Problem Sheet 1: Solutions

1. Taking the suggested dot product and using the summation convention we have that

$$
\begin{aligned}
k_i \int_C (\mathbf{f}\wedge\mathbf{t})_i \, \mathrm{d}s = \int_C k_i(\mathbf{f}\wedge\mathbf{t})_i \, \mathrm{d}s &= -\int_C (\mathbf{f}\wedge\mathbf{k})_i t_i && \text{[Scalar triple product]}\\
&= -\int_S n_i\big[\nabla\wedge(\mathbf{f}\wedge\mathbf{k})\big]_i && \text{[Stokes' Theorem]}\\
&= -\int_S n_i\varepsilon_{ijk}\partial_j[\varepsilon_{klm}f_l k_m] \, \mathrm{d}S\\
&= -\int_S n_i k_m(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\partial_j f_l \, \mathrm{d}S && \text{[Using the hint and } \mathbf{k} \text{ constant]}\\
&= -\int_S n_i(k_j\partial_j f_i - k_i\partial_j f_j) \, \mathrm{d}S\\
&= k_i\int_S n_i\partial_j f_j \, \mathrm{d}S - k_i\int_S n_j\partial_i f_j \, \mathrm{d}S && \text{[letting } i\leftrightarrow j\text{].}
\end{aligned}
$$

Since $\mathbf{k}$ is an arbitrary vector we must, in fact, have the desired result.

Now, letting $\mathbf{f}=\gamma\mathbf{n}$ we have

$$
\begin{aligned}
-\left[\gamma\int_C \boldsymbol{\nu}\, \mathrm{d}s\right]_i = \left[\int_C \gamma(\mathbf{n}\wedge\mathbf{t}) \, \mathrm{d}s\right]_i = \left[\int_C \mathbf{f}\wedge\mathbf{t} \, \mathrm{d}S\right]_i\\
&= \int_S \left[n_i\nabla\cdot(\gamma\mathbf{n}) - n_j\partial_i(\gamma n_j)\right] \, \mathrm{d}S && \text{[Using the earlier result]}\\
&= \int_S \{n_i\left[\gamma\partial_j n_j + (n_j\partial_j\gamma = 0)\right] && \text{[Since } \gamma \text{ is constant normal to } S\text{]}\\
&- n_j(n_j\partial_i\gamma + \gamma\partial_i n_j)\} \, \mathrm{d}S\\
&= \int_S \left[n_i\gamma(\nabla\cdot\mathbf{n}) - \partial_i\gamma - \gamma\partial_i(n_j n_j/2)\right] \, \mathrm{d}S\\
&= \int_S \left[n_i\gamma(\nabla\cdot\mathbf{n}) - \partial_i\gamma\right] \, \mathrm{d}S && \text{[Since } n_j n_j = \mathbf{n}\cdot\mathbf{n} = 1\text{]}\\
&= \int_S \left[\mathbf{n}\gamma(\nabla\cdot\mathbf{n}) - \nabla\gamma\right]_i \, \mathrm{d}S,
\end{aligned}
$$

as required.

2. We are given that

$$\ell_c^2 \frac{h_{xx}}{(1+h_x^2)^{3/2}} = h,$$

which we multiply by h_x and integrate once to obtain

$$A\ell_c^2 - \ell_c^2(1+h_x^2)^{-1/2} = \tfrac{1}{2}h^2,$$

for some constant of integration A.

Now, as $x \to +\infty$, $h \to 0$, $h_x \to 0$ so that $A = 1$ and

$$\tfrac{1}{2}h^2 = \ell_c^2\left[1 - (1+h_x^2)^{-1/2}\right]. \tag{1}$$

At $x = 0$, $h_x = -\cot\theta$ so that

$$\tfrac{1}{2}h_0^2 = \ell_c^2\left[1 - (1+\cot^2\theta)^{-1/2}\right] = \ell_c^2(1-\sin\theta),$$

i.e.

$$h_0 = \pm\ell_c\left[2(1-\sin\theta)\right]^{1/2},$$

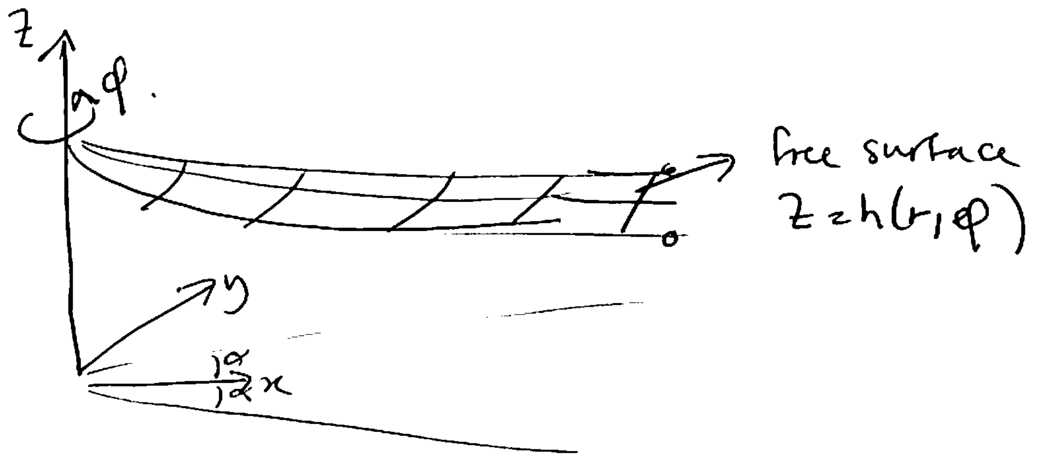
as desired. Based on simple geometry, we expect to take the positive square root if $\theta < \pi/2$ and the negative root if $\theta > \pi/2$, though of course this depends on the chosen sign convention for h!

The area of displaced liquid is given by

$$\begin{aligned} A &= \int_0^\infty h \, \mathrm{d}x = \int_{-\cot\theta}^{0} \frac{\ell_c^2}{(1+h_x^2)^{3/2}} \, \mathrm{d}(h_x) && \text{[Using the Laplace–Young equation]} \\ &= \ell_c^2 \int_{-\theta}^{-\pi/2} \frac{-1/\sin^2 u}{1/|\sin u|^3} \, \mathrm{d}u && \text{[Letting } h_x = \cot u\text{]} \\ &= \ell_c^2 \cos\theta. \end{aligned}$$

The weight of liquid displaced is $\rho g A = \gamma\cos\theta$. The vertical force provided by surface tension acting at the contact line is also $\gamma\cos\theta$. This result is thus a special case of the generalized Archimedes' principle discussed in lectures.

3. A schematic sketch is shown below.



We have the general, coordinate independent, form of the Laplace–Young equation

$$\rho g h = \gamma \kappa = -\gamma \nabla \cdot \mathbf{n}.$$

The equation of the free surface is $0 = z - h(r, \phi)$ and so the unit normal vector is

$$\mathbf{n} = \frac{(-h_r, -h_\phi/r, 1)}{\left(1 + h_r^2 + h_\phi^2/r^2\right)^{1/2}}.$$

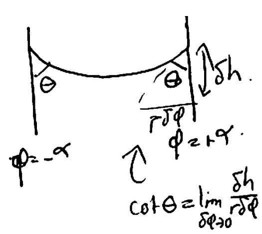
For small deflections (linearising) we have $\mathbf{n} \approx (-h_r, -h_\phi/r, 1)$ so that

$$\nabla \cdot \mathbf{n} = -\frac{1}{r}\frac{\partial}{\partial r}\left(r h_r\right) - \frac{1}{r^2}\frac{\partial^2 h}{\partial \phi^2} = -\nabla^2 h$$

and we immediately have that the Laplace–Young equation takes the form

$$h = \ell_c^2 \nabla^2 h$$

where $\ell_c = (\gamma/\rho g)^{1/2}$ is the capillary length, as usual.



To obtain the correct contact angle condition we take a cross-section through the interface, at constant r, say and look along the $-\mathbf{i}$ direction (back towards the origin). We see immediately from the figure above that

$$\left.\frac{1}{r}\frac{\partial h}{\partial \phi}\right|_{\phi = \pm\alpha} = \pm \cot\theta.$$

In our derivation, we have assumed that the the meniscus slope is small, in particular, $|\nabla h|^2 \ll 1$. Given the above boundary condition and this requirement, we also need to ensure that $\cot\theta \ll 1$ and hence $|\theta - \frac{\pi}{2}| \ll 1$, as required.

Uniqueness We will first show that, if we can find a solution, it must be unique.

As usual, we proceed by contradiction assuming that there are two distinct solutions of the problem, $h_1 \neq h_2$ both satisfying the Laplace–Young equation and the relevant boundary conditions (at $\theta = \pm\alpha$ or at $\theta = \pm\pi/4$ — it doesn't matter which). Letting $w = h_2 - h_1$ it is obvious that

$$\nabla^2 w = w/\ell_c^2 \tag{2}$$

and

$$\frac{1}{r}\frac{\partial w}{\partial \phi}\bigg|_{\phi=\pm\alpha} = 0 \tag{3}$$

and, finally, that $w \to 0$ far from the walls (i.e. as $r \to \infty$ with $\phi \neq \pm\alpha$).

Letting S be the projection of the interface onto the (x, y) plane, which is bounded by the curve C, we consider the integral

$$\begin{aligned}
\int_S w\nabla^2 w \,\mathrm{d}S &= \int_S w^2/\ell_c^2 \geq 0 && \text{[Using (2)]}\\
&= \int_S \left[\nabla\cdot(w\nabla w) - (\nabla w)^2\right] \,\mathrm{d}S\\
&= \int_C (\mathbf{n}\cdot\nabla w = 0) \,\mathrm{d}s - \int_S (\nabla w)^2 \,\mathrm{d}S && \text{[Using the boundary condition (3)]}\\
&= -\int_S (\nabla w)^2 \,\mathrm{d}S \leq 0,
\end{aligned} \tag{4}$$

which is the contradiction we sought. Hence the solution must be unique.

Finding a solution To find a solution, it is enough to check that the solution given satisfies the Laplace–Young equation and the boundary conditions.

Another approach is to introduce rotated coordinates (X, Y) so that the $90°$ wedge coincides with the X and Y axes, i.e. we let

$$X = \frac{x - y}{\sqrt{2}}, \quad Y = \frac{x + y}{\sqrt{2}}.$$

Then the Laplace–Young equation for the interface shape $H(X, Y)$ becomes

$$H_{XX} + H_{YY} = H/\ell_c^2$$

with boundary conditions

$$H_X(X=0) = -\cot\theta, \quad H_Y(Y=0) = -\cot\theta. \tag{5}$$

and decay conditions far away from the wall.

Searching for separable solutions of the form $H(X,Y) = \xi(X)\eta(Y)$ we find that

$$\frac{\xi''}{\xi} + \frac{\eta''}{\eta} = 1/\ell_c^2$$

which gives solutions of the form

$$H = A\exp(-\alpha X/\ell_c)\exp(-\beta Y/\ell_c)$$

where $1 = \alpha^2 + \beta^2$. Applying the boundary conditions (5) we find that we must combine two solutions of this form: one with $\alpha = 1, \beta = 0$ and the other with $\alpha = 0, \beta = 1$. We therefore have

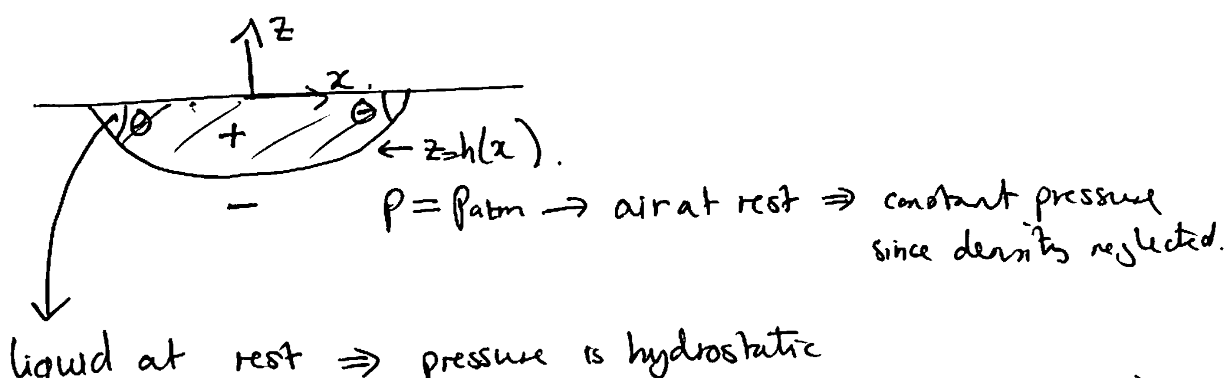
$$H = \ell_c \cot\theta \left\{ e^{-X/\ell_c} + e^{-Y/\ell_c} \right\}.$$

Noting that $X = (x-y)/\sqrt{2} = r\sin(\pi/4 - \phi)$ and that $Y = (x+y)/\sqrt{2} = r\sin(\phi + \pi/4)$ we therefore have that

$$h(r,\phi) = \ell_c \cot\theta \left\{ e^{-r\sin(\pi/4-\phi)/\ell_c} + e^{-r\sin(\phi+\pi/4)/\ell_c} \right\},$$

as required.

4. The scenario is as shown in the figure below.



Since the liquid is static, the pressure within it is hydrostatic, i.e.

$$p = p_0 - \rho g z$$

where the z coordinate is measured vertically upwards and p_0 is some reference pressure ($p_0 \neq p_{\text{atm}}$).

Because of surface tension, there is a pressure jump across the interface:

$$
\begin{aligned}
(p_+ - p_-)|_{z=h(x)} &= \gamma\kappa \approx \gamma h_{xx} \\
&= p_0 - \rho g h(x) - p_{\text{atm}} \qquad \text{[Using hydrostatic pressure in the liquid]}
\end{aligned}
$$

from which we immediately have

$$
h + \ell_c^2 h_{xx} = \frac{p_0 - p_{\text{atm}}}{\rho g} \tag{6}
$$

with $\ell_c^2 = \gamma/\rho g$ as usual. Note that (6) is slightly different from the usual Laplace–Young equation in that it has a source term on the RHS *and* the solutions of the homogeneous problems are oscillatory rather than the usual exponential decay.

Differentiating (6) with respect to x we obtain the required third order ODE; solving either this ODE or (6) we have solutions of the form

$$
h(x) = A + B\sin x/\ell_c + C\cos x/\ell_c.
$$

The coefficients A, B, C are to be determined from the boundary conditions

$$
\begin{aligned}
h(\pm x_0) &= 0 \\
h_x(\pm x_0) &= \pm\tan\theta \approx \pm\theta \qquad \text{[Since } \theta \ll 1\text{]}
\end{aligned}
$$

From the first boundary condition, we have

$$
A - B\sin x_0/\ell_c + C\cos x_0/\ell_c = A + B\sin x_0/\ell_c + C\cos x_0/\ell_c = 0
$$

from which either $B = 0$ or $x_0/\ell_c = n\pi$.

From the second boundary condition, we have

$$
\pm\ell_c\theta = B\cos x_0/\ell_c \mp C\sin x_0/\ell_c
$$

from which either $B = 0$ or $x_0/\ell_c = (n + 1/2)\pi$.

For consistency between the two sets of boundary conditions, we must have $B = 0$ (i.e. the drop is symmetric) and we immediately find that

$$
h(x) = \theta\ell_c\left[\cot x_0/\ell_c - \frac{\cos x/\ell_c}{\sin x_0/\ell_c}\right]
$$

For this solution, $|h_x| = \theta|\sin x/\ell_c|/\sin x_0/\ell_c \leq \theta/\sin x_0/\ell_c$ so the small slope approximation is valid provided that $\theta \ll \sin x_0/\ell_c$.

The area of the drop is

$$
A = \int_{-x_0}^{x_0} -h\ \mathrm{d}x = 2\theta\ell_c^2\left[1 - \frac{x_0}{\ell_c}\cot x_0/\ell_c\right].
$$

As $x_0/\ell_c \to \pi$, $A \to \infty$. This suggests that infinitely large droplets can be supported beneath a horizontal plate. Intuitively, we expect that droplets should fall off the plate if they become too large. The problem with our linearised analysis is that as $x_0/\ell_c \to \pi$ there are no values of θ for which our linearised analysis is self-consistent — the small-slope approximation breaks down in this limit.

5 We are given that

$$h_t + \left[\frac{h^3}{3\text{Ca}}\left(h_{xxx} - \text{Bo }h_x\right)\right]_x = 0.$$

Letting $h(x,t) = h_0 + \delta h_1(x,t)$ with h_0 constant we find that

$$0 = \delta\frac{\partial h_1}{\partial t} + \left[\frac{h_0^3 + 3\delta h_0^2 h_1}{3\text{Ca}}\left(\delta h_{1,xxx} - \delta\text{ Bo }h_{1,x}\right)\right]_x + O(\delta^3).$$

Examining the $O(\delta)$ terms we immediately see that

$$0 = \frac{\partial h_1}{\partial t} + \frac{h_0^3}{3\text{Ca}}\left(h_{1,xxxx} - \text{Bo }h_{1,xx}\right),$$

as required.

Letting $h_1 = \Re\left[e^{\sigma t + ikx}\right]$ we have

$$\sigma = -\frac{h_0^3}{3\text{Ca}}\left[(ik)^4 - \text{Bo}(ik)^2\right] = -\frac{h_0^3}{3\text{Ca}}\left(k^4 + \text{Bo }k^2\right),$$

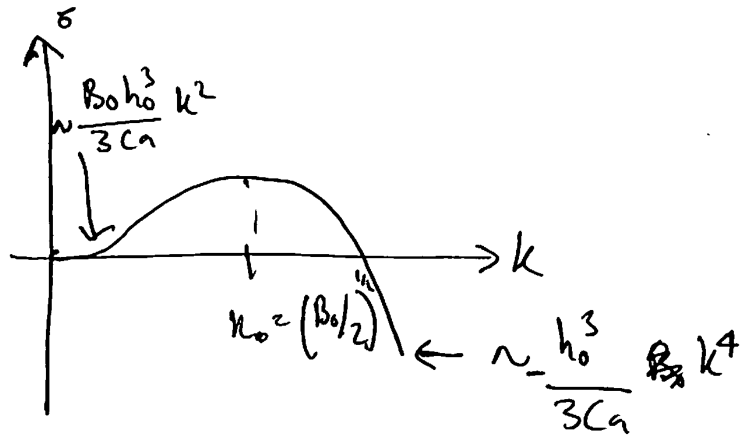
as required. Clearly $\sigma < 0$ for all $k \neq 0$; disturbances therefore decays with time and the film is stable to perturbations of any wavelength.

For $k = 0$, $\sigma = 0$ and so a uniform perturbation is neutrally stable — it neither grows nor decays.

When the film is beneath (rather than above) the plate, gravity acts in the opposite direction, $\text{Bo} \to -\text{Bo}$ and so we have

$$\sigma = -\frac{h_0^3}{3\text{Ca}}\left(k^4 - \text{Bo }k^2\right),$$

A sketch is shown below:



We see that the surface tension term $(-k^4)$ is stabilising since it makes σ more negative, while gravity $(+\text{Bo } k^2)$ is destabilising since it acts to increase σ.

The situation is unstable whenever $\sigma > 0$, i.e. for

$$0 < k < \text{Bo}^{1/2}.$$

The maximally unstable mode is that for which σ is maximised. We have

$$\sigma'(k) = -\frac{h_0^3}{3\text{Ca}}(4k^3 - 2\text{Bo } k)$$

and so the maximally unstable wavelength is $\lambda = 2\pi/k_*$ where $k_* = (\text{Bo}/2)^{1/2}$.