Problem Sheet 3

1. Section A. Double diffusive convection – The Hopf bifurcation. From the notes on double diffusive convection, recall that we have

$$A\sigma^2 + B\sigma + C = 0, \tag{1}$$

where

$$A = (\alpha^2 + m^2\pi^2)\mathrm{Le},$$
$$B = (\alpha^2 + m^2\pi^2)^2[1 + \mathrm{Le}] + \alpha^2(\mathrm{Ra}_s - \mathrm{Ra}\ \mathrm{Le}),$$

and

$$C = (\alpha^2 + m^2\pi^2)^3 + [\mathrm{Ra}_s - \mathrm{Ra}]\,\alpha^2(\alpha^2 + m^2\pi^2),$$

and instability occurs if either of the two roots of Eqn. (1) has positive real part.

Suppose a bifurcation to instability occurs with $\mathrm{Re}(\sigma) = 0$ as the complex conjugates values of σ cross the imaginary axis at $\pm i\Omega$, say, thus inducing an oscillatory instability.

(a) Show that $B = 0,\ C > 0$ are required for the system to be at an oscillatory bifurcation point.

(b) Hence show that

$$\mathrm{Le}\ \mathrm{Ra}_s > \mathrm{Ra}$$

is required for the system to be at an oscillatory bifurcation point.

A more detailed picture and discussion of the geophysical relevance of the oscillatory bifurcation can be found J. S. Turner, *Buoyancy Effects in Fluids*, CUP. There it is noted that the observation of oscillatory instability is generally rare.

2. Section B. *Washburn's law*

   Consider a dry sponge, one end of which is dipped into a bath of stationary liquid. Model the sponge as a one-dimensional porous medium and assume that the liquid is sucked up into the sponge so that a liquid interface is situated at $z = h(t)$. Assume that the pressure just beneath the interface, i.e. at $z = h(t)^-$, is $p = -\gamma\kappa$ relative to atmospheric pressure (where κ is a typical curvature within the porous medium) and assume that the pressure at the base of the sponge matches that at the surface of the liquid, i.e. it is at atmospheric pressure. What is the pressure distribution within the (wetted) porous medium? Write down a first order ode for the evolution of $h(t)$. Show that initially

   $$h(t) \approx (2\gamma k\kappa t/\mu\phi)^{1/2}$$

   and that after a long time $h(t) \approx h_\infty = \gamma\kappa/\rho g$.

   Show further that the time t_s taken to reach $h(t_s) = s$ is given by

   $$t_s = -\frac{\mu\phi}{k\rho g}\left[s + h_\infty \log\left(1 - s/h_\infty\right)\right].$$

3. Section B. *The Saffman–Taylor instability*

   Consider the two-dimensional motion of the interface between two liquids of differing viscosity in a porous medium of permeability k. The interface is positioned at $x = \zeta$; the fluid in $x < \zeta$ has viscosity μ_1 and that in $x > \zeta$ has viscosity μ_2. You may neglect the influence of gravity throughout this question.

   (a) If the interface moves at constant speed V, so that $\zeta = Vt$, show that the pressure in each liquid is given by

   $$p_i^{(0)} = \frac{\mu_i V}{k}(Vt - x)$$

   for $i = 1, 2$.

   (b) Consider now a sinusoidal perturbation to the interface, so that $\zeta = Vt + \epsilon e^{\sigma t}\sin\kappa y$ with $\epsilon e^{\sigma t} \ll Vt$. Determine the linearized pressure perturbation in fluids 1 and 2.

   (c) The surface tension between the two fluids, γ, gives rise to a pressure jump across the interface proportional to the macroscopic curvature of the interface. Show that the growth rate σ is given by

   $$\sigma = \frac{\mu_2 - \mu_1}{\mu_1 + \mu_2}V\kappa - \frac{\gamma k}{\mu_1 + \mu_2}\kappa^3.$$

   Deduce that if $(\mu_2 - \mu_1)V < 0$ the interface is always stable but that if $(\mu_2 - \mu_1)V > 0$ the interface is unstable to perturbations with wavenumber $0 < \kappa < \kappa_0$ where

   $$\kappa_0 = \left[\frac{(\mu_2 - \mu_1)V}{\gamma k}\right]^{1/2}.$$

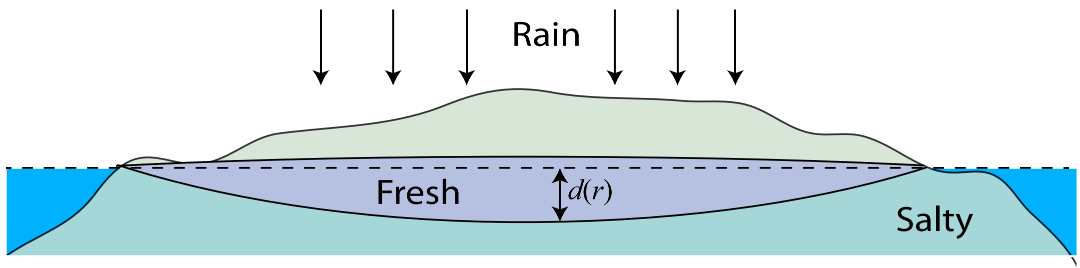
   (d) Explain why we might ultimately expect to see an instability with wavelength

   $$\lambda = 12^{1/2}\pi k^{1/2}\mathrm{Ca}^{-1/2}$$

where $\mathrm{Ca} = (\mu_2 - \mu_1)V/\gamma$ is a capillary number.

(e) Is this model likely to be realistic for the Saffman–Taylor instability in a porous medium?

4. Section B. *Freshwater lenses*



Consider a permeable island situated within a salty body of water (as illustrated in the figure above). Because of the infiltration of fresh (rain) water of density ρ_0 at a constant flux w_r per unit area a lens of fresh water forms 'floating' above and displacing the more dense salt water (density ρ_s) that permeates the porous medium beneath sea level.

Under what circumstances may we assume that the pressure within the lens is hydrostatic? Making this assumption and setting the pressure at the sea's surface to be 0 show that the pressure a depth $z(r)$ below sea level within the lens is given by

$$p = (\rho_s - \rho_0)gd + \rho_0 gz,$$

where $d(r)$ is the depth of the lower surface of the lens beneath sea level.

Using Darcy's law together with the balance between the outflow of fresh water, caused by this pressure, and the infiltration of fresh water, due to rain, and assuming axisymmetry show that

$$d(r) = \left[\frac{\mu w_r}{2(\rho_s - \rho_0)gk}\right]^{1/2} \left(R^2 - r^2\right)^{1/2}$$

where $r = R$ is the radius at which the lens meets sea level.

What is the profile of the upper surface of the lens above sea level?

5. Section C. *Generalized gravity currents*

Consider again the governing equation for a gravity-driven current in a porous medium, eq. (3.62) of the printed lecture notes, namely

$$\frac{\partial h}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(rh\frac{\partial h}{\partial r}\right).$$

However, now we allow the volume of the current to increase as a power law in time so that the current must satisfy a modified (3.63):

$$2\pi \int_0^{a(t)} rh \;\mathrm{d}r = t^{\alpha}$$

with $h[a(t), t] = 0$.

(a) Show by a scaling analysis that it is appropriate to look for a similarity solution of the form

$$h(r,t) = t^{(\alpha-1)/2} f(\eta)$$

where $\eta = rt^{-(\alpha+1)/4}$ and that the function $f(\eta)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta f\frac{\mathrm{d}f}{\mathrm{d}\eta}\right) + \frac{\alpha+1}{4}\eta^2\frac{\mathrm{d}f}{\mathrm{d}\eta} + \frac{1-\alpha}{2}\eta f = 0,$$

with

$$\int_0^{\eta_N} \eta f \;\mathrm{d}\eta = 1/2\pi, \quad f(\eta_N) = 0.$$

(b) By letting $\eta = \eta_N(1-\delta)$ with $\delta \ll 1$ show that, in similarity variables, the edge of the current has a constant contact angle and that, in particular,

$$f(\eta) \sim \frac{\alpha+1}{4}\eta_N(\eta_N - \eta), \quad \eta \to \eta_N.$$

Comments and corrections to Eamonn Gaffney