-discrete basis $\mathcal{B} = \bigcup_n \mathcal{B}_n$ (where each $\mathcal{B}_n$ is discrete).*

For every open $U \subseteq X$ there is a continuous $f \colon X \to [0,1]$ such that $U = f^{-1}((0,1])$.

Proof. Let U be an open subset of X and for each $n \in \mathbb{N}$ let $U_n = \bigcup \left\{ B \in \mathcal{B}_n \colon \overline{B} \subseteq U \right\}$. Since $\mathcal{B}$ is a basis and X is regular, $U = \bigcup_n U_n$. But as each $\mathcal{B}_n$ is discrete and hence locally finite, $\overline{U_n} = \bigcup \left\{ \overline{B} \colon \overline{B} \subseteq U \right\} \subseteq U$ so that $\bigcup_n \overline{U_n} = U$. Thus every open set is a countable union of closed sets (and, by duality, every closed set is a countable intersection of open sets).

Now apply Urysohn's Lemma for each $n \in \mathbb{N}$ to choose continuous $f_n \colon X \to [0, 2^{-n}]$ such that $f_n\left(\overline{U_n}\right) \subseteq \{2^{-n}\}$ and $f_n(X \setminus U) \subseteq \{0\}$. By the M-test $f = \sum_n f_n$ is continuous and $f(X \setminus U) \subseteq \{0\}$ and $f(U) \subseteq (0,1]$ as claimed. □

Definition 8.14. Suppose κ is any cardinal.

The hedgehog of spininess κ, H_κ is the quotient of $[0,1] \times \kappa$ obtained by identifying all the points $(0,\alpha), \alpha \in \kappa$.

Lemma 8.15. *The hedgehog of spininess κ is metrizable with metric*

$$d((x,\alpha),(y,\beta)) = \begin{cases} y & x = 0 \\ x & y = 0 \\ |x-y| & \alpha = \beta \\ x+y & \alpha \neq \beta \end{cases}.$$

Hence a countable product of hedgehogs is metrizable.

Proof. Straightforward verification. □

Theorem 8.16. *Every normal space in with a σ-discrete basis is a subspace of a product of countably many hedgehogs and thus metrizable.*

Proof. Suppose X is a normal space with a σ-discrete basis $\mathcal{B} = \bigcup_n \mathcal{B}_n$ where each $\mathcal{B}_n$ is a discrete family of open sets and write $\kappa_n = |\mathcal{B}_n|$.

By the lemma above, for each $B \in \mathcal{B}_n$ we can find a continuous $f_B \colon X \to [0,1]$ such thath $B = {f_B}^{-1}((0,1])$.

Define the functions $f_n \colon X \to H_{\kappa_n}$ by

$$f_n(x) = \begin{cases} f_B(x) & x \in B \in \mathcal{B}_n \\ 0 & \text{otherwise} \end{cases}.$$

First note that by discreteness of $\mathcal{B}_n$, f_n is well-defined (each x belongs to at most one $B \in \mathcal{B}_n$).

Next, f_n is continuous: fix $x \in X$ and open $U \ni x$ such that U meets at most one $B \in \mathcal{B}_n$. If U meets no $B \in \mathcal{B}_n$ then $f_n = 0$ on U and hence is continuous at x. If U meets a unique $B \in \mathcal{B}_n$ then $f_n = f_B$ on U and again f_n is continuous at x.

By definition of f_n, we have $\mathcal{B}_n \subseteq \left\{ {f_n}^{-1}((0,1] \times \{\alpha\}) : \alpha \in \kappa_n \right\}$ and hence the family $f_n, n \in \mathbb{N}$ satisfies the conditions of the Embedding Lemma (noting that X is T_1).

Thus $\Delta_n f_n \colon X \to \prod_n H_{\kappa_n}$ is an embedding of X into the product of countably many hedgehogs. □

We can summarize the result of this section by:

Theorem 8.17. *TFAE:*

1. *X is metrizable*
2. *X is normal and has a σ-discrete base.*
3. *X is a subspace of a product of countably many hedgehogs.*

Examining the proofs carefully, we can do better: we can replace σ-discrete by σ-locally finite and we can insist that each of the hedgehogs has spininess $w(X)$.

Finally, with a bit of work we can show that regular spaces that have a σ-locally finite base are normal (essentially having a σ-locally finite base implies that the space is paracompact) and hence we can replace normality in the second condition by regularity to obtain:

Theorem 8.18 (Bing-Nagata-Smirnov Metrization Theorem)**.** *A topological space is metrizable if and only if it is regular and has a σ-locally finite base.*

9 Filters

9.1 Abstract Filters

Definition 9.1. Suppose X is a set.

A filter on X is a set $\mathcal{F} \subseteq \mathcal{P}(X)$ such that

- $\emptyset \notin \mathcal{F} \neq \emptyset$;
- $\mathcal{F}$ is closed under finite intersection;
- $\mathcal{F}$ is closed under supersets.

A filter $\mathcal{U}$ which is maximal with respect to $\subseteq$ is called an ultrafilter.

Two collections $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}(X)$ mesh, written $\mathcal{F}\#\mathcal{G}$, if and only if $\forall F \in \mathcal{F} \forall G \in \mathcal{G}\ F \cap G \neq \emptyset$. In this case we also use $\mathcal{F}\#\mathcal{G}$ for $\{F \cap G \colon F \in \mathcal{F}, G \in \mathcal{G}\}$. If $\mathcal{F}$ and/or $\mathcal{G}$ consist are singletons, we may omit the curly braces, i.e. if $\mathcal{F} = \{F\}$, we write $F\#\mathcal{G}$ instead of $\mathcal{F}\#\mathcal{G}$.

A basis for a filter $\mathcal{F}$ is a collection $\mathcal{B} \subseteq \mathcal{F}$ such that $\mathcal{F} = \{F \subseteq X \colon \exists B \in \mathcal{B}\ B \subseteq F\}$.

A collection $\mathcal{C} \subseteq \mathcal{P}(X)$ has the finite intersection property if and only if for every finite $\mathcal{C}' \subseteq \mathcal{C}$ we have $\bigcap \mathcal{C}' \neq \emptyset$.

The following lemmas are elementary:

Lemma 9.2. *Suppose X is a set. A non-empty collection $\mathcal{B} \subseteq X$ is a filter basis for a unique filter $\mathcal{F}$ on X if and only if $\emptyset \notin \mathcal{B}$ and $\forall A, B \in \mathcal{B} \exists C \in \mathcal{B}\ C \subseteq A \cap B$, in which case $\mathcal{F} = \{A \subseteq X \colon \exists B \in \mathcal{B}\ B \subseteq A\}$. We call $\mathcal{F}$ the filter generated by $\mathcal{B}$.*

Lemma 9.3. *Suppose X is a set. A non-empty collection $\mathcal{S} \subseteq X$ is contained in a smallest (wrt $\subseteq$) filter $\mathcal{F}$ if and only if $\emptyset \notin \mathcal{S}$ and $\mathcal{S}$ has the finite intersection property, in which case $\mathcal{F}$ is the filter generated by the filter basis $\{\bigcap \mathcal{S}' \colon \mathcal{S}' \text{ finite } \subseteq \mathcal{S}\}$. We call $\mathcal{F}$ the filter generated by $\mathcal{S}$.*

Lemma 9.4. *Suppose X is a set.*

Two non-empty collections $\mathcal{F}, \mathcal{G}$ of subsets of X mesh if and only if there is a filter containing both of them.

In the following we make no explicit distinction between a filter and a basis generating it.

Lemma 9.5. *Suppose X is a set, $\mathcal{F}$ a filter on X and $\mathcal{G} \subseteq \mathcal{P}(X)$ closed under finite intersections. If $\mathcal{G}\#\mathcal{F}$ then $\mathcal{G}\#\mathcal{F}$ is a filter basis on X (and we also use $\mathcal{G}\#\mathcal{F}$ for the filter generated by this basis.*

Lemma 9.6. *Suppose X is a set. A filter $\mathcal{U}$ on X is an ultrafilter on X if and only if for every $A \subseteq X$ we have $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.*

Proof. Suppose $\mathcal{U}$ is maximal wrt $\subseteq$ and $A \subseteq X$ such that $A \notin \mathcal{U}$. If $U \in \mathcal{U}$ then $U \not\subseteq A$ so that $U\#(X \setminus A)$. Hence $(X \setminus A)\#\mathcal{U}$ is a filter containing $\mathcal{U}$ and by maximality the two must be equal, giving $X \setminus A \in \mathcal{U}$.

Conversely, let $\mathcal{U}$ be a filter on X satisfying the condition and suppose $\mathcal{G}$ is a strictly bigger filter on X. Then $A \in \mathcal{G} \setminus \mathcal{U}$ for some $A \subseteq X$. By the condition $X \setminus A \in \mathcal{U} \subseteq \mathcal{G}$ so that $(X \setminus A) \cap A = \emptyset \in \mathcal{G}$, a contradiction. $\square$

Lemma 9.7. *Suppose X is a set. A filter $\mathcal{U}$ on X is an ultrafilter on X if and only if for every $A, B \subseteq X$, if $A \cup B \in \mathcal{U}$ then $A \in \mathcal{U}$ or $B \in \mathcal{U}$ if and only if for every $A, B \subseteq X$, if $A, B \notin \mathcal{U}$ then $A \cup B \notin \mathcal{U}$.*

Proof. Suppose $\mathcal{U}$ is an ultrafilter on X and $A \cup B \in \mathcal{U}$. If $A \notin \mathcal{U}$ then $X \setminus A \in \mathcal{U}$ and hence $\mathcal{U} \ni (A \cup B) \cap X \setminus A \subseteq B$ so that $B \in \mathcal{U}$.

Conversely, suppose $\mathcal{U}$ is a filter and the condition holds. For $A \subseteq X$, note that since $\mathcal{U} \ni X = A \cup (X \setminus A)$, one of A or $X \setminus A$ belongs to $\mathcal{U}$.

The last condition is simply the contrapositive of the previous one. $\square$

Example 9.8. Suppose X is a set, $x \in X$. The following are filters on X:

- $\mathcal{P}_x = \{A \subseteq X : x \in A\}$, the principal filter at x (it is in fact an ultrafilter);
- $\mathrm{cof} = \{X \setminus F : F \text{ finite } \subseteq X\}$ the cofinite (or Frechet) filter on X;
- if X is a topological space with topology τ then $\mathcal{N}_x = \{N \subseteq X : \exists U \in \tau \; x \in U \subseteq N\}$ is the neighbourhood filter at x.

Theorem 9.9 (Ultrafilter Extension Lemma)**.** *Suppose X is a set. Then every filter can be extended to an ultrafilter.*

Proof (not examinable). Either well-order $\mathcal{P}(X)$ and construct an ultrafilter by recursion on this well-order: given a filter $\mathcal{F}$ recursively define $\mathcal{F}_A$ for $A \in \mathcal{P}(X)$ by $\mathcal{F}\#A$ if $\mathcal{F}\#A$ and $\mathcal{F}\#(X \setminus A)$ otherwise.

Alternatively, observe that the union of an increasing chain of filters is a filter (which is an upper bound) and apply Zorn's Lemma. $\square$

The following fact is non-examinable: We write ZF for the axioms of set theory without the Axiom of Choice.

Theorem 9.10. *It is consistent with* ZF *that the only ultrafilters (on any set) are the principal ultrafilters. In particular, it is consistent with* ZF *that the cofinite filter on ω cannot be extended to an ultrafilter.*

Lemma 9.11. *Suppose X, Y are sets, $f : X \to Y$ is a function.*

If $\mathcal{F}$ is a filter on X, then $f(\mathcal{F}) = \{f(F) : F \in \mathcal{F}\}$ is a filter-basis for a filter $\mathcal{G}$ on X (which we also, in general, denote by $f(\mathcal{F})$) and $\mathcal{G} = \left\{B \subseteq Y : f^{-1}(B) \in \mathcal{F}\right\}$. Moreover, if $\mathcal{F}$ is an ultrafilter on X then $\mathcal{G}$ is an ultrafilter on Y.

Proof. Computing $f(F_1) \cap f(F_2) \supseteq f(F_1 \cap F_2)$ easily implies that $f(\mathcal{F})$ is a filter basis.

Next, if $B \subseteq Y$ such that $f^{-1}(B) \in \mathcal{F}$ then $f\left(f^{-1}(B)\right) \subseteq B$ so that $B \in \mathcal{G}$. Conversely, if $B \in \mathcal{G}$ then there is $F \in \mathcal{F}$ with $f(F) \subseteq B$ so that $F \subseteq f^{-1}(B)$ and hence $f^{-1}(B) \in \mathcal{F}$.

Finally, if $\mathcal{F}$ is an ultrafilter on X and $B \subseteq Y$ then since $f^{-1}(B) \cup f^{-1}(Y \setminus B) = X$ one of B or $Y \setminus B$ belongs to $\mathcal{G}$ as required. $\square$

9.2 Topological filters

Example 9.12. Suppose X is a set and $a : \mathbb{N} \to X; k \mapsto a_k$ is a sequence in X.

Writing $T_n = \{a_k : k \geq n\}$ for the n-tail of (a_n), we have that $\{T_n : n \in \mathbb{N}\}$ is a filter basis for a filter $\mathcal{F}_{(a_n)}$ on X.

We note that if X is a metric space then $a_n \to x \in X$ if and only if $\mathcal{N}_x \subseteq \mathcal{F}_{(a_n)}$ since $a_n \to x$ means that every open $U \ni x$ contains some tail T_n.

Definition 9.13. Suppose X is a topological space, $\mathcal{F}$ a filter on X and $x \in X$.

We say that $\mathcal{F}$ *converges to* x, written $\mathcal{F} \to x$, if and only if $\mathcal{N}_x \subseteq \mathcal{F}$.

We write $\lim \mathcal{F} = \{x \in X : \mathcal{F} \to x\}$ and abuse notation to write $\lim \mathcal{F} = x$ for $\mathcal{F} = \{x\}$.

Lemma 9.14. *Suppose X is a topological space, $\mathcal{F}, \mathcal{G}$ are filters on X and $x \in X$.*

If $\mathcal{F} \to x$ and $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{G} \to x$.

Lemma 9.15. *Suppose X is a topological space, $A \subseteq X$ and $x \in X$.*

TFAE:

1. *$x \in \overline{A}$.*
2. *$\mathcal{N}_x \# A$.*
3. *There is a filter $\mathcal{F}$ on X containing A that converges to X.*
4. *There is an ultrafilter $\mathcal{U}$ on X containing A that converges to X.*

Proof. Since

$$\overline{A} = \{x \in X : \forall \text{ open } U \ni x \; U \cap A \neq \emptyset\}$$

the first two conditions are equivalent.

Equivalence of the other three conditions is then obvious (using the Ultrafilter Extension Lemma). □

Lemma 9.16. *Suppose X, Y are topological space and $f : X \to Y$ is a function.*

TFAE:

1. *f is continuous.*
2. *For every filter $\mathcal{F}$ on X and $x \in \lim \mathcal{F}$, $f(x) \in \lim f(\mathcal{F})$.*
3. *For every ultrafilter $\mathcal{U}$ on X and $x \in \lim \mathcal{F}$, $f(x) \in \lim f(\mathcal{F})$.*

Proof. Note that if f is continuous (at x) then $f(\mathcal{N}_x) \subseteq \mathcal{N}_{f(x)}$. Hence if $\mathcal{N}_x \subseteq \mathcal{F}$ then $f(F) \supseteq \mathcal{N}_{f(x)}$.

The second implication is obvious.

Finally if $x \in \overline{A}$ then we can find an ultrafilter $\mathcal{U} \to x$ that contains A. Hence $f(\mathcal{U}) \to f(x)$ and $f(A) \in \mathcal{U} f$ so that $x \in \overline{f(A)}$. □

Lemma 9.17. *A topological space is Hausdorff if and only if every filter converges to at most one point if and only if every ultrafilter converges to at most one point.*

Proof. Suppose X is a topological space.

If $\mathcal{F}$ is a filter on X converging to two distinct points $x, y \in X$ then $\mathcal{N}_x, \mathcal{N}_y \subseteq \mathcal{F}$ which implies $\mathcal{N}_x \# \mathcal{N}_y$ which in turn implies that any two open sets containing x and y respectively must meet.

Conversely if X is not Hausdorff then let $x, y \in X$ be distinct such that any two open sets containing x and y meet. Then $\mathcal{N}_x \# \mathcal{N}_y$ and hence the generated filter converges to x and y. □

10 Compactness

10.1 Prerequisites

I assume that this material has been seen before.

Definition 10.1. Suppose X is a set.

A cover of X is a collection $\mathcal{U}$ of subsets of X such that $\bigcup \mathcal{U} \supseteq X$.

A subcover of a cover $\mathcal{U}$ of X is a family $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V}$ is a cover of X.

We use adjectives to specify the kinds of sets which appear in a cover. For example, we will use 'open cover' to refer to a cover consisting of open sets.

Definition 10.2. X is compact if and only if every open cover $\mathcal{U}$ has a finite subcover, i.e. there is a finite $\mathcal{V} \subseteq \mathcal{U}$ such that $X \supseteq \bigcup \mathcal{V}$.

A subset A of X is compact as a subset if and only if every X-open cover of A has a finite subcover.

Lemma 10.3. *Suppose X is a topological space.*

X is compact if and only if every family of closed subsets with the f.i.p. has non-empty intersection.

Proof. Duality together with the fact that $\mathcal{U}$ is a cover of X if and only if $\bigcap_{U \in \mathcal{U}} X \setminus U = \emptyset$. □

In view of the following lemma, we drop the distinction between 'compact as a subset' and 'compact (with respect to the subspace topology)'.

Lemma 10.4. *Suppose X is a topological space and A is a subset of X.*

A is compact as a subset if and only if A is compact with respect to the subspace topology.

Proof. Suppose A is compact as a subset. Let $\mathcal{U}$ be an A-open cover. For each $U \in \mathcal{U}$ choose X-open V_U such that $U = V_U \cap A$. Then $\mathcal{V}_{\mathcal{U}}$ is an X-open cover of X so has a finite subcover which we can write as $\{V_{U_i} : i = 0, \ldots, n\}$ for some n and $U_i \in \mathcal{U}$. Then $\{U_i : i = 0, \ldots, n\}$ is the required subcover of $\mathcal{U}$.

For the converse suppose A is compact with respect to the subspace topology. Let $\mathcal{U}$ be an X-open cover of A. Then $\{U \cap A : U \in \mathcal{U}\}$ is an open (wrt the subspace topology) cover of A and thus has a finite subcover which we can write as $\{U_i \cap A : i = 0, \ldots, n\}$ for some $U_i \in \mathcal{U}$. Then $\{U_i : i = 0, \ldots, n\}$ is the required subcover of $\mathcal{U}$. □

Lemma 10.5. *A closed subset of a compact space is compact.*

Proof. Suppose X is a compact space and $C \subseteq X$ is closed.

Suppose $\mathcal{U}$ is an X-open cover of C. Then $\mathcal{U} \cup \{X \setminus C\}$ is an open cover of X and hence has a finite subcover $\mathcal{V}$. We then note that $\mathcal{V} \cap \mathcal{U}$ is the required finite subcover of $\mathcal{U}$. $\square$

10.2 Compactness via ultrafilters

Lemma 10.6. *A topological space is compact if and only if every ultrafilter on X converges.*

Proof. Suppose X is a topological space.

$\Rightarrow$: Suppose $\mathcal{U}$ is an ultrafilter on X that does not converge. For each $x \in X$ we can thus choose X-open $U_x \ni x$ such that $U_x \notin \mathcal{U}$. Then $\{U_x : x \in X\}$ is an open cover of X so has a finite subcover $\mathcal{V}$. By finiteness $X = \bigcup \mathcal{V} \notin \mathcal{U}$, a contradiction.

$\Leftarrow$: Suppose that every ultrafilter on X converges. Let $\mathcal{C}$ be a family of closed sets with the f.i.p.. Then $\mathcal{C}$ generates a filter which can be extended to an ultrafilter $\mathcal{U}$ converging to some $x \in X$. As each $C \in \mathcal{C}$ belongs to $\mathcal{U}$ we have that $x \in \overline{C} = C$ for each $C \in \mathcal{C}$ and thus $x \in \bigcap \mathcal{C} \neq \emptyset$ as required. $\square$

Theorem 10.7 (Tychonoff's Theorem)**.** *A product of compact spaces is compact.*

Proof. Suppose $X_i, i \in I$ are compact spaces and let $\mathcal{U}$ be an ultrafilter on $\prod_i X_i$. Then $\mathcal{U}_i = \pi_i(\mathcal{U})$ are ultrafilters on X_i and hence converge to some $x_i \in X_i$.

Let $x = (x_i)_{i \in I} \in \prod_i X_i$. For every $i \in I$ and X_i-open $U_i \ni x_i$ we have $U_i \in \mathcal{U}_i$ so that ${\pi_i}^{-1}(U_i) \in \mathcal{U}$. Hence all subbasic open sets containing x belong to $\mathcal{U}$ and thus $\mathcal{N}_x \subseteq \mathcal{U}$ so that $\mathcal{U} \to x$ as required. $\square$

10.3 Tychonoff's Theorem implies the Axiom of Choice

This subsection is non-examinable.

Note that we employed the Axiom of Choice twice in the proof of Tychonoff's Theorem. First to show that compactness is equivalent to ultrafilters converging (Ultrafilter Extension Lemma) and secondly to choose some $x_i \in \lim \pi_i(\mathcal{U})$. This second 'choice' is not needed in Hausdorff spaces (since then $\lim \pi_i(\mathcal{U})$ is at most a singleton anyway).

To understand the proof, you need to accept (prove) that 'finitely many choices do not require the Axiom of Choice'.

Theorem 10.8. *Tychonoff's Theorem implies the Axiom of Choice.*

Proof. Suppose $Y_i, i \in I$ are non-empty sets. For each i, let $X_i = Y_i \cup \{\star_i\}$ where $\star_i \notin Y_i$ (e.g. $\star_i = Y_i$ to avoid choice). The topology on X_i is $\{\emptyset, Y_i, \{\star_i\}, X_i\}$.

Clearly each X_i is compact and Y_i is a closed subset of X_i. By Tychonoff's Theorem, $\prod_i X_i$ is compact and $\mathcal{C} = \left\{{\pi_i}^{-1}(Y_i) : i \in I\right\}$ is thus a collection of closed sets. We need to verify the f.i.p., i.e.

$$\forall F \text{ finite } \subseteq I,\ C_F = \bigcap_{i \in F} {\pi_i}^{-1}(Y_i) \neq \emptyset.$$

So fix a finite $F \subseteq I$ and note that by induction $\prod_{i \in F} Y_i \neq \emptyset$ (as $A, B \neq \emptyset \to A \times B \neq \emptyset$). Let $y \in \prod_{i \in F} Y_i$ and define $x \in \prod_i X_i$ by $x = y \cup \{i \mapsto \star_i \colon i \in I \setminus F\} \in C_F$ (the 'choice' of y does not need the Axiom of Choice).

By compactness $\bigcap \mathcal{C} \neq \emptyset$ and it is easy to verify that $\bigcap \mathcal{C} \subseteq \prod_i Y_i$. $\square$

11 Compactifications

11.1 An Example

11.1.1 The Space

Let

$$\beta\omega = \{\mathcal{U} \colon \mathcal{U} \text{ is an ultrafilter on } \mathbb{N}\}.$$

Two ultrafilters could be considered to be similar if they contain lots of the same subsets of $\mathbb{N}$, so if for $A \subseteq \mathbb{N}$, we define

$$A^\star = \{\mathcal{U} \in \beta\omega \colon A \in \mathcal{U}\}$$

the collection $\{A^\star \colon A \subseteq \mathbb{N}\}$ is a reasonable subbasis for a topology τ on $\beta\omega$.

Note that since the $\mathcal{U}$ are filters, for $A, B \subseteq \mathbb{N}$,

$$A^\star \cap B^\star = \{\mathcal{U} \in \beta\omega \colon A, B \in \mathcal{U}\} = \{\mathcal{U} \in \beta\omega \colon A \cap B \in \mathcal{U}\} = (A \cap B)^\star.$$

Thus

$$\mathcal{B} = \{A^\star \colon A \subseteq \mathbb{N}\}$$

is in fact a basis for τ.

Also note that

$$\beta\omega \setminus A^\star = (\mathbb{N} \setminus A)^\star$$

since the $\mathcal{U}$ are ultrafilters on $\mathbb{N}$ (i.e. $A \notin \mathcal{U} \iff \mathbb{N} \setminus A \in \mathcal{U}$), so the basic open sets are in fact clopen.

11.1.2 $\beta\omega$ is Hausdorff

Next, if $\mathcal{U}, \mathcal{V}$ are distinct ultrafilters then there is $A \subseteq \mathbb{N}$ such that wlog $A \in \mathcal{U} \setminus \mathcal{V}$. As they are ultrafilters this gives $\mathbb{N} \setminus A \in \mathcal{V} \setminus \mathcal{U}$ so that $A^\star$ and $(\mathbb{N} \setminus A)^\star$ are disjoint open sets containing $\mathcal{U}$ and $\mathcal{V}$ respectively.

11.1.3 $\beta\omega$ is compact

Note that the basic closed sets (complements of basic open sets) are

$$A_\star = \{\mathcal{U} \in \beta\omega \colon A \notin \mathcal{U}\} = (\mathbb{N} \setminus A)^\star.$$

Let $\mathcal{C}_\star = \left\{C^i_\star \colon i \in I\right\}$ be a collection of basic closed sets with the finite intersection property. As $C^i_\star \cap C^j_\star = (C^i \cap C^j)_\star$ we see that $\mathcal{C} = \left\{C^i \colon i \in I\right\}$ is a family of subsets of $\mathbb{N}$ with the finite intersection property. Thus $\mathcal{C}$ is a subbasis for a filter on $\mathbb{N}$ which can be extended to an ultrafilter $\mathcal{U} \supseteq \mathcal{C}$. Then $\mathcal{U} \in \bigcap \mathcal{C}_\star$ showing that $\bigcap \mathcal{C}_\star \neq \emptyset$.

As it is enough to consider basic open covers (and hence families of basic closed sets with the f.i.p.) to prove that a space is compact, this gives the claim.

11.1.4 $n \mapsto \mathcal{P}_n$ is a dense homemorphic embedding

Let $\beta \colon \mathbb{N} \to \beta\omega$ be given by $n \mapsto \mathcal{P}_n$. This is a continuous ($\mathbb{N}$ is discrete so every map from $\mathbb{N}$ is continuous) injection. By noting that

$$\beta(A) = A^\star \cap \beta(\mathbb{N})$$

we see that β is open onto its image and hence a homeomorphic embedding.

Now let $A^\star$ be a non-empty basic open subset of $\beta\omega$. Then $A \neq \emptyset$ and picking $n \in A$ we see that $\mathcal{P}_n \in A^\star \cap \beta(\mathbb{N})$ showing that $\beta(\mathbb{N})$ is dense as claimed.

11.1.5 Summary

We have constructed a compact Hausdorff space $\beta\omega$ which contains (a copy of) $\mathbb{N}$ as a dense subspace. The extra points we added (the elements of the remainder $\beta\omega \setminus \beta(\mathbb{N})$) correspond to the non-converging ultrafilters on $\mathbb{N}$.

11.2 General Theory of Compactifications

Definition 11.1. Suppose X is a topological space.

A *compactification* of X is a pair (h, Y) such that Y is a compact Hausdorff topological space and $h \colon X \to Y$ is a homeormorphic embedding such that $h(X)$ is dense in Y.

The *remainder* of a compactification (h, Y) of X is the subspace $Y \setminus h(X)$ of Y.

We sometimes emphasize the fact that Y is Hausdorff by writing 'Hausdorff compactification' instead of 'compactification'.

Definition 11.2. Suppose X is a topological space and (h, Y), (g, Z) are compactifications of X.

We say that (g, Z) is larger than (h, Y), written $(h, Y) \leq (g, Z)$, if and only if there is a continuous function $\pi \colon Z \to Y$ such that $\pi \circ g = h$.

We say that (h, Y) and (g, Z) are equivalent compactifications of X, written $(h, Y) \sim (g, Z)$ if and only if there is a homeomorphism $\pi \colon Z \to Y$ such that $\pi \circ g = h$.

Lemma 11.3. *Suppose X is a topological space and (h, Y), (g, Z) are compactifications of X.*

If $(h, Y) \leq (g, Z)$ and $(g, Z) \leq (h, Y)$ then $(g, Z) \sim (h, Y)$.

Thus $\leq$ induces a partial order on the collection of equivalence classes (wrt $\sim$) of compactifications of X.

Proof. Let $\pi \colon Z \to Y$ witness $(h, Y) \leq (g, Z)$ and $\rho \colon Y \to Z$ witness $(g, Z) \leq (h, Y)$. We claim that π and ρ are inverses of each other so that both are homeomorphisms as required.

To show this, it is sufficient to show that $\pi \circ \rho = \mathrm{id}_Y$ and $\rho \circ \pi = \mathrm{id}_Z$ (which will follow by symmetry).

Since $g(X)$ is dense in Y and Y is Hausdorff, it is sufficient to show that $\pi \circ \rho$ is the identity on $g(X)$. But $\pi(\rho(g(x)) = \pi(h(x)) = g(x)$ as required. $\square$

Lemma 11.4. *Suppose X is a topological space and (h, Y) and (g, Z) are compactifications of X. If $\pi \colon Z \to Y$ witnesses that $(h, Y) \leq (g, Z)$ then π maps the remainder onto the remainder, i.e.*

$$\pi(Z \setminus g(X)) = Y \setminus h(X).$$

Proof. We first note that the image of the compact space Z under π contains the dense subset $h(X)$ of Y and hence must be all of Y (as compact subsets are closed in Hausdorff spaces).

In the following we will use that if $A \subseteq Z$ then $g(X) \cap A = g\left(g^{-1}(A)\right)$ which follows from injectivity of g

Now assume that there is $z \in Z \setminus g(X)$ such that $\pi(z) \in h(X)$ and fix $x \in X$ such that $\pi(z) = h(x)$. Since $z \neq g(x)$ we can find an Z-open $U \ni z$ such that $g(x) \notin \overline{U}$. Since $g(X)$ is dense in Z, $z \in \overline{U \cap g(X)} \not\ni g(x)$ and $x \notin \overline{g^{-1}(U)}$ since g is a homeomorphic embedding (i.e. if $x \in \overline{g^{-1}(U)}$ then $g(x) \in \overline{g(g^{-1}(U))} = \overline{U \cap g(X)}$). Since h is a homeomorphic embedding, $h(x) \notin \overline{h(g^{-1}(U))}^{h(X)} = h(X) \cap \overline{h(g^{-1}(U))}^{Y}$. But

$$h(x) = \pi(z) \in \pi\left(\overline{U \cap g(X)}\right) \subseteq \overline{\pi(U \cap g(X))} = \overline{\pi(g(g^{-1}(U)))} = \overline{h(g^{-1}(U))}$$

since $\pi \circ g = h$ and we have the required contradiction. $\square$

Theorem 11.5. *A topological space has a Hausdorff compactification if and only if it is Tychonoff.*

Proof. If a topological space X has a Hausdorff compactification, it is a homeomorphic to a subset of a compact Hausdorff space. But compact Hausdorff spaces are normal, hence functionally normal (Urysohn's Lemma) hence Tychonoff and subspaces of Tychonoff spaces are Tychonoff.

Conversely if X is Tychonoff then there is a homeomorphic embedding $h \colon X \to [0, 1]^\kappa$ for some κ. Thus $(h, \overline{h(X)})$ is a Hausdorff compactification of X. $\square$

Theorem 11.6. *The partially ordered set of (equivalence classes of) compactifications of a Tychonoff space has suprema.*

Proof. Suppose $(g_i, Y_i), i \in I$ are compactifications of X. Let $\Delta = \Delta_i g_i \colon X \to \prod_i Y_i$ be the diagonal. Since one (in fact every) g_i is a homeomorphic embedding, Δ is a homeomorphic embedding. Writing $S = \overline{\Delta(X)}$ and Δ_S for the map Δ with co-domain restricted to S, we see that (Δ_S, S) is a compactification of X. Clearly each $\pi_i \colon S \to Y_i$ witnesses that $(g_i, Y_i) \le (\Delta_S, S)$.

Now suppose that (h, Z) is a compactification of X such that for all $i \in I, (g_i, Y_i) \le (h, Z)$ as witnessed by maps $\sigma_i \colon Z \to Y_i$. Then $H = \Delta_i \sigma_i \colon Z \to \prod_i Y_i$ is continuous and if $x \in X$ then for each $i \in I$, $\Delta_S(x)_i = g_i(x) = \sigma_i(h(x)) = H(h(x))_i$ so that $\Delta_S = H \circ h$ as required. It remains to show that H maps Z into S: for this we note that

$$H(Z) = H\left(\overline{h(X)}\right) \subseteq \overline{H(h(X))} = \overline{\Delta_S(X)} = S.$$

Hence $(\Delta_S, S) \le (h, Z)$ and thus $(\Delta_S, S) = \sup_i (g_i, Y_i)$. □

11.3 The Stone-Čech Compactification

Lemma 11.7. *Suppose Y is a Hausdorff topological space.*

If X is a dense subspace of Y then $|Y| \le 2^{2^{|X|}}$.

Proof. Let $N \colon Y \to \mathcal{P}(\mathcal{P}(X))$ be given by $N(y) = \{U \cap X : y \in U \text{ open } \subseteq Y\}$. Since X is dense in Y, each non-empty Y-open set meets X. Hence each element of $N(y)$ is non-empty and each $N(y)$ is a filter basis. Since distinct points of Y have disjoint open sets containing them, N is an injection as required. □

Theorem 11.8. *Suppose X is a Tychonoff topological space.*

Then X has a greatest compactification, denoted by $(\beta, \beta X)$ and called the Stone-Čech-compactifcation of X.

Proof. By the previous lemma every equivalence class of compactifications has a representative where the space has cardinality $2^{2^{|X|}}$. Choose one representative for each equivalence class and taking the supremum over all of these representatives gives the result. □

Definition 11.9. Suppose X is a topological space, (g, Y) a compactification of X and $\mathcal{C}$ a class of topological spaces.

(g, Y) satisfies the *Stone-Čech property* for $\mathcal{C}$ with respect to $\mathcal{C}$ if and only if for every $Z \in \mathcal{C}$ and every continuous $f \colon X \to Z$ there is a continuous $F \colon Y \to Z$ such that $f = F \circ g$. We say that F *extends* f.

Usually we drop the 'with respect to X' as this is clear from the context.

Theorem 11.10. *Suppose X is a Tychonoff topological space.*

The Stone-Čech-compactification of X is the unique compactification satisfying the Stone-Čech property for compact Hausdorff spaces.

Proof. We first show uniqueness: suppose (g, Y) and (h, Z) are compactications for X that satisfy the Stone-Čech property for compact Hausdorff spaces. Since (g, Y) satisfies the Stone-Čech property for compact Hausdorff spaces there is a continuous $H\colon Y \to Z$ such that $h = H \circ g$. Hence $(h, Z) \leq (g, Y)$. By symmetry $(g, Y) \leq (h, Z)$ and by a previous lemma (g, Y) and (h, Z) are thus equivalent compactifications of X.

We next show that the Stone-Čech-compactification satisfies the Stone-Čech-property for compact Hausdorff spaces: let Z be compact Hausdorff and $f\colon X \to Z$ be continuous. Since $\beta\colon X \to \beta X$ is an embedding, so is $\beta\Delta f\colon X \to \beta X \times Z$ and hence $(\beta\Delta f, \overline{\beta\Delta f\,(X)})$ is a compactification of X. Thus there is $\pi\colon \beta X \to \overline{\beta\Delta f\,(X)}$ with $\beta\Delta f = \pi \circ \beta$. Then $\pi_Z \circ \pi\colon \beta X \to Z$ is continuous and extends f. □

Theorem 11.11. *Suppose X is a topological space.*

The Stone-Čech-compactification of X is the unique compactification satisfying the Stone-Čech property for $\{[0,1]\}$.

Proof. It is sufficient to prove that the Stone-Čech property for $\{[0,1]\}$ implies the Stone-Čech property for compact Hausdorff spaces.

So let (g, Y) be a compactification of X satisfying the Stone-Čech property for $\{[0,1]\}$, Z a compact Haudorff space and $f\colon X \to Z$ a continuous map.

Since Z is normal so Tychonoff, there is a homeomorphic embedding $h\colon Z \to [0,1]^\kappa$ for some κ. For $i \in \kappa$ we write $h_i = \pi_i \circ h$ and note that $h_i \circ f\colon X \to [0,1]$ is continuous so that there is a continuous $F_i\colon Y \to [0,1]$ with $h_i \circ f = F_i \circ g$. Thus $F = \Delta_i F_i\colon Y \to [0,1]^\kappa$ is continuous and $\pi_i \circ F \circ g = h_i \circ f$ showing that F extends f.

We need to check that $F\,(Y) \subseteq Z$: for this we compute

$$F\,(Y) = F\left(\overline{X}g\right) \subseteq \overline{F\,(g\,(X))} \subseteq \overline{Z} = Z$$

since Z is a compact subset of the Hausdorff space $[0,1]^\kappa$ and hence closed in $[0,1]^\kappa$. □

11.4 The One-point Compactification

Theorem 11.12. *Suppose X is a Tychonoff space.*

TFAE:

1. *$X^\star = \beta X \setminus \beta\,(X)$ is closed in βX.*
2. *There is a compactification (g, Y) of X such that $Y \setminus g\,(X)$ is closed in Y.*
3. *For every compactification (g, Y) of X, $Y \setminus g\,(X)$ is closed in Y.*

Proof. We prove (3) ⇒ (2) ⇒ (1) ⇒ (3).

(3) ⇒ (2): Since X is Tychonoff, it has a compactification which by (3) has the required property.

(2) $\Rightarrow$ (1): Suppose (g, Y) is as in (2). Let $\pi\colon \beta X \to Y$ witness $(g, Y) \leq (\beta, \beta X)$ and note that since remainders are mapped to remainders (and of course the image of X to the image of X) we have $\pi^{-1}(Y \setminus g(X)) = X^\star$. As π is continuous, the result follows.

(1) $\Rightarrow$ (3): Suppose (g, Y) is a compactification of X. Let $\pi\colon \beta X \to Y$ witness $(g, Y) \leq (\beta, \beta X)$ and note that since remainders are mapped to remainders (and of course the image of X to the image of X) and π is onto, we have $Y \setminus g(X) = \pi(X^\star)$. As π is continuous, it preserves compactness and since Y is Hausdorff and compact subsets of Hausdorff spaces are closed the result follows. $\square$

Definition 11.13. Suppose X is a non-compact Hausdorff topological space with topology τ and $\infty \notin X$.

The *one-point* (not necessarily Hausdorff) *compactification* of X is the set $\omega X = X \cup \{\infty\}$ with topology

$$\sigma = \tau \cup \{\omega X \setminus C\colon C \text{ compact } \subseteq X\}$$

and dense embedding $\omega\colon X \to \omega X; x \mapsto x$.

Proof. We need to check that σ is a topology on ωX. This is straightforward using that compact subsets of X are closed in X and the intersection of compact sets in a Hausdorff space is compact.

We also need to check that ω is a dense embedding. Again, using that compact subsets of X are closed in X this is straightforward. Density follows from non-compactness of X.

Finally, we need to check that ωX is compact. So let $\mathcal{U}$ be an open cover. Find $U \in \mathcal{U}$ such that $\infty \in U$ so that $U = \omega X \setminus C$ for some compact $C \subseteq X$. As C is compact (both wrt τ and wrt σ since $C \subseteq X$) there is a finite $\mathcal{V} \subseteq \mathcal{U}$ such that $\bigcup \mathcal{V} \supseteq C$. Then $\mathcal{V} \cup \{\omega X \setminus C\}$ is the required finite subcover of $\mathcal{U}$. $\square$

Definition 11.14. Suppose X is a topological space.

X is locally compact if and only if every neighbourhood filter has a filter basis of compact sets, i.e. for every $x \in X$ and open $U \ni x$ there is open V and compact K such that $x \in V \subseteq K \subseteq U$.

Lemma 11.15. *A compact Hausdorff space is locally compact.*

Proof. Suppose X is compact Hausdorff and $x \in U$ open $\subseteq X$.

As X is compact Hausdorff, it is regular so there is open V and compact K such that $x \in V \subseteq K \subseteq U \subseteq X$ as required. $\square$

Lemma 11.16. *An open subspace of a locally compact Hausdorff space is locally compact.*

Proof. Suppose X is locally compact Hausdorff and Y is an open subspace of X. Let $x \in U$ where U is a Y-open subset of Y. As Y is open in X, U is in fact X-open and hence there are X-open V and compact K with $x \in V \subseteq K \subseteq U \subseteq Y$. But then $V = V \cap Y$ is Y-open and $K = K \cap Y$ is compact as required. $\square$

Theorem 11.17. *Suppose X is a Hausdorff topological space.*

The one-point compactification ωX of X is Hausdorff if and only if X is locally compact.

Proof. We note that X is open in its one-point compactification.

$\Leftarrow$: Let $x, y \in \omega X$ be distinct. If $x, y \in X$ then Hausdorffness of X gives disjoint X-open sets $U \ni x, V \ni y$ which are ωX-open by definition of ωX. So assume that $y = \infty$. By local compactness (with $U = X$) find X-open V and compact $K \subseteq X$ such that

$$x \in V \subseteq K \subset X.$$

Then $V \ni x$ and $\omega X \setminus K \ni \infty$ are the required disjoint Y-open sets.

$\Rightarrow$: Since X is open in ωX and ωX is compact Hausdorff, X is locally compact by the two previous lemmas. $\square$

Corollary 11.18. *A locally compact Hausdorff space is Tychonoff.*

Corollary 11.19. *Suppose X is a Hausdorff space.*

X is locally compact if and only if one (and hence all) of the three conditions from 11.12 holds.

Theorem 11.20. *Suppose X is a Tychonoff space and $\infty \notin X$.*

X is locally compact if and only if X has a smallest compactfication.

In this case, the smallest compactification is the one-point compactification.

Proof. We prove both directions separately

$\Rightarrow$: Suppose X is locally compact. We will show that the one-point compactification $(\omega, \omega X)$ is the smallest compactification of X. So assume (g, Y) is any compactification of X and define

$$\pi \colon Y \to \omega X; y \mapsto \begin{cases} \omega\left(g^{-1}(y)\right), & y \in g(X) \\ \infty, & \text{otherwise} \end{cases}.$$

Clearly $\omega = \pi \circ g$ so that we only need to verify that π is continuous. So let U be open in ωX. If $\infty \notin U$ then U is an open subset of X and hence $\pi^{-1}(U) = g(U)$ is open in $g(X)$ which is open in Y (since X is locally compact Hausdorff). If $\infty \in U$ then $U = \omega X \setminus C$ for some compact subset of X. Hence $\pi^{-1}(U) = Y \setminus g(C)$ and C is compact so $g(C)$ is compact so closed in Y.

$\Leftarrow$: Suppose (g, Y) is the smallest compactification of Y. We will show that $Y \setminus g(X)$ is a singleton, giving the result. So assume for a contradiction that there are distinct $y_1, y_2 \in Y \setminus g(X)$. Then $Y' = Y \setminus \{y_1, y_2\}$ is an open subset of Y so is locally compact Hausdorff and hence has a one-point compactification (h, Z) which is smaller than the two-point compactification (id, Y) of Y' (in the lattice of compactifications of Y') as witnessed by some $\pi \colon Y \to Z$.

Since
$$Z = \overline{h(Y')} \subseteq \overline{h\left(\overline{g(X)}^{Y'}\right)} \subseteq \overline{\overline{h(g(X))}} = \overline{h \circ g(X)}$$
we can see that $(h \circ g, Z)$ is a compactification of X. But then π witnesses that $(h \circ g, Z) \leq (g, Y)$. As (g, Y) is the smallest compactification of X it must be equivalent to $(h \circ g, Z)$ and by the proof of Lemma **??** π has to be a homeomorphism. But π maps both of y_1 and y_2 to ∞, a contradiction. □

11.5 Čech-completeness

Definition 11.21. Suppose Z is a topological space and $X \subseteq Z$.

X is a G_δ-subset of Z if and only if X is a countable intersection of Z-open sets, i.e. there are Z-open U_n, $n \in \mathbb{N}$, such that $X = \bigcap_{n\in\mathbb{N}} U_n$.

X is an F_σ-subset of Z if and only if X is a countable union of Z-closed sets, i.e. there are Z-closed C_n, $n \in \mathbb{N}$, such that $X = \bigcup_{n\in\mathbb{N}} C_n$.

Lemma 11.22. *Suppose Z is a topological space and $X \subseteq Z$.*

X is a G_δ-subset of Z if and only if $Z \setminus X$ is an F_σ-subset of Z.

Proof. Dualtiy. □

Definition 11.23. A topological space X is Čech-complete if and only if X is $T_{3.5}$ and $\beta X \setminus \beta(X)$ is an F_σ-subset of βX.

Lemma 11.24. *Suppose X is a $T_{3.5}$ topological space.*

TFAE:

1. *X is Čech-complete.*
2. *For every compactification (h, Y) of X, $Y \setminus h(X)$ is an F_σ-subset of Y.*
3. *For some compactificaiton (h, Y) of X, $Y \setminus h(X)$ is an F_σ-subset of Y.*

Proof. As for locally compact: (2) implies (3) follows since X has a compactification.

For (3) implies (1) note that if $g\colon \beta X \to Y$ witnesses $(h, Y) \leq (\beta, \beta X)$ and $Y \setminus h(X) = \bigcup_n C_n$ for closed C_n, then $\beta X \setminus \beta(X) = g^{-1}(Y \setminus h(X)) = \bigcup_n g^{-1}(C_n)$. Hence by continuity of g the result follows.

For (1) implies (2) note that if $g\colon \beta X \to Y$ witnesses $(h, Y) \leq (\beta, \beta X)$ and $\beta X \setminus \beta(X) = \bigcup_n C_n$ for closed and hence compact C_n, then $Y \setminus h(X) = g\left(\bigcup_n C_n\right) = \bigcup_n g(C_n)$. Hence by continuity of g, the fact that images of compact sets under continuous maps are compact and the fact that compact subsets of Hausdorff spaces are closed, the result follows. □

Definition 11.25. A metric space (X, d) is complete if and only if every Cauchy sequence in X converges.

A topological space X is completely metrizable if and only if there is a metric d on X such that (X, d) is complete and d induces the topology on X.

Lemma 11.26. *A countable product of completely metrizable spaces is completely metrizable.*

A closed subset of a completely metrizable space is completely metrizable.

Proof. Sheet 0 and standard Part A result. □

Lemma 11.27. *Suppose Z is a complete metric space and X is a G_δ-subset of Z.*

Then X is completely metrizable.

Proof. Let d be a complete metric on Z inducing its topology.

Let $Z \setminus X = \bigcup_n C_n$ with C_n closed in Z. Then each $d_{C_n} \colon X \to [0,\infty); x \mapsto \inf\{d(x,c) \colon c \in C_n\}$ is continuous and hence by the Embedding Lemma and its Corollary $D = \mathrm{id}\,\Delta\Delta_n d_{C_n} \colon Z \to Z \times [0,\infty)^{\mathbb{N}} = P$ is a homeomorphic embedding of Z and hence of X.

We now claim that $D(X) = D(Z) \cap Z \times \pi_{[0,\infty)^{\mathbb{N}}}{}^{-1}\left((0,\infty)^{\mathbb{N}}\right) = P_1$: if $x \in D$ then $d_{C_n}(x) > 0$ for all $n \in \mathbb{N}$ giving $\subseteq$. On the other hand if $z \in Z \setminus D$ and $d_{C_n}(z) > 0$ then $z \notin \bigcup_n C_n$ so $z \in X$ giving $\supseteq$.

Next, $D(Z)$ is closed in P: Write $\pi_n \colon Z \times [0,\infty)^{\mathbb{N}} \to [0,\infty)$ for the projection onto the $(n+1)^{st}$ coordinate (i.e. the map $(z, r_1, r_2, \dots) \mapsto r_n$) and π_Z for the projection onto Z. Then $d_{C_n} \circ \pi_Z$ and π_n are continuous for each n so that

$$\{d_{C_n} \circ \pi_Z = \pi_n\} = \{p \in P \colon d_{C_n}(\pi_Z(p)) = \pi_n(p)\}$$

is closed (as $[0,\infty)$ is Hausdorff). Now note that

$$D(Z) = \bigcap_n \{d_{C_n} \circ \pi_Z = \pi_n\}.$$

Hence P_1 is closed in $Z \times (0,\infty)^{\mathbb{N}}$ and the latter the completely metrizable as $(0,\infty)$ if homeomorphic to $\mathbb{R}$.

Hence X is homeomorphic to a closed subspace of a completely metrizable space and hence completely metrizable. □

Lemma 11.28. *For every metric space (X,d) there is complete metric space $(Y,\hat{d})$ and a dense isometric embedding $h \colon X \to Y$.*

Proof. Let $Y = \left\{(x_n) \in X^{\mathbb{N}} \colon (x_n) \text{ is Cauchy}\right\} / \sim$ where $(x_n) \sim (y_n) \iff d(x_n, y_n) \to 0$. Note that $\sim$ is indeed an equivalence relation on the set of Cauchy sequences of X. Define $\hat{d}([(x_n)],[(y_n)]) = \lim d(x_n,y_n)$, noting that this is well defined. Check that $\hat{d}$ is indeed a metric on Y and that it is complete (this is a bit fiddly). Finally observe that $h \colon X \to Y; h(x) = (x)$ where (x) is the sequence with constant value x is an isometric embedding. Either note that $h(X)$ is dense or take $\overline{h(X)}$ to complete the proof. □

We call (h, Y) a completion of X (and could show that Y is unique up to the natural definition of equivalence) and identify X with its image under h.

Lemma 11.29. *If a metrizable space X is Čech-complete then it is completely metrizable.*

Proof. Let Y be a completion of X and $(\beta, \beta Y)$ its Stone-Čech compactification. As X is dense in Y, $\beta(X)$ will be dense in βY, so $(\beta|_X, \beta Y)$ is also a compactification of X. Thus $\beta(X)$ is a G_δ-subset of βY and hence a G_δ-subset of $\beta(Y)$ (simply intersect the witnessing open sets with $\beta(Y)$). Hence X is a G_δ-subset of Y and thus completely metrizable. □

Lemma 11.30. *If a metric space (X, d) is complete and $\mathcal{F}$ is a filter on X that contains sets of arbitrarily small diameter, then $\mathcal{F}$ converges (to a point of X).*

Proof. For each $n \in \mathbb{N}$ choose $C_n \in \mathcal{F}$ with $\operatorname{diam}(C_n) \leq 2^{-n}$. Without loss of generality C_n is closed and (by taking finite intersections) the C_n for a decreasing sequence of closed sets. For each $n \in \mathbb{N}$, we can thus choose $x_n \in C_n$ and observe that if $n \leq m$ then $x_n, x_m \in C_n$ so that $d(x_n, x_m) \leq 2^{-n}$. Hence (x_n) is Cauchy and thus converges to some $\hat{x} \in X$. But then $\hat{x} \in \bigcap C_n$ (since each C_n is closed and a tail of (x_m) belongs to C_n). Therefore $B_{2^{-n}}(x) \supseteq C_n$ and thus $\mathcal{F} \to \hat{x}$ as required. □

Lemma 11.31. *If a space X is completely metrizable then it is Čech-complete.*

Proof. Let d be a complete metric on X inducing the original topology on X. Without loss of generality we may assume that d is bounded by 1 (otherwise take $\min\{d, 1\}$ which will still be complete). For each $x \in X$, let $d_x \colon X \to [0, 1]$ be given by $d_x(y) = d(x, y)$, which is clearly continuous so extends to a continuous function $\beta d_x \colon \beta X \to [0, 1]$.

For each $x \in X$, let $V_{n,x} = \beta d_x{}^{-1}([0, 2^{-n}))$ and observe that $X \cap V_{n,x} = B_{2^{-n}}(x)$ which has diameter $\leq 2^{-n+1}$.

For each $n \in \mathbb{N}$ we let

$$U_n = \bigcup_{x \in X} V_{n,x}$$

which is open in βX and claim $X = \bigcap_n U_n$.

That $X \subseteq \bigcap_n U_n$ is clear.

Now let $z \in \bigcap_n U_n$. Then $X \# \mathcal{N}_z \to z$ (by density of X in βX).

On the other hand for each $n \in \mathbb{N}$, some $X \cap V_{n,x} \in X \# \mathcal{N}_z$ so that interpreting $X \# \mathcal{N}_z$ as a filter on X, this filter contains sets of arbitrarily small diameter and thus converges to some $\hat{x} \in X$. As βX is Hausdorff, $X \# \mathcal{N}_z$ has a unique limit and thus $z = \hat{x} \in X$ as required. □

We summarize:

Theorem 11.32. *A metrizable space X is completely metrizable if and only if X is Čech-complete.*

11.6 The Baire Category Theorem

Theorem 11.33. *Suppose X is a compact Hausdorff space.*

If $U_n, n \in \mathbb{N}$ is a countable family of dense open sets, then $\bigcap_n U_n$ is dense.

Proof. First note that the intersection of two dense open sets is dense open: if U_1, U_2 are dense open and V is non-empty open then $V \cap U_1$ is non-empty and open and hence $V \cap U_1 \cap U_2$ is non-empty. Thus we may assume wlog (replacing them with $\bigcap_{k \leq n} U_k$) that the U_n are decreasing.

Now, fix a non-empty open V and inductively non-empty open W_n such that $\overline{W_{n+1}} \subseteq U_{n+1} \cap W_n$ with $W_0 \subseteq V \cap U_0$. This is possible: $V \cap U_0$ is non-empty by density of U_0 so pick $x \in V \cap U_0$ and then use regularity of X to obtain W_0 as required. For the inductive step, having defined W_n, we note again that $W_n \cap U_{n+1}$ is non-empty by density of U_{n+1} so pick $x \in W_n \cap U_{n+1}$ and W_{n+1} by regularity of X.

Now observe that

$$\bigcap_{n \geq 0} W_n \subseteq \bigcap_{n \geq 0} \overline{W_n} \subseteq \bigcap_{n \geq 1} \overline{W_n} \subseteq \bigcap_{n \geq 1} W_n \subseteq \bigcap_{n \geq 0} W_n$$

where the last $\subseteq$ follows from $W_1 \subseteq W_0$. Thus $\bigcap_n W_n = \bigcap_n \overline{W_n}$ and the latter is an intersection of a decreasing family of non-empty closed sets in a compact space and hence non-empty. Finally observe that since $W_0 \subseteq V$ and $\bigcap_n W_n \subseteq \bigcap_n U_n$ we have $\bigcap_n U_n \cap V \neq \emptyset$ as required. $\square$

Definition 11.34. Supppose X is a topological space.

X is Baire if and only if a countable intersection of dense open sets is dense.

Dually, a countable union of closed, co-dense subsets of X is co-dense.

Theorem 11.35. *A dense G_δ subset of a Baire space is Baire.*

Proof. Suppose X is a dense G_δ subset of the Baire space Y. Let $V_n, n \in \mathbb{N}$ be Y-open sets such that $X = \bigcap_n V_n$ and $W_n, n \in \mathbb{N}$ be X-dense, X-open subsets of X. Then each W_n is Y-dense (as X is dense in Y) and we can find Y-open U_n such that $W_n = X \cap U_n$. Then $\{U_n \colon n \in \mathbb{N}\} \cup \{V_n \colon n \in \mathbb{N}\}$ is a countable family of dense open sets.

Now assume that W is X-open and non-empty. Find Y-open U such that $W = X \cap U$. As Y is Baire,

$$\emptyset \neq U \cap \bigcap_n V_n \cap \bigcap_n U_n = U \cap X \cap \bigcap_n U_n = W \cap \bigcap_n X \cap U_n = W \cap \bigcap_n W_n$$

as required. $\square$

Corollary 11.36. *Čech-complete Tychonoff spaces and hence completely metrizable spaces are Baire.*

Proof. By definition a Čech-complete Tychonoff space is homeomorphic to a dense G_δ-subset of a compact Hausdorff space. $\square$

Corollary 11.37. *$\mathbb{Q}$ is not a G_δ-subset of $\mathbb{R}$.*

Proof. Note that $\mathbb{Q}$ is dense in $\mathbb{R}$ which is completely metrizable, so Baire. If $\mathbb{Q}$ were a G_δ-subset of $\mathbb{R}$ it would be Baire. Enumerating $\mathbb{Q}$ as $\{q_n : n \in \mathbb{N}\}$ and considering the closed co-dense sets $\{q_n\}, n \in \mathbb{N}$ gives a contradiction. □

Lemma 11.38. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function.*

The set of continuity of f, $\{x \in \mathbb{R} : f$ is continuous at $x\}$ is a G_δ-subset of $\mathbb{R}$.

Proof. Let $A_n = \{x \in \mathbb{R} : \exists \delta > 0\ f(B_\delta(x)) \subseteq B_{2^{-n}}(f(x))\}$. Clearly the A_n form a decreasing sequence of subsets of $\mathbb{R}$ and $\bigcap_n A_n = \bigcap_n A_{n+1}$ is the set of continuity of f.

Now, if $x \in A_{n+1}$ and $\delta > 0$ satisfies $f(B_\delta(x)) \subseteq B_{2^{-n}}(f(x))$ then $B_\delta(x) \subseteq A_n$. Thus there is an open set V_n such that $A_{n+1} \subseteq V_n \subseteq A_n$. Thus

$$\bigcap_n A_n \subseteq \bigcap_n A_{n+1} \subseteq \bigcap_n V_n \subseteq \bigcap_n A_n$$

so the result follows. □

Corollary 11.39. *There is no function $f : \mathbb{R} \to \mathbb{R}$ whose set of continuity is exactly $\mathbb{Q}$.*

11.7 A Combinatorial Theorem using $\beta\mathbb{N}$ - Not lectured

Theorem 11.40 (van der Waerden's Theorem)**.** *If $A_1, \ldots, A_n$ is a partition of $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $l \in \mathbb{N}$ then there is $k \leq n$ such that A_k contains an arithmetic partition of length l.*

Proof based on work by Dona Strauss. We first will define a semi-group operation $\oplus$ on $\beta\mathbb{N}$ which is left-continuous, i.e. for fixed $p \in \beta\mathbb{N}$ the map $.\oplus p : \beta\mathbb{N} \to \beta\mathbb{N}; q \mapsto q \oplus p$ is continuous: to do so, fix $n \in \mathbb{N}$ and note that $n+. : \mathbb{N} \to \mathbb{N}; m \mapsto n + m$ is continuous and hence extends to a continuous map $n\hat{+}. : \beta\mathbb{N} \to \beta\mathbb{N}$. Now fix $p \in \beta\mathbb{N}$ and extend the continuous map $.\hat{+}p : \mathbb{N} \to \beta\mathbb{N}; n \mapsto n\hat{+}p$ to $.\oplus p$. Using limits, one can easily show that $\oplus$ is a semi-group operatrion (i.e. is associative) and by construction it is left-continuous and restricts to the usual addition on $\mathbb{N}$.

We will now write $+$ instead of $\oplus$.

Next, we will show that $(\beta\mathbb{N}, +)$ contains a smallest, non-empty compact ideal K: an ideal is collection $I \subseteq \beta\mathbb{N}$ such that for all $p \in \beta\mathbb{N}$, $p + I \subseteq I$ and $I + p \subseteq I$. Clearly $\beta\mathbb{N}$ is an ideal and the intersection of all ideals is a possibly empty ideal. If $I_1, \ldots, I_k$ are ideals then $I_1 + \cdots + I_n \subseteq \bigcap_1^n I_k$ so the collection of compact ideals has the finite intersection property and hence non-empty intersection.

Finally, we will show that K contains an idempotent, i.e. some p such that $p + p = p$: K is a compact sub-semi-group of $(\beta\mathbb{N}, +)$. Now, note that the intersection of a decreasing chain of non-empty closed sub-semi-groups of K is a non-empty sub-semi-group, so that by Zorn's Lemma, there is a minimal closed sub-semi-group S of K. Let $p \in S$: then $\{q+p\colon q \in S\}$ is a sub-semi-group of S so by minimality equals S. Hence $Z = \{q \in S\colon q+p = p\} = . \oplus p^{-1}(\{p\})$ is non-empty and closed (by continuity). It is easy to check that Z is a sub-semi-group of S so again by minimality $Z = S$, giving that $p+p = p$.

So, to summarize: $(\beta\mathbb{N}, +)$ is a semi-group containing a smallest ideal K which contains an idempotent p.

Now let $P = (\beta\mathbb{N})^l$ with pointwise operation $+$, another semi-group which contains $\mathbb{N}^l$ as a dense sub-semi-group. Let

$$A = \{(a, a+d, a+2d, \ldots, a+(l-1)d)\colon a, d \in \mathbb{N}\} \subseteq \mathbb{N}^l \subseteq P$$

and

$$D = \{(x, \ldots, x)\colon x \in \mathbb{N}\} \cup A \subseteq \mathbb{N}^l \subseteq P.$$

Note that D is a sub-semi-group of P and A is an ideal in D. By taking limits we get that $\overline{D}$ is a sub-semi-group of P and that $\overline{A}$ is an ideal in $\overline{D}$. Clearly $\hat{p} = (p, \ldots, p) \in \overline{D}$ and letting $K(\overline{D})$ be the smallest closed ideal of $\overline{D}$ (as above) we get $\pi_k(K(\overline{D}))$ as a compact (so closed) ideal of $\pi_k(\overline{D}) = \overline{\pi_k(D)} = \beta\mathbb{N}$ and hence $\hat{p} \in K(\overline{D}) \subseteq \overline{A}$.

To summarize, we have a $p \in \beta\mathbb{N}$ such that $\hat{p} \in \overline{A}$. Now, since $\mathcal{N}_p \cap \mathbb{N}$ is an ultrafilter on $\mathbb{N}$ (see the third example sheet), we have some $A_k \in \mathcal{N}_p \cap \mathbb{N}$. Hence $\overline{A_k}$ is a neighbourhood of p and thus $\overline{A_k}^l \cap A \neq emptyset$. Since $A \subseteq \mathbb{N}^l$ we get that $A_k^l \cap A \neq \emptyset$, as required. $\square$

12 Connectedness

Definition 12.1. Suppose X is a topological space.

A disconnection of X is a partition of X into two non-empty open sets (these are then necessarily disjoint proper subsets of X which cover X and are also closed).

A space is disconnected if and only if it admits a disconnection.

A space is connected if and only if it does not admit a disconnection.

Note that a subset of X is disconnected if and only if it is disconnected in the subspace topology.

We need a few results from Part A:

Lemma 12.2. *A space X is connected if and only if the only clopen subsets of $\emptyset$ and X.*

Proof. If $\emptyset \neq D \neq X$ is clopen then $D, X \setminus D$ is a disconnection of X. $\square$

Lemma 12.3. *A space is connected if and only if every continuous $\{0,1\}$-valued map is constant.*

Proof. If U, V is a disconnection, then the indicator function 1_U on U is a continuous non-constant map.

Conversely, if $f : X \to \{0, 1\}$ is a continuous surjection, then $f^{-1}(0)$ and $f^{-1}(1)$ form a disconnection. $\square$

We will write 2 for $\{0, 1\}$ (with the discrete topology).

Corollary 12.4. *The one point space is connected.*

Lemma 12.5. *Suppose X is a topological space and $A, A_i, i \in I$ are connected subsets of X.*

If $\forall i \in I\ A_i \cap A \neq \emptyset$ then $A \cup \bigcup_i A_i$ is connected.

Proof. Wlog $X = A \cup \bigcup_i A_i$. Consider a continuous $f : X \to 2$. Note that $f|_A$ and $f|_{A_i}$ $(i \in I)$ are continuous and thus constant (by assumption), with values c and $c_i, i \in I$. Fix $i \in I$, $a \in A \cap A_i$ and observe that $c = f(a) = c_i$. Hence f is constant. $\square$

Definition 12.6. Suppose X is a topological space and $x \in X$.

The component of x (in X) is

$$C_X(x) = \bigcup \{A \subseteq X : x \in A \text{ and } A \text{ is connected}\}.$$

The quasicomponent of x (in X) is

$$Q_X(x) = \bigcap \{C \subseteq X : x \in C \text{ and } C \text{ is clopen}\}.$$

Lemma 12.7. *Suppose X is a topological space and $x \in X$.*

The component of x is connected and contained in the quasicomponent of X and the quasicomponent of x is closed in X.

Proof. Write

$$C_X(x) = \{x\} \cup \bigcup \{A \subseteq X : x \in A \text{ and } A \text{ is connected}\}$$

and apply the earlier lemma to see that $C_X(x)$ is connected.

If D is clopen in X and $x \in D$ then $C_X(x) \cap D$ is clopen in $C_X(x)$ and non-empty so must be all of $C_X(x)$. Hence $C_X(x) \subseteq D$, giving $C_X(x) \subseteq Q_X(x)$.

Finally $Q_X(x)$ is an intersection of closed sets, so closed. $\square$

Lemma 12.8. *If X is compact Hausdorff, $\mathcal{C}$ is a family of closed subsets of X and U is open such that $\bigcap \mathcal{C} \subseteq U$ then there is finite $\mathcal{C}' \subseteq \mathcal{C}$ such that $\bigcap \mathcal{C}' \subseteq U$.*

Proof. Suppose $\mathcal{C}$ and U are as in the lemma.

Note that $\mathcal{C} \cup \{X \setminus U\}$ is a family of closed subsets of X with empty intersection so that there is a finite subfamily $\mathcal{C}' \cup \{X \setminus U\}$ with empty intersection. But this gives $\bigcap \mathcal{C}' \subseteq U$ as required. $\square$

Theorem 12.9 (Sura-Bura Lemma)**.** *In a compact Hausdorff space, components and quasicomponents coincide.*

Proof. Suppose X is compact Hausdorff and $x \in X$. Write $Q = Q_X(x)$.

It is sufficient to show that Q is connected. So assume that A, B form a Q-disconnection, i.e. A, B are disjoint, non-empty, Q-open and $Q = A \cup B$. Since Q is X-closed, A, B are X-closed and as X is compact Hausdorff so normal, there are disjoint X-open U, V such that $A \subseteq U, B \subseteq V$. Note that in particular $\overline{U} \cap V = \emptyset = U \cap \overline{V}$. Then

$$Q = \bigcap \{C \subseteq X\} \, x \in C \text{ clopen } = A \cup B \subseteq U \cup V$$

so by the preceding lemma there is a finite collection $\mathcal{C}$ of clopen sets containing x such that $A \cup B \subseteq C = \bigcap \mathcal{C} \subseteq U \cup V$. Note that C is X-clopen (as a finite intersection of X-clopen sets).

Hence

$$\overline{U \cap C} \subseteq \overline{U} \cap C = \overline{U} \cap ((U \cup V) \cap C) = U \cap C$$

so that $U \cap C$ and (similarly) $V \cap C$ are X-clopen. Since these are disjoint one of them contains x, wlog $x \in U \cap C$. But then $B \subseteq V$ so that $B \cap U \cap C = \emptyset$. Hence $B = B \cap Q \subseteq B \cap U \cap C = \emptyset$, contradicting non-emptyness of B. □

13 Disconnectedness

Definition 13.1. Suppose X is a topological space.

X is totally disconnected if and only if every component of X is a singleton.

X is zero-dimensional if and only if X has a basis of clopen sets.

Lemma 13.2. *In a compact Hausdorff space, total disconnectedness and zero-dimensionality are equivalent.*

Proof. Suppose X is compact Hausdorff.

If X is zero-dimensional, let $\mathcal{B}$ be a clopen basis and for each $x \in X$ let $\mathcal{B}_x = \{B \in \mathcal{B} \colon x \in B\}$. As X is Hausdorff we have

$$Q_X(x) \subseteq \bigcap \mathcal{B}_x = \{x\}$$

and since $C_X(x) \subseteq Q_X(x)$ X is totally disconnected.

Conversely, assume that X is totally disconnected. We claim that

$$\{C \subseteq X \colon C \text{ clopen } \}$$

is a basis for X.

Let $x \in X$ and $U \ni x$ be open. By the Sura-Bura Lemma

$$Q_X(x) = C_X(x) = \{x\} \subseteq U$$

and recalling that Q_X is an intersection of clopen sets in the compact Hausdorff space, there are finitely many clopen sets $\mathcal{C}_x$ (each containing x) with $x \in C = \bigcap \mathcal{C}_x \subseteq U$ and C is clopen. □

13.1 Stone Duality - an Outline

Definition 13.3. A Boolean Algebra is a set B together with partial order $\leq$ such that

1. B has a maximal element 1 and a minimal element 0;
2. B has binary (and hence finitary) suprema (written $a \vee b$) and infima (written $a \wedge b$);
3. there is a negation operation $\neg\colon B \to B$ satisfying $\forall b \in B\ (\neg b) \vee b = 1$ and $\forall b \in B\ (\neg b) \wedge b) = 0$.

If B and C are Boolean Algebras, a homomorphism from B to C is a function $f\colon B \to C$ preserving $\vee, \wedge, 0, 1$ and (hence) $\neg$.

Lemma 13.4. *Suppose X is a topological space. Then $\mathcal{B}_X = \{A \subseteq X\colon A \text{ is clopen in } X\}$ with partial order $\subseteq$ is a Boolean Algebra with $\vee = \cup$, $\wedge = \cap$ and $\neg A = X \setminus A$.*

Moreover if $f\colon X \to Y$ is a continuous between two topological spaces then $f_\star\colon \mathcal{B}_Y \to \mathcal{B}_X$ given by $f_\star(A) = f^{-1}(A)$ is a Boolean Algebra homomorphism (and compositions work).

Proof. Straightforward. □

Definition 13.5. Suppose B is a Boolean Algebra. A filter $\mathcal{F}$ on B is a subset of B such that

1. $0 \notin \mathcal{F} \neq \emptyset$;
2. $\forall a, b \in \mathcal{F}\ a \wedge b \in \mathcal{F}$;
3. $\forall a \in \mathcal{F}\ \forall b \in B\ (a \leq b \implies b \in \mathcal{F})$.

An ultrafilter on B is a maximal filter (wrt $\subseteq$).

Lemma 13.6. *Suppose B is a Boolean Algebra and $\mathcal{U}$ is a filter on B.*

The following are equivalent:

1. *$\mathcal{U}$ is an ultrafilter.*
2. *$\forall b \in B$ exactly one of $b \in \mathcal{U}$ and $\neg b \in \mathcal{U}$.*
3. *$\forall a, b \in B$, if $a \vee b \in \mathcal{U}$ then $a \in \mathcal{U}$ or $b \in \mathcal{U}$.*

Proof. As for ultrafilters on topological spaces (note that $\mathcal{P}(X)$ with $\subseteq$ is a Boolean Algebra). □

Theorem 13.7. *Every filter on a Boolean Algebra can be extended to an ultrafilter.*

Proof. Apply Zorn's Lemma, noting that a union of an increasing chain of filters is a filter. □

Lemma 13.8. *Suppose B is a Boolean algebra.*

Let $X_B = \{\mathcal{U}\colon \mathcal{U}$ is an ultrafilter on $B\}$.

Writing

$$U_b = \{\mathcal{U} \in X_B\colon b \in \mathcal{U}\}$$

we have that

$$\{U_b\colon b \in B\}$$

is a basis for a compact Hausdorff zero-dimensional topology on X.

We call X_B with this topology the Stone space of B.

Proof. As in 11.1. □

Lemma 13.9. *Suppose $f\colon B \to C$ is a Boolean Algebra homomorphism.*

If $\mathcal{U}$ is an ultrafilter on C then

$$f^\star(U) := \mathcal{V} = \{b \in B\colon f(b) \in \mathcal{U}\}$$

is an ultrafilter on B.

Proof. Since $f(0_B) = 0_B \notin \mathcal{U}$ and $f(1_B) = 1_C \in \mathcal{U}$ we have $0_B \notin \mathcal{V} \neq \emptyset$.

Next, if $b_1, b_2 \in \mathcal{V}$ then $f(b_1 \wedge b_2) = f(b_1) \wedge f(b_2) \in \mathcal{U}$ so that $\mathcal{V}$ is closed under binary infima.

Finally if $b_1 \in \mathcal{V}$ and $b_1 \leq b_2 \in B$ then $f(b_1) \leq f(b_2)$ so that $b_2 \in \mathcal{V}$.

Hence $\mathcal{V}$ is a filter.

To see that it is an ultrafilter, note that for each $b \in B$, $f(\neg b) = \neg f(b)$ and exactly one of $f(b)$ and $\neg f(b)$ belongs to $\mathcal{U}$. □

Lemma 13.10. *Suppose $f\colon B \to C$ is a Boolean Algebra homomorphism.*

Then $f^\star\colon X_C \to X_B$ is a continuous map (and compositions work).

Proof. Straightforward. □

Theorem 13.11 (Stone Duality)**.** *The contravariant functors $X \mapsto \mathcal{B}_X$ and $B \mapsto X_B$ defined above are 'inverses'(up to isomorphism) of each other for the class of compact Hausdorff zero-dimensional spaces.*