Problem Sheet 2: Solutions

1. From the vertical component of the momentum equation (after applying the lubrication approximation) we have that

$$\frac{\partial p}{\partial z} = -\rho g$$

so that the pressure distribution within the drop is hydrostatic, i.e.

$$p = p_0 + \rho g \left[h(r,t) - z\right] \tag{1}$$

where we have neglected the pressure jump due to surface tension and p_0 is the constant atmospheric pressure.

The horizontal component of the momentum equation is, in the lubrication approximation,

$$\mu \frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial r} = \rho g \frac{\partial h}{\partial r}.$$

Integrating twice subject to $u(z=0) = 0$ (no-slip) and $u_z(z=h) = 0$ (no stress) we find that

$$u = \frac{\rho g}{2\mu} z(z - 2h) \frac{\partial h}{\partial r},$$

and

$$\bar{u} = -\frac{\rho g}{3\mu} h^2 \frac{\partial h}{\partial r}.$$

At this point is acceptable to quote the general result that

$$\frac{\partial h}{\partial t} + \nabla \cdot \mathbf{q} = 0$$

where $\mathbf{q} = h\bar{u}\mathbf{e}_r$ is the fluid flux, though it would be reasonable to ask for this to be derived:

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}\left(r \int_0^h u \ \mathrm{d}z\right) &= \int_0^h \frac{1}{r}\frac{\partial}{\partial r}(ru) \ \mathrm{d}z + h_r u(h) \\
&= -\int_0^h \frac{\partial w}{\partial z} \ \mathrm{d}z + u(h) h_r \qquad \text{[Using } \nabla \cdot \mathbf{u} = 0\text{]} \\
&= -w(h) + w(0) + u(h) h_r \\
&= -\left[h_t + u(h) h_r\right] + u(h) h_r \qquad \text{Using the k.b.c. and } w(0) = 0 \\
&= -h_t.
\end{aligned}$$

Thus we have

$$h_t = \frac{\rho g}{3\mu}\frac{1}{r}\frac{\partial}{\partial r}\left(rh^3\frac{\partial h}{\partial r}\right), \tag{2}$$

as desired.

We are told that the volume of the drop is a given constant V and so we must have

$$V = \int_0^{a(t)} 2\pi r h \ \mathrm{d}r.$$

In scaling terms we may write volume conservation as $V \sim R^2 H$ where R is a typical radial scale and H a typical vertical scale at time T. In scaling terms the governing pde (2) reads

$$\frac{H}{T} \sim \frac{\rho g}{\mu}\frac{H^4}{R^2}$$

from which we have

$$T \sim \frac{\mu}{\rho g}R^2 \left(H^{-3} \sim R^6/V^3\right).$$

Hence the typical radial scale R at time T must scale according to

$$R \sim \left(\frac{\rho g V^3}{\mu}T\right)^{1/8}$$

so that the radius $a(t)$ of the droplet must scale in the same way.

To progress further we non-dimensionalize lengths using $V^{1/3}$ and time using $\mu/\rho g V^{1/3}$ so that we wish to solve

$$h_t = \frac{1}{3r}\frac{\partial}{\partial r}\left(rh^3\frac{\partial h}{\partial r}\right), \tag{3}$$

subject to the volume constraint

$$\int_0^{a(t)} rh \ \mathrm{d}r = 1/(2\pi). \tag{4}$$

Based on the scalings discussed above it is natural to seek a similarity solution of the form $h(r,t) = t^{-1/4}\Theta(\eta)$ where $\eta = rt^{-1/8}$ and we expect the drop to occupy the region $0 \leq \eta \leq \eta_*$ where $\eta_* = a(t)/t^{1/8}$ is the position of the edge of the drop; i.e. $h(a(t),t) = 0$.

[*Note that it is better to use the above similarity ansatz than the alternative t/r^8 since we take more spatial derivatives than time derivatives. However, this alternative approach does work if one is sufficiently careful.*]

Substituting this similarity form into (3) we have

$$-\frac{1}{8}(2\Theta + \eta\Theta') = \frac{1}{3\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\Theta^3\Theta'\right).$$

This may be integrated once to give

$$-\frac{3}{8}\eta^2\Theta = \eta\Theta^3\Theta'$$

where the constant of integration must vanish to ensure that the solution is well-behaved as $\eta \to 0$. A further integration gives

$$\Theta = \left(\frac{3}{4}\right)^{2/3}(\eta_*^2 - \eta^2)^{1/3}$$

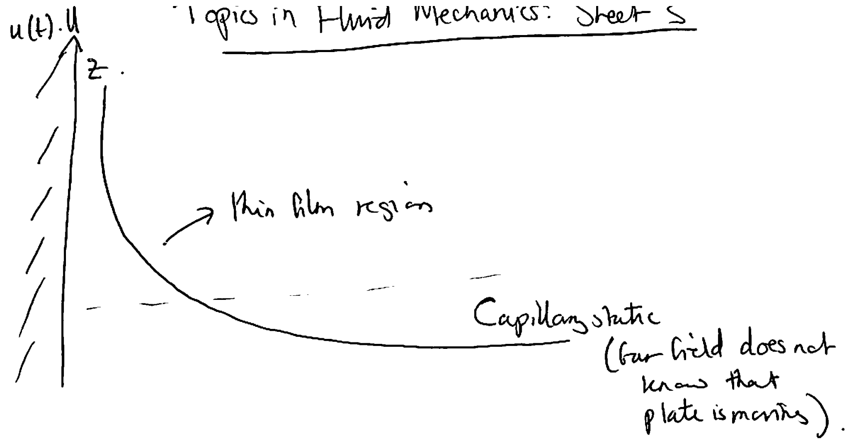
where we have applied the boundary condition that $\Theta(\eta_*) = 0$.

To determine the value of η_* we return to the similarity form of (4), which reads

$$\begin{aligned}\frac{1}{2\pi} &= \int_0^{\eta_*} \eta\Theta \; \mathrm{d}\eta = \left(\frac{3}{4}\right)^{2/3}\int_0^{\eta_*} \eta(\eta_*^2 - \eta^2)^{1/3} \; \mathrm{d}\eta \\ &= \left(\frac{3}{4}\right)^{2/3}\frac{3}{8}\eta_*^{8/3}\end{aligned}$$

from which we immediately have $\eta_* = (2^{10}/3^5\pi^3)^{1/8}$ and the result for $a(t)$ follows.

2. The scenario is as shown below



We would like to make use of the analysis presented in lectures. However, this was presented in the frame of reference of the stationary plate. We propose moving into a frame of reference defined by

$$\bar{z} = z - \int_0^t u(t')\ \mathrm{d}t'$$

with $\bar{t} = t$ and $\bar{h}(\bar{z}, \bar{t}) = h(z, t)$. In this frame of reference, the plate is stationary, so that $\bar{w}(\bar{x} = 0, t) = 0$ and we can quote the result from lectures that

$$\frac{\partial \bar{h}}{\partial \bar{t}} + \frac{\partial}{\partial \bar{z}}\left[\frac{\bar{h}^3}{3\mathrm{Ca}}\left(\bar{h}_{\bar{z}\bar{z}\bar{z}} - \mathrm{Bo}\right)\right] = 0.$$

To convert this back into the lab frame, we use the chain rule noting that $\partial/\partial\bar{z} = \partial/\partial z$ and

$$\frac{\partial}{\partial \bar{t}} = \frac{\partial}{\partial t} + u(t)\frac{\partial}{\partial z}$$

so that

$$\frac{\partial h}{\partial t} + u(t)\frac{\partial h}{\partial z} + \frac{\partial}{\partial z}\left[\frac{h^3}{3\mathrm{Ca}}\left(h_{zzz} - \mathrm{Bo}\right)\right] = 0.$$

As in lecture notes, we take $\mathrm{Bo} = 1$ as the definition of the horizontal length scale, i.e. $R = \ell_c$, so that the outer solution has

$$H \sim \frac{(z - z_0)^2}{\sqrt{2}} \quad \text{and} \quad z_0 = \sqrt{2}.$$

To examine the region close to $z = z_0$, we let $z = z_0 + \epsilon\tilde{z}$ where $\tilde{z}$ is an inner variable. We find that close to the apparent contact point $h \sim \epsilon^2\tilde{z}^2/\sqrt{2}$ and so it is natural to let $h = \epsilon^2\tilde{h}$. Substituting into the lubrication equation, we find that

$$\epsilon^2\frac{\partial \tilde{h}}{\partial t} + \epsilon u(t)\frac{\partial \tilde{h}}{\partial \tilde{z}} + \frac{1}{\epsilon}\frac{\partial}{\partial \tilde{z}}\left[\frac{\epsilon^6\tilde{h}^3}{3\mathrm{Ca}}\left(\epsilon^{-1}\tilde{h}_{\tilde{z}\tilde{z}\tilde{z}} - 1\right)\right] = 0$$

so that, upon choosing $\epsilon = \mathrm{Ca}^{1/3}$ (as in lectures), we have

$$\epsilon \frac{\partial \tilde{h}}{\partial t} + u(t) \frac{\partial \tilde{h}}{\partial \tilde{z}} + \frac{\partial}{\partial \tilde{z}} \left[\frac{\tilde{h}^3}{3} \left(\tilde{h}_{\tilde{z}\tilde{z}\tilde{z}} - \epsilon \right) \right] = 0. \tag{5}$$

Examining the leading order (in ϵ) problem, we find that

$$u(t) \frac{\partial \tilde{h}}{\partial \tilde{z}} + \frac{\partial}{\partial \tilde{z}} \left(\frac{\tilde{h}^3}{3} \tilde{h}_{\tilde{z}\tilde{z}\tilde{z}} \right) = 0,$$

which integrates to give

$$u(t)\tilde{h} + \frac{\tilde{h}^3}{3} \tilde{h}_{\tilde{z}\tilde{z}\tilde{z}} = f(t).$$

As $\tilde{z} \to \infty$, we expect that $\tilde{h} \to \tilde{h}_0$, a constant and so

$$u(t)\tilde{h} + \frac{\tilde{h}^3}{3} \tilde{h}_{\tilde{z}\tilde{z}\tilde{z}} = u(t)\tilde{h}_0 \tag{6}$$

subject to $\tilde{h} \sim \tilde{z}^2/\sqrt{2}$ as $\tilde{z} \to -\infty$.

Following the lecture notes, we let $\tilde{h} = \tilde{h}_0 g(\zeta)$, $\tilde{z} = z_* \zeta$ so that (6) becomes

$$g + \frac{\tilde{h}_0^3}{u(t) z_*^3} \tfrac{1}{3} g^3 g_{\zeta\zeta\zeta} = 1.$$

We choose $z_* = \tilde{h}_0 / u(t)^{1/3}$ so that we obtain the Landau–Levich equation

$$g + \tfrac{1}{3} g^3 g_{\zeta\zeta\zeta} = 1$$

with boundary conditions $g \to 1$ as $\zeta \to \infty$ and

$$g \sim \frac{\tilde{h}_0}{u(t)^{2/3}} \frac{\zeta^2}{\sqrt{2}}$$

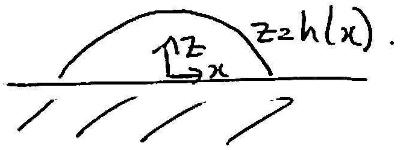
as $\zeta \to -\infty$.

From the notes, we recall that the numerical solution of the Landau–Levich equation has $g \sim 0.67\zeta^2$ as $\zeta \to -\infty$ and so we have

$$\frac{\tilde{h}_0}{\sqrt{2} u(t)^{2/3}} = 0.67$$

and so $\tilde{h}_0 \approx 0.95 u(t)^{2/3}$, as required.

Q3



The kinematic boundary condition is:

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = w - E \qquad (1)$$

In deriving lubrication flows for droplets used length R timescale R/U and velocity scale U.

Using these scales here, we find:

$$\frac{\partial \tilde{h}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{h}}{\partial \tilde{x}} = \tilde{w} - E/U$$

But U is thus far arbitrary, so we choose $U = E$

$$\Rightarrow \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = w - 1 \quad \text{on } z=h(x).$$

(This is consistent with scaling used for lubrication theory but have $Ca = \frac{\mu E}{\gamma}$ specified).

In the fluid we have the dimensionless lubrication equations:

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial x}$$

$$0 = \frac{\partial p}{\partial z} \Rightarrow p = f(x)$$

$$= -\frac{1}{Ca}\frac{\partial^2 h}{\partial x^2} \quad \text{on } z=h(x)$$

$$\Rightarrow p = -\frac{1}{Ca}\frac{\partial^2 h}{\partial x^2} \quad \text{everywhere.}$$

$$\therefore u = \frac{1}{2}\frac{\partial p}{\partial x} z(z-2h) \qquad \left[\because \begin{array}{l} u=0 \text{ on } z=0, \\ u_z=0 \text{ on } z=h \end{array}\right]$$

$$\Rightarrow h\bar{u} = \frac{1}{2}\frac{\partial p}{\partial x}\left[\frac{h^3}{3} - h^3\right] = -\frac{h^3}{3}\frac{\partial p}{\partial x} = \frac{h^3}{3Ca}\frac{\partial^3 h}{\partial x^3}.$$

Now:

$$\frac{\partial}{\partial x}\int_0^h u\,dz = h_x u(h) + \int_0^h \left(\frac{\partial u}{\partial x} = -\frac{\partial w}{\partial z}\right) dz$$

$$= h_x u(h) - \left[w(h) = 1 + \frac{\partial h}{\partial t} + h_x u(h)\right]$$

$$= -1 - \frac{\partial h}{\partial t}.$$

$$\therefore \quad \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(\bar{u}h) = -1$$

i.e. $$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\frac{h^3}{3Ca}\frac{\partial^3 h}{\partial x^3}\right) = -1. \quad (*).$$

$Ca \ll 1 \Rightarrow$ @ leading order:

$$\frac{\partial}{\partial x}\left(\frac{h_0^3}{3}\frac{\partial^3 h_0}{\partial x^3}\right) = 0 \Rightarrow \frac{\partial^3 h_0}{\partial x^3} = 0$$

$[h_0^3 h_{0,xxx} = \text{cst} = 0$ as $h_0 = 0$ at $x = \pm s]$

$$\Rightarrow h_0 = k(t)(s^2 - x^2)$$

To ensure $A(t) = \int_{-s}^{s} h_0\,dx = k(t) \cdot \frac{4s^3}{3}$

$$\Rightarrow k(t) = \frac{3}{4s^3} A(t).$$

Integrating the film eqn $(*)$ we have:

$$\frac{d}{dt}\int_{-s}^{s} h\,dx = -\int_{-s}^{s} 1\,dx = -2s$$

So, at leading order:

$$-2s = \frac{d}{dt}\int_{-s}^{s} h_0\,dx = \frac{d}{dt}A \Rightarrow \dot{A} = -2s,$$ as desired.

Let $h = h_0 + Ca\, h_1 + Ca^2 h_2 + \cdots$

and then calculate:

$$Ca\,\bar{u} = \frac{h^2}{3} h_{3x} = \frac{h_0^2 h_{0,3x}}{3} \overset{\nearrow 0}{} + Ca\left(\frac{h_0^2 h_{1,3x}}{3} + \frac{2h_0 h_1 h_{0,3x}}{3}\right) + \cdots$$

$$= \bar{u}_0 \overset{\nearrow 0}{} + Ca\,\bar{u}_1 + \cdots$$

Sub this into evolution equation (*):

$$-1 = \frac{\partial h_0}{\partial t} + \frac{Ca}{Ca}\frac{\partial}{\partial x}\left(h_0 \bar{u}_1 + h_1 \bar{u}_0\right)$$

$h_1 \bar{u}_0 \to 0$ since $\bar{u}_0 = 0$

$$\Rightarrow \quad h_0 \bar{u}_1 = -\int_0^x 1 + \frac{\partial h_0}{\partial t}\, dx' \qquad \left[\because \bar{u}_1(x=0) = 0 \text{ by symmetry}\right].$$

$$= -x - \int_0^x -\frac{3}{2s^2}(s^2 - x^2)\, dx'$$

$$= -x + \frac{3}{2s^2}\left(s^2 x - \frac{x^3}{3}\right)$$

$$= \frac{x}{2s^2}\left(s^2 - x^2\right).$$

$$\Rightarrow \bar{u}_1 = \frac{\frac{x}{2s^2}(s^2 - x^2)}{\frac{3A(t)}{4s^3}(s^2 - x^2)}$$

$$\bar{u} = \bar{u}_1 + Ca\,\bar{u}_2 + \cdots$$

$$= \frac{2s}{3}\frac{x}{A(t)} + O(Ca) \qquad \text{as desired.}$$

If gravity is present then we have:

$$\frac{1}{3} h^2 \left(h_{xxx} - B_0 h_x \right) = \bar{u}_0 + Ca\, \bar{u}_1 + \cdots$$

The analysis above will be unchanged provided that

$$h_{xxx} - B_0 h_x \simeq h_{xxx}$$

Now if we choose $L = S$ so that gradients of h_0 etc are all $O(1)$ then we require $B_0 \ll 1$

and hence: $\frac{\rho g}{\gamma} S^2 \ll 1$,

4. We have the usual lubrication equation for the horizontal velocity, u, i.e.

$$\mu\frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial x}, \tag{7}$$

while at leading order the vertical component gives

$$\frac{\partial p}{\partial z} = -\rho g.$$

Integrating the latter and taking the pressure in the atmosphere to be 0, we have

$$p = \rho g\left[h(x,t) - z\right] - \gamma\frac{\partial^2 h}{\partial x^2}. \tag{8}$$

Integrating (7) we have that

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}z^2 + Az + B$$

where A and B are constants to be determined from the boundary conditions of the problem:

- The condition of zero shear stress at the free surface gives

$$\left.\frac{\partial u}{\partial z}\right|_{z=h} = 0 \qquad \Longrightarrow \; A = -\frac{1}{\mu}\frac{\partial p}{\partial x}h.$$

- The slip condition $u = \lambda u_z$ at $z = 0$ gives

$$B = \lambda A = -\frac{\lambda}{\mu}\frac{\partial p}{\partial x}h$$

  and hence

$$u = -\frac{1}{\mu}\frac{\partial p}{\partial x}\left[\tfrac{1}{2}z^2 - zh - \lambda h\right].$$

  Now

$$\begin{aligned}\bar{u} = \frac{1}{h}\int_0^h u \; \mathrm{d}z &= \frac{1}{\mu h}\frac{\partial p}{\partial x}\left[\tfrac{1}{6}h^3 - \tfrac{1}{2}h^3 - \lambda h^2\right].\\ &= -\frac{1}{\mu}\frac{\partial p}{\partial x}\left[\tfrac{1}{3}h^2 + \lambda h\right].\end{aligned} \tag{9}$$

  From (8) we have that

$$\frac{\partial p}{\partial x} = \rho g\frac{\partial h}{\partial x} - \gamma\frac{\partial^3 h}{\partial x^3}$$

  and so

$$\bar{u} = \frac{\gamma}{\mu}\left(\frac{\partial^3 h}{\partial x^3} - \ell_c^{-2}\frac{\partial h}{\partial x}\right)\left(\tfrac{1}{3}h^2 + \lambda h\right),$$

where $\ell_c^2 = \gamma/\rho g$, as usual.

The associated thin film equation is eqn (2.27) from the notes and so we have

$$0 = \frac{\partial h}{\partial t} + \frac{\gamma}{\mu}\frac{\partial}{\partial x}\left[\left(\frac{\partial^3 h}{\partial x^3} - \ell_c^{-2}\frac{\partial h}{\partial x}\right)\left(\tfrac{1}{3}h^3 + \lambda h^2\right)\right]$$

[*It is worth emphasizing again that the general statement of conservation of mass (having used the kinematic boundary condition) is of the form*

$$\frac{\partial h}{\partial t} + \nabla \cdot \mathbf{q} = 0,$$

since this comes up frequently in such problems.]

5. We return to eqn (5) of Q2 above:

$$\epsilon\frac{\partial \tilde{h}}{\partial t} + u(t)\frac{\partial \tilde{h}}{\partial \tilde{z}} + \frac{\partial}{\partial \tilde{z}}\left[\frac{\tilde{h}^3}{3}\left(\tilde{h}_{\tilde{z}\tilde{z}\tilde{z}} - \epsilon\right)\right] = 0.$$

We wish to understand the possible dominant balances between the four terms in this equation.

We anticipate that in the far field, above the turnaround region, the many derivatives in the third term will decay since we expect the film to tend to a constant thickness.

We rescale by letting $y = \gamma\tilde{z}$ with $\gamma \ll 1$ yet to be determined (using y just to avoid introducing another z variable). We find that the governing pde then becomes

$$\epsilon\frac{\partial \tilde{h}}{\partial t} + u(t)\gamma\frac{\partial \tilde{h}}{\partial y} + \frac{\partial}{\partial y}\left[\frac{\tilde{h}^3}{3}\left(\gamma^4\tilde{h}_{yyy} - \epsilon\gamma\right)\right] = 0.$$

Clearly the fourth term is $O(\epsilon\gamma)$ and so is sub-dominant to the first two terms, which are $O(\epsilon)$ and $O(\gamma)$, respectively. We also have that the third term is sub-dominant to the second, since we chose $\gamma \ll 1$. The only remaining terms are the first two and so, to make these balance, we choose $\gamma = \epsilon$ leading to

$$\frac{\partial \tilde{h}}{\partial t} + u(t)\frac{\partial \tilde{h}}{\partial y} = 0, \tag{10}$$

as required; note that $y = z - z_0$.

Matching this solution back into the turnaround region (from above), we expect that $\tilde{h} = 0.948u(t)^{2/3}$ as $y \to 0$. We therefore have that

$$\tilde{h}(0,t) = 0.948u(t)^{2/3}.$$

We now turn to the solution of (10). We have that

$$e^t\frac{\partial \tilde{h}}{\partial t} + \frac{\partial \tilde{h}}{\partial z} = 0,$$

and would like to write this as

$$0 = \frac{\mathrm{d}\tilde{h}}{\mathrm{d}s} = \frac{\partial \tilde{h}}{\partial t}\frac{\partial t}{\partial s} + \frac{\partial \tilde{h}}{\partial z}\frac{\partial z}{\partial s},$$

using the chain rule for some $s(z,t)$. Comparing coefficients of $\partial\tilde{h}/\partial t$ and $\partial\tilde{h}/\partial z$ we have that $\partial t/\partial s = e^t$, $\partial z/\partial s = 1$, i.e. $s = -e^{-t} + f(z) = z + g(t)$. We find therefore that on curves of constant $\tilde{h}$ (since $\mathrm{d}\tilde{h}/\partial s = 0$) we must have $z + e^{-t} = a$ for some constant a. In turn, we conclude that $\tilde{h} = G(a)$, i.e.

$$\tilde{h}(z,t) = G(z + e^{-t}),$$

for some function $G(a)$ to be determined.

The boundary condition $\tilde{h}(0,t) = 0.948\exp(-2t/3)$ for $t > 0$ gives that

$$G(e^{-t}) = 0.948\exp(-2t/3).$$

In general, therefore, we have

$$\tilde{h}(z,t) = 0.948(z + e^{-t})^{2/3}, \quad 0 < z + e^{-t} < 1,$$

and hence, as $t \to \infty$,

$$\tilde{h}(z,t) \to 0.948z^{2/3}, \quad 0 < z < 1,$$

as required.