Problem Sheet 4

As required in this example sheet, you may use the fact that the Stokes flow equations with velocity boundary conditions have a unique solution, up to an additive constant in the pressure. See Pozrikidis, Ch. 1, Boundary integral and singularity methods for linearized viscous flow, CUP.

1. Section A. **Rotating Sphere in Stokes Flow**

   With $\hat{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}_0$, $r = |\boldsymbol{x} - \boldsymbol{x}_0|$ and

   $$G_{ij} = \frac{\delta_{ij}}{r} + \frac{\hat{x}_i \hat{x}_j}{r^3},$$

   define the rotational dipole $\boldsymbol{G}^c$ by

   $$G^c_{im} := \frac{1}{2}\epsilon_{mlj}\frac{\partial G_{ij}}{\partial x_{0,l}},$$

   where

   $$\epsilon_{mlj} := \begin{cases} +1 & \text{if} \quad (m,l,j) = (1,2,3) \text{ or } (3,1,2) \text{ or } (2,3,1) \\ -1 & \text{if} \quad (m,l,j) = (1,3,2) \text{ or } (2,1,3) \text{ or } (3,2,1) \\ 0 & \text{if} \quad \text{any of } i,j,k \text{ are equal} \end{cases}$$

   $\boldsymbol{G}^c$ is also known as a rotlet or couplet.

   (a) Show that

   $$G^c_{im} = \epsilon_{iml}\frac{\hat{x}_l}{r^3}.$$

   (b) Show the solution for the Stokes flow associated with a sphere of radius a centred at $\boldsymbol{x}_0$ and rotating with angular velocity $\boldsymbol{\Omega}$ is given by $a^3 G^C_{im}\Omega_m$.

   (c) With an origin at the centre of the sphere, determine the torque exerted on the on the sphere by the fluid.

2. Section B. Ciliary Pumping.

   Blake [1] considered a more general metachronal wave which, after non-dimensionalisation, takes the form

   $$x_e = x + \epsilon\sum_{n=1}^{\infty}(a_n \sin(n[x+t]) - b_n\cos(n[x+t]), \qquad y_e = \epsilon\sum_{n=1}^{\infty}(c_n \sin(n[x+t]) - d_n\cos(n[x+t]).$$

(a) Using the result in the lecture notes, without detailed calculation, show that at leading non-trivial order

$$U = \frac{1}{2}\epsilon^2 \sum_{n=1}^{\infty} n^2 [c_n^2 + d_n^2 - a_n^2 - b_n^2 + 2(a_n d_n - c_n b_n)].$$

(b) Blake also found the non-dimensional power required per unit surface area of envelope at leading non-trivial order:

$$P = \epsilon^2 \sum_{n=1}^{\infty} n^3 [c_n^2 + d_n^2 + a_n^2 + b_n^2].$$

Find the metachronal wave/waves that maximises/maximise the modulus of U, the velocity of the far field flow, for a fixed power per unit area, P.

3. Section B. Ciliate Motility. Consider a ciliate, that is a cell covered in beating cilia, which induce movement (Fig. 1). As a simple model, suppose that the cell is spherical and that the ciliary envelope is such that the cell radius does not deform (at least approximately). Alternatively, this could represent the streaming velocity around a spherical Janus particle; see Fig. 1.

   In particular suppose the cell moves its surface in a tangential direction such that, in the non-dimensionalised system, the radius of the cell is $r = 1$ and during the deformation the polar angle θ (such that $z = r\cos\theta$) is mapped to Θ with

$$\Theta = \theta + \epsilon\beta_1(t)\sin\theta.$$

   By working in a frame comoving with the cell, show that its swimming speed is

$$\frac{2}{3}\epsilon\dot{\beta}_1(t).$$

   Hint *Without loss, suppose a reference frame comoving with the swimmer is oriented such that the direction of the swimmer velocity is given by* $\boldsymbol{U} = U\boldsymbol{e}_z$. *Write down the non-dimensional Stokes equations and boundary conditions and show that*

$$\mathbf{u} = \left[-U(t) + \frac{Q(t)}{r^3}\right]\cos\theta \boldsymbol{e}_r + \left[U(t) + \frac{P(t)}{r^3}\right]\sin\theta \boldsymbol{e}_\theta, \qquad p = Const$$

   is a solution of the Stokes equation for $Q(t) = 2P(t)$ *and an appropriate choice of* $P(t)$. *Symbolic algebra, such as the use of Mathematica, is recommended for detailed calculation.*

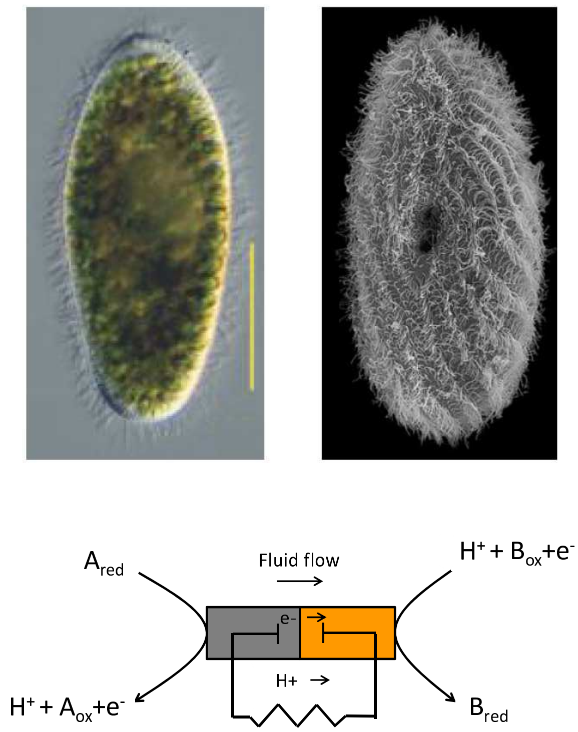

Figure 1: Images of the ciliate, *Paramecium (Viridoparamecium) chlorelligerum*. Upper Left, (a). An image from flash photomicrography of freely motile specimen after disturbance. Scale bar, 50μm. Upper Right, (b). A scanning electron microscope image of the dorsal side, illustrating the density of cilia on the cell surface. From Kreutz et. al [2]. Lower (c). A schematic of a prospective self-electrophoretic propulsion mechanism for a conducting Janus particle within an acidic environment, whereby a slip-velocity is induced by the ion flows generated by a simultaneous catalytic oxidation of a fuel present in the solute, A, on one side of the particle and a catalytic reduction of a fuel, B, on the other. From Paxton et. al [3].

4. Section B. Resistive Force theory. Throughout this question, one can work with the leading non-trivial order of the parameter ϵ, where the flagellum location in the cell fixed frame is given by $y = \epsilon h(s,t)$.

   In the lecture notes any possible movement of the cell in the y-direction was neglected. With the movement of the cell body given by $\boldsymbol{U} = (U, V)$ and the velocity of a flagellar element given by $(U, V + \epsilon h_t)$ write down a set of simultaneous equations for U, V. Hence find U and determine conditions on $h(s,t)$ where the neglect of V in the notes is a good approximation.

5. Section C. Resistive Force theory. Consider a planar beating filament, with the filament beating in a co-moving reference frame described by $y = \epsilon h(s,t)$. However, now we assume the filament is no longer moving a cell body. In addition, we assume that the filament is now moving in a background shear flow $\boldsymbol{v} = -\gamma y \boldsymbol{e}_x$ and it starts such that its midline is along $y = 0$, where the shear velocity is zero. Determine the filament's horizontal swimming speed up to and including $O(\epsilon)$ in terms of its beat pattern, $h(s,t)$, its length L, its radius a, the fluid shear rate, γ, and the fluid viscosity μ.

   You may assume that the filament does not drift in the y-direction so that its displacement in the y-direction is always $O(\epsilon)$ and that resistive force theory generalises to give the drag force per unit length, $\boldsymbol{f}$, on the filament via

$$\boldsymbol{f} = -C_N(\boldsymbol{I} - \boldsymbol{e}_T\boldsymbol{e}_T)(\boldsymbol{U} - \boldsymbol{v}) - C_T(\boldsymbol{e}_T \cdot (\boldsymbol{U} - \boldsymbol{v}))\boldsymbol{e}_T.$$

   where $\boldsymbol{U}$ is the velocity of the filament, and the resistance coefficients C_N, C_T are independent of $\boldsymbol{v}$.

References

[1] J. R. Blake. Infinite models for ciliary propulsion. *J. Fluid Mech.*, 49:209–222, 1971.

[2] M. Kreutz, T. Stoeck, and W. Foissner. Morphological and molecular characterization of paramecium (viridoparamecium nov. subgen.) chlorelligerum kahl (ciliophora). *J. Eukaryot. Microbiol.*, 59:548, 2012.

[3] W. Paxton, A. Sen, and T. Mallouk. Motility of catalytic nanoparticles through self-generated forces. *Chem. Eur. J.*, 11:6462, 2005.