BO1.1 History of Mathematics
Lecture IV
The beginnings of calculus, part 2: quadrature

MT23 Week 2

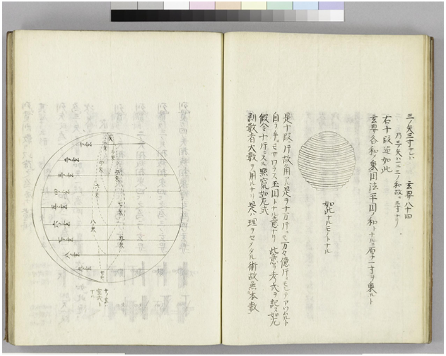
Summary

- *Enri*: a non-Western prelude
- Quadrature (finding areas)
- Indivisibles
- Infinitesimals
- The contributions of Newton & Leibniz

Seki and *enri*

The development of calculus is a largely Western story, but similar ideas did appear elsewhere

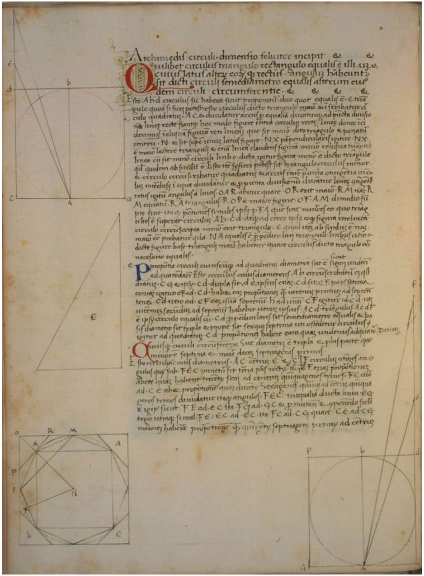
For example, in the late 17th century, Seki Takakazu and his school developed *enri* 円理 ('circle principles'), which concerned the calculation of arc lengths, areas, and volumes



One result of *enri* was the determination of the volume of a sphere via 'slicing'

But *enri* was much narrower in scope than calculus

Archimedes: *Κύκλου μέτρησις* (*Measurement of a circle*)

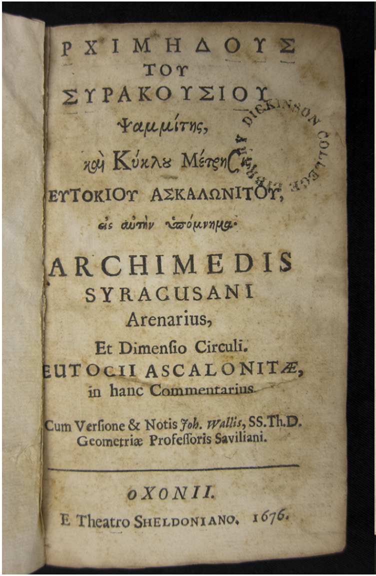

Archimedis circuli dimensio feliciter incipit

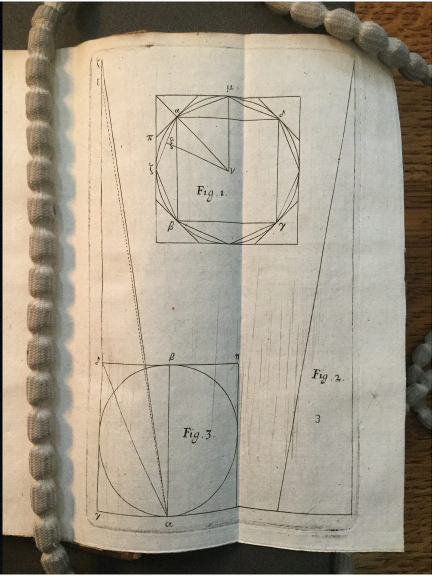
Translated into Latin as *Dimensio circoli* by Jacobus Cremonensis, c. 1450–1460

Illustrated by Piero della Francesca

Available online with other texts by Archimedes

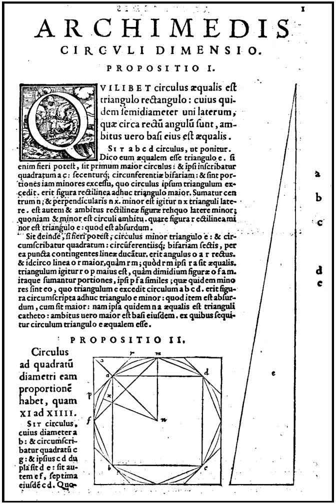
Archimedes: *Κύκλου μέτρησις* (*Measurement of a circle*)





Edition by John Wallis, Oxford, 1676

Archimedes: *Κύκλου μέτρησις* (*Measurement of a circle*)



I

ARCHIMEDIS

CIRCVLI DIMENSIO.

PROPOSITIO I.

QVILIBET circulus æqualis eſt triangulo rectangulo: cuius quidem ſemidiameter uni laterum, quæ circa rectū angulū ſunt, ambitus uero baſi eius eſt æqualis.

SIT a b c d circulus, ut ponitur. Dico eum æqualem eſſe triangulo e. ſi enim fieri poteſt, ſit primum maior circulus: & ipſi inſcribatur quadratum a c: ſeceturq; circunferentiæ bifariam: & ſint portiones iam minores exceſſu, quo circulus ipſum triangulum excedit. erit figura rectilinea adhuc triangulo maior. Sumatur centrum n; & perpendicularis n x. minor eſt igitur n x trianguli latere. eſt autem & ambitus rectilineæ figuræ reliquo latere minor; quoniam & minor eſt circuli ambitu. quare figura rectilinea minor eſt triangulo e: quod eſt abſurdum.

Sit deinde, ſi fieri poteſt, circulus minor triangulo e: & circumſcribatur quadratum: circūferentiisq; bifariam ſectis, per ea puncta contingentes lineæ ducātur. erit angulus o a r rectus. & idcirco linea o r maior, quàm r m; quòd r m ipſi r a ſit æqualis. triangulum igitur r o p maius eſt, quàm dimidium figuræ o f a m. itaque ſumantur portiones, ipſi p f a ſimiles; quæ quidem minores ſint eo, quo triangulum e excedit circulum a b c d. erit figura circumſcripta adhuc triangulo e minor: quod item eſt abſurdum, cum ſit maior: nam ipſa quidem n a æqualis eſt trianguli catheto: ambitus uero maior eſt baſi eiuſdem. ex quibus ſequitur circulum triangulo e æqualem eſſe.

PROPOSITIO II.

Circulus ad quadratū diametri eam proportionē habet, quam XI ad XIIII.

SIT circulus, cuius diameter a b: & circumſcribatur quadratū c g: & ipſius c d dupla ſit d e: ſit autem e f, ſeptima eiuſdē c d. Quo-

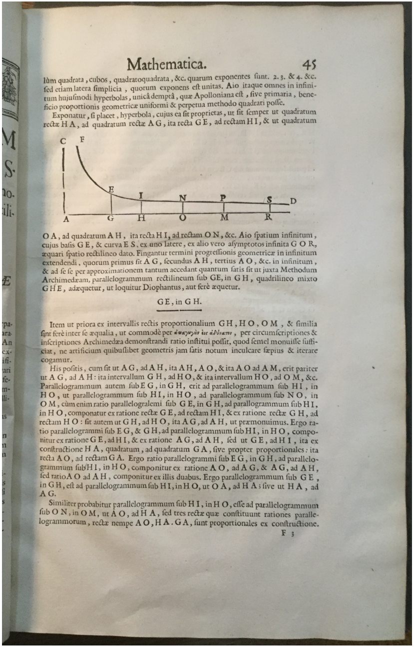
(*Archimedis opera*, edited by Commandino, 1558) — see *Mathematics emerging*, §1.2.3

A circle is equal to a right-angled triangle with height equal to the circumference of the circle and base equal to the radius.

Proof by exhaustion and double contradiction

Later: the ratio of the circumference to the diameter is greater than $3\frac{10}{71}$ and less than $3\frac{1}{7}$.

Fermat's quadrature of a hyperbola (c. 1636)



Mathematica. 45

lùm quadrata, cubos, quadratoquadrata, &c. quarum exponentes ſunt. 2. 3. & 4. &c. ſed etiam latera ſimplicia, quorum exponens eſt unitas. Aio itaque omnes in infinitum hujuſmodi hyperbolas, unicâ demptâ, quæ Apolloniana eſt, ſive primaria, beneficio proportionis geometricæ uniformi & perpetua methodo quadrari poſſe.

Exponatur, ſi placet, hyperbola, cujus ea ſit proprietas, ut ſit ſemper ut quadratum rectæ H A, ad quadratum rectæ A G, ita recta G E, ad rectam H I, & ut quadratum

C F E I N P S D A G H O M R

O A, ad quadratum A H, ita recta H I, ad rectam O N, &c. Aio ſpatium infinitum, cujus baſis G E, & curva E S, ex uno latere, ex alio vero aſymptotos infinita G O R, æquari ſpatio rectilineo dato. Fingantur termini progreſſionis geometricæ in infinitum extendendi, quorum primus ſit A G, ſecundus A H, tertius A O, &c. in infinitum, & ad ſe ſe per approximationem tantum accedant quantum ſatis ſit ut juxta Methodum Archimedæam, parallelogrammum rectilineum ſub GE, in G H, quadrilineo mixto G H E, adæquetur, ut loquitur Diophantus, aut ferè æquetur.

G E, in G H.

Item ut priora ex intervallis rectis proportionalium G H, H O, O M, & ſimilia ſint ferè inter ſe æqualia, ut commodè per ἀπαγωγὴν εἰς ἀδύνατον, per circumſcriptiones & inſcriptiones Archimedæa demonſtrandi ratio inſtitui poſſit, quod ſemel monuiſſe ſufficiat, ne artificium quibuſlibet geometris jam ſatis notum inculcare ſæpius & iterare cogamur.

His poſitis, cum ſit ut A G, ad A H, ita A H, A O, & ita A O ad A M, erit pariter ut A G, ad A H: ita intervallum G H, ad H O, & ita intervallum H O, ad O M, &c. Parallelogrammum autem ſub E G, in G H, erit ad parallelogrammum ſub H I, in H O, ut parallelogrammum ſub H I, in H O, ad parallelogrammum ſub N O, in O M, cùm enim ratio parallelogralemi ſub G E, in G H, ad parallogrammum ſub H I, in H O, componatur ex ratione rectæ G E, ad rectam H I, & ex ratione rectæ G H, ad rectam H O: ſit autem ut G H, ad H O, ita A G, ad A H, ut præmonuimus. Ergo ratio parallelogrammi ſub E G, & G H, ad parallelogrammum ſub H I, in H O, componitur ex ratione G E, ad H I, & ex ratione A G, ad A H, ſed ut G E, ad H I, ita ex conſtructione H A, quadratum, ad quadratum G A, ſive propter proportionales: ita recta A O, ad rectam G A. Ergo ratio parallelogrammi ſub E G, in G H, ad parallelogrammum ſub H I, in H O, componitur ex ratione A O, ad A G, & A G, ad A H, ſed ratio A O ad A H, componitur ex illis duabus. Ergo parallelogrammum ſub G E, in G H, eſt ad parallelogrammum ſub H I, in H O, ut O A, ad H A; ſive ut H A, ad A G.

Similiter probabitur parallelogrammum ſub H I, in H O, eſſe ad parallelogrammum ſub O N, in O M, ut A O, ad H A, ſed tres rectæ quæ conſtituunt rationes parallelogrammorum, rectæ nempe A O, H A. G A, ſunt proportionales ex conſtructione.

F 3

Worked out c. 1636, but only published posthumously in *Varia opera mathematica*, 1679.

In modern terms, this is the curve described by $y = \dfrac{1}{x^2}$.

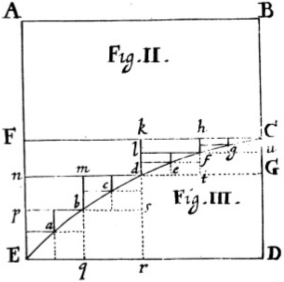
See *Mathematics emerging*, §3.2.1.

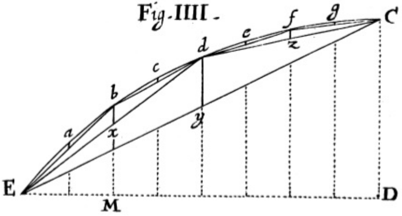
The rectangular (or 'Apollonian') hyperbola

In modern notation, $y = \dfrac{1}{x}$

- Quadrature evaded Fermat
- Partial results obtained by Grégoire de Saint Vincent, c. 1625, published in *Opus geometricum*, 1647
- Empirical observation that if $A(x)$ is the area under the hyperbola from 1 to x, then $A(\alpha\beta) = A(\alpha) + A(\beta)$ (cf. logarithms)
- Problem solved in early 1650s by William Brouncker; published in 1668 in volume 3 of *Philosophical Transactions of the Royal Society*

Brouncker's quadrature of the hyperbola (1668)





To put this into modern terms, take A as the origin, and AB, AE as the x- and y-axes, respectively. Then the diagram represents the area under $\dfrac{1}{1+x}$ from $x = 0$ to $x = 1$.

(See *Mathematics emerging*, §3.2.2.)

Brouncker's article of 1668

PHILOSOPHICAL TRANSACTIONS.

Monday, *April* 13. 1668

The Contents.

The Squaring of the Hyperbola, *by an infinite series of Rational Numbers, together with its Demonstration, by that* Eminent *Mathematician, the Right Honourable the Lord Viscount* Brouncker.

WHat the Acute Dr. *John Wallis* had intimated, some years since, in the Dedication of his Answer to M. *Meibomius de proportionibus*, *vid.* That the World one day would learn from the Noble Lord *Brounker*, the *Quadrature* of the *Hyperbole*; the Ingenious Reader may see performed in the subjoyned operation, which its Excellent Author was now pleased to communicate, as followeth in his own words;

My Method for Squaring the Hyperbola *is this*:

LEt AB be one *Asymptote* of the Hyperbola EdC; and let AE and BC be parallel to th'other: Let also AE be to BC as 2 to 1; and let the Parallelogram ABDE equal 1. See *Fig.* 1. And note, that the Letter x every where stands for Multiplication.

Supposing the Reader knows, that EA. αζ. KH. βν. dθ. γκ. δλ. εμ. CB.&c. are in an *Harmonic* series, or a *series reciproca primanorum seu arithmeticè proportionalium* (otherwise he is referr'd for satisfaction to the 87, 88, 89, 90, 91, 92, 93, 94, 95, prop. *Arithm. Infinitor. Wallisij*:)

I say $ABCdEA = \frac{1}{1\times2} + \frac{1}{3\times4} + \frac{1}{5\times6} + \frac{1}{7\times8} + \frac{1}{9\times10}$ &c.

$EdCDE = \frac{1}{2\times3} + \frac{1}{4\times5} + \frac{1}{6\times7} + \frac{1}{8\times9} + \frac{1}{10\times11}$ &c. *in infinitum.*

$EdCyE = \frac{1}{2\times3\times4} + \frac{1}{4\times5\times6} + \frac{1}{6\times7\times8} + \frac{1}{8\times9\times10}$ &c.

For (in Fig. 2, & 3) the Parallelog.		*And (in Fig. 4.) the Triangl.*		*Note.*
$CA = \frac{1}{1\times2}$		$EdC = \frac{1}{2\times3\times4}$	$= \frac{\square dD - \square dF}{2}$
$dD = \frac{1}{2\times3}$	$dF = \frac{1}{3\times4}$	$Ebd = \frac{1}{4\times5\times6}$	$= \frac{\square br - \square bn}{2}$	$\square CA = dD + dF$
$br = \frac{1}{4\times5}$	$bn = \frac{1}{5\times6}$	$dfC = \frac{1}{6\times7\times8}$	$= \frac{\square fG - \square fk}{2}$	$\square dD = br + bn$
$fG = \frac{1}{6\times7}$	$fk = \frac{1}{7\times8}$	$Eab = \frac{1}{8\times9\times10}$	$= \frac{\square aq - \square ap}{2}$	$\square dF = fG + fk$
$aq = \frac{1}{8\times9}$	$ap = \frac{1}{9\times10}$	$bcd = \frac{1}{10\times11\times12}$	$= \frac{\square cs - \square cm}{2}$	$\square br = aq + ap$
$cs = \frac{1}{10\times11}$	$cm = \frac{1}{11\times12}$	$def = \frac{1}{12\times13\times14}$	$= \frac{\square et - \square el}{2}$	$\square bn = cs + cm$
$et = \frac{1}{12\times13}$	$el = \frac{1}{13\times14}$	$fgC = \frac{1}{14\times15\times16}$	$= \frac{\square gu - \square gh}{2}$	$\square fG = et + el$
$gu = \frac{1}{14\times15}$	$gh = \frac{1}{15\times16}$	*&c.*		$\square fk = gu + gh$
&c.	*&c.*			*&c.*

New methods: indivisibles and infinitesimals

Indivisibles: geometric objects making up a higher-dimensional object (e.g., points $\rightarrow$ line, lines $\rightarrow$ plane)

Infinitesimal: arbitrarily small but nonzero quantity

But distinction often blurred

During the 17th century, both concepts saw much use — despite the fact that they appeared to contradict Euclidean principles

Indivisibles

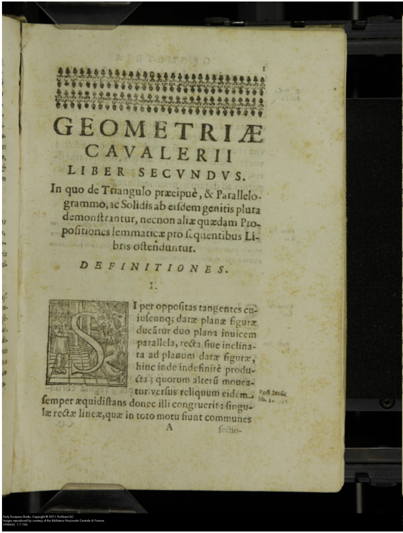
Early treatments by de Saint Vincent in c. 1623 (but not published until 1647) and Roberval in c. 1628–34 (but not published until 1693).

First published treatment by Bonaventura Cavalieri (1598–1647) in *Geometria indivisibilibus continuorum nova quadam ratione promota* [*Geometry advanced in a new way by the indivisibles of the continua*] (1635).

Used by Evangelista Torricelli (1608–1647) in 1644 to calculate the volume of an infinite hyperboloid of revolution.

Developed by John Wallis (1616–1703) and others.

Cavalieri's *Geometria*



1

GEOMETRIÆ CAVALERII

LIBER SECVNDVS.

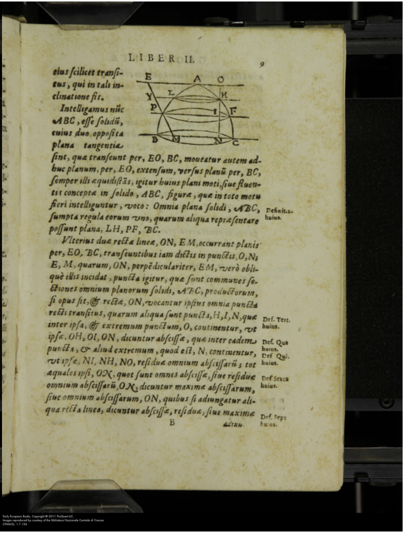
In quo de Triangulo præcipuè, & Parallelogrammo, ac Solidis ab eisdem genitis plura demonstrantur, necnon aliæ quædam Propositiones lemmaticæ pro sequentibus Libris ostenduntur.

DEFINITIONES.

I.

SI per oppositas tangentes cuiuscunq; datæ planæ figuræ ducātur duo plana inuicem parallela, recta, siue inclinata ad planum datæ figuræ, hinc inde indefinitè producta; quorum alterū moueatur versus reliquum eidem semper æquidistans donec illi congruerit: singulæ rectæ lineæ, quæ in toto motu fiunt communes Post Secūd. lib. 1.

A sectio-



eius scilicet transitus, qui in tali inclinatione fit.

Intelligamus nūc, ABC, esse solidū, cuius duo opposita plana tangentia sint, quæ transeunt per, EO, BC, moueatur autem adhuc planum, per, EO, extensum, versus planū per, BC, semper illi æquidistās, igitur huius plani moti, siue fluentis conceptæ in solido, ABC, figuræ, quæ in toto motu fieri intelliguntur, voco: Omnia plana solidi, ABC, sumpta regula eorum vno, quarum aliqua repræsentare possunt plana, LH, PF, BC. Definit. 2. huius.

Vlterius duæ rectæ lineæ, ON, EM, occurrant planis per, EO, BC, transeuntibus iam dictis in punctis, O, N; E, M, quarum, ON, perpēdiculariter, EM, verò obliquè illis incidat, puncta igitur, quæ sunt communes sectiones omnium planorum solidi, ABC, productorum, si opus sit, & rectæ, ON, vocantur ipsius omnia puncta recti transitus, quarum aliqua sunt puncta, H, I, N, quæ inter ipsa, & extremum punctum, O, continentur, vt ipsæ, OH, OI, ON, dicuntur abscissæ, quæ inter eadem puncta, & aliud extremum, quod est, N, continentur, vt ipsæ, NI, NH, NO, residuæ omnium abscissarū; tot æquales ipsi, ON, quot sunt omnes abscissæ, siue residuæ omnium abscissarū, ON, dicuntur maximæ abscissarum, siue omnium abscissarum, ON, quibus si adiungatur aliqua recta linea, dicuntur abscissæ, residuæ, siue maximæ Def. Tert. huius. Def. Qua huius. Def Qui. huius. Def. Sexta huius. Def. Sept huius.

B adiun-

Torricelli's hyperbolic solid (*Opera geometrica*, 1644)

Problema Secundum. 115

Lemma V.

Cviuſcunque cylindri ghil *intra ſolidum acutum deſcripti (vt in pręcedenti figura) ſuperficies ſine baſibus æqualis eſt circulo cuius ſemidiameter ſit linea* df. *nempe ſemiaxis, ſiue ſemilatus verſum ipſius hyperbolæ. Hoc enim in ipſo progreſſu præcedentis lemmatis demonſtratum eſt.*

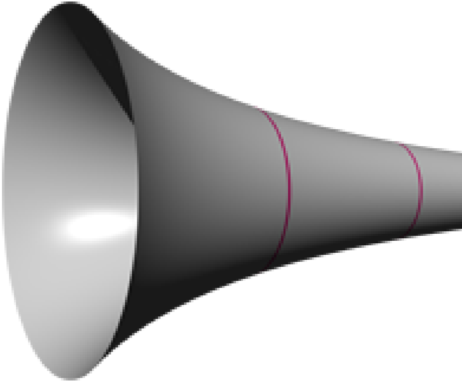
Theorema.

Solidum acutum hyperbolicum infinitè longum, ſectum plano ad axem erecto, vnà cum cylindro ſuæ baſis, æquale eſt cylindro cuidam recto, cuius baſis diameter ſit latus verſum, ſiue axis hyperbolæ, altitudo verò ſit æqualis ſemidiametro baſis ipſius acuti ſolidi.

Eſto hyperbola cuius aſymptoti *ab*, *ac* angulum rectum contineant; ſumptoq; in hyperbola quolibet puncto *d*, ducatur *dc* æquidiſtans ipſi *ab*, & *dp* æquidiſtans *ac*. Tũ conuertatur vniuerſa figura circa axẽ *ab*. ità vt fiat ſolidum acutum byperbolicum *ebd*, vna cum cylindro ſuæ baſis *fedc*. Poducatur *ba* in *h*. ita vt *ah*. æqualis ſit integro axi, ſiue lateri verſo hyperbolæ. Et circa diametrum *ah* intelligatur circulus erectus ad aſymptoton *ac*: & ſuper baſi *ah* concipiatur cylindrus rectus *acgh*, cuius altitudo ſit *ac*, nempe ſemidiameter baſis acuti ſolidi. Dico ſolidum vniuerſum *febdc*, quanquam ſine fine longum, æquale tamen eſſe cylindro *acgh*.

Accipiatur in recta *ac* quodlibet punctum *i*, & per *i* intelligatur ducta ſuperficies cylindrica *onli* in ſolido acuto

P 2 com-



(See *Mathematics emerging*, §3.3.1.)

John Wallis (1616–1703)

Studied at Emmanuel College, Cambridge (BA 1637, MA 1640)

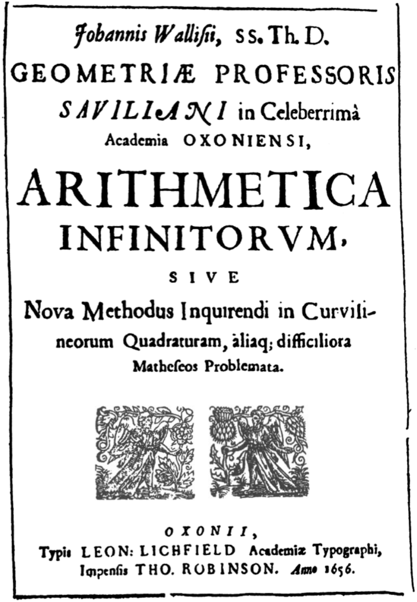
1643–1649: scribe for Westminster Assembly

1644–1645: Fellow of Queens' College, Cambridge

1643–1689: cryptographer to Parliament, then to the Crown

1649–1703: Savilian Professor of Geometry in Oxford

Arithmetica infinitorum


Johannis Wallisii, SS. Th. D.
GEOMETRIÆ PROFESSORIS
SAVILIANI in Celeberrimâ
Academia OXONIENSI,
ARITHMETICA
INFINITORUM,
SIVE
Nova Methodus Inquirendi in Curvilineorum Quadraturam, àliaq; difficiliora
Matheseos Problemata.
OXONII,
Typis LEON: LICHFIELD Academiæ Typographi,
Impensis THO. ROBINSON. Anno 1656.

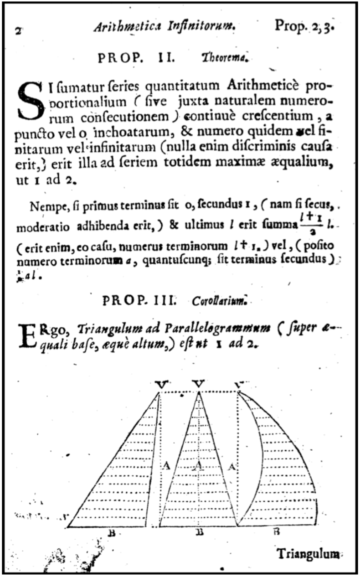
John Wallis,
Arithmetica infinitorum
(*The arithmetic of infinitesimals*)
Oxford, 1656

Translation by
Jacqueline A. Stedall
Springer, 2004

Arithmetica infinitorum

- Arithmetical methods rather than geometrical, but repeatedly appealed to geometry for justification
- Investigation of sums of sequences of powers (or ratios of these to a known fixed quantity) — usually decreasing
- Fixed an endpoint, dividing interval into infinite number of arbitrarily small subintervals — these are the 'infinitesimals' of Wallis' title

Wallis and indivisibles



PROP. II. *Theorema.*

SI ſumatur ſeries quantitatum Arithmeticè proportionalium (ſive juxta naturalem numerorum conſecutionem) continuè creſcentium, a puncto vel o inchoatarum, & numero quidem vel finitarum vel infinitarum (nulla enim diſcriminis cauſa erit,) erit illa ad ſeriem totidem maximæ æqualium, ut 1 ad 2.

Nempe, ſi primus terminus ſit o, ſecundus 1, (nam ſi ſecus, moderatio adhibenda erit,) & ultimus l erit ſumma $\frac{l+1}{2}l$. (erit enim, eo caſu, numerus terminorum $l+1$.) vel, (poſito numero terminorum a, quantuſcunq; ſit terminus ſecundus) $\frac{1}{2}al$.

PROP. III. *Corollarium.*

ERgo, *Triangulum ad Parallelogrammum (ſuper æquali baſe, æquè altum,) eſt ut* 1 *ad* 2.

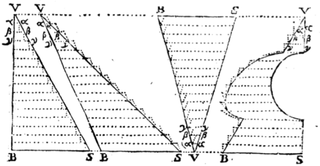
Triangulum

For the triangle … consists of an infinite number of parallel lines in arithmetic proportion …

(See *Mathematics emerging*, §2.4.2.)

Wallis and indivisibles?

ralis initium. Quamvis enim Sectorum illorum numero infinitorum aggregatum, ipſi figuræ lineis recta & Spirali terminatæ, (juxta methodum Indiviſibilium) æquale ponatur; non tamen illud de omnium Arcubus cum ipſa Spirali (propriè dicta) comparatis obtinebit. Tantundem enim eſſet, acſi quis, dum infinita numero parallelogramma triangulo inſcripta (aut etiam circumſcripta) toti triangulo VBS æqualia videat, inde



concluderet eorum omnium latera rectæ VS adjacentia (rectæ VB parallela) ipſi VS ſimul æqualia eſſe, vel quæ rectæ VB adjacent (ipſi VS parallela) æqualia ſimul eſſe toti VB. (Quod ſiquando verum eſſe contingat, puta in triangulo iſoſceli, non tamen id univerſaliter concludendum erit.) Atq; hoc quidem eo potius admonendum duxi, quod viderim etiam viros doctos nonnunquam ſpecioſa ejuſmodi veriſimilitudine in lapſum proclives eſſe. Cur autem omiſſa Spirali genuina, ſpuriam hanc peripheriæ comparaverim; cauſa eſt, quod huic poſſim, non autem illi, æqualem peripheriam aſſignare.

PROP. XIV. *Corollarium.*

ET propterea etiam *ſegmenta ſpiralis, a principio ſpiralis exorſa, ſunt ad rectas conterminas, ſicut Parabolæ Diametri interceptæ, ad ordinatim-applicatas.*

For it amounts to the same thing as if, when an infinite number of parallelograms are inscribed in (or circumscribed about) a triangle, it seems that they equal to complete triangle ...

(See *Mathematics emerging*, §2.4.2.)

Sums of powers

Wallis' method depended upon the summation rule

$$\sum_{a=0}^{A} a^n \approx \frac{A^{n+1}}{n+1}$$

A version of this was developed by Ibn al-Haytham (Alhazen) in 11th-century Egypt, and it was certainly known to Fermat, Roberval, and Cavalieri in the 1630s for positive integers n, but in the 1650s Wallis extended it to negative and fractional n.

See: Victor J. Katz, 'Ideas of calculus in Islam and India', *Mathematics Magazine* 68(3) (1995), 163–174

Simple ‘integrals’

Using the summation rule we can ‘integrate’

$$x^2, \quad x^3, \quad ..., \quad x^{1/3}, \quad ..., \quad x^{-4}, \quad ...$$

and

$$(1+x)^3 \quad \text{or} \quad (1+x^2)^5 \quad \text{or} \quad ...$$

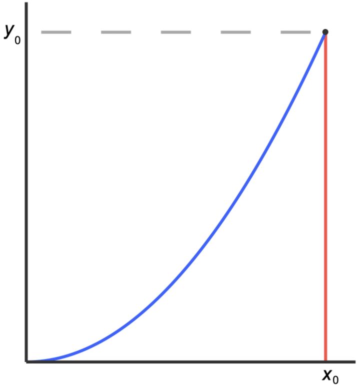
but what about

$$(1-x^2)^{1/2} \quad \text{[for a circle]}$$

or

$$(1+x)^{-1} \quad \text{[for a hyperbola]}\,?$$

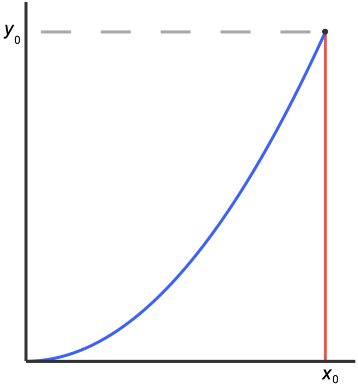
Wallis and the quadrature of a parabola



Wallis sought the area under the parabola $y = x^2$ between $x = 0$ and $x = x_0$

He used the language of ratio, hence sought to calculate the ratio of the area A under the curve to that of the corresponding rectangle $(x_0 y_0)$, which we may think of as the fraction $\frac{A}{x_0 y_0}$

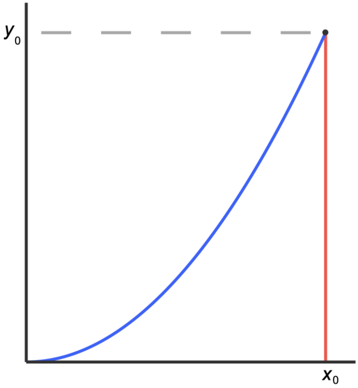
Wallis and the quadrature of a parabola



Wallis considered an area to be the sum of the lengths of the lines contained within it (makes sense?), so

- A is the sum of the values of x^2 as x ranges from 0 to x_0
- $x_0 y_0$ is the sum of as many copies of x_0^2 (?)

Wallis and the quadrature of a parabola



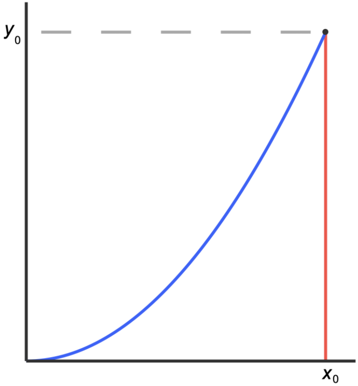
Break $(0, x_0)$ into n subintervals, suppose that x only takes the values at the endpoints of these, and consider the ratio

$$R = \frac{0^2 + 1^2 + 2^2 + \cdots + n^2}{n^2 + n^2 + n^2 + \cdots + n^2}$$

As we make n larger, this ratio will become a closer approximation to $\frac{A}{x_0 y_0}$

[Note that we are deliberately avoiding the terminology of limits, and that some x_0^2s have been cancelled, thanks to the use of ratios]

Wallis and the quadrature of a parabola

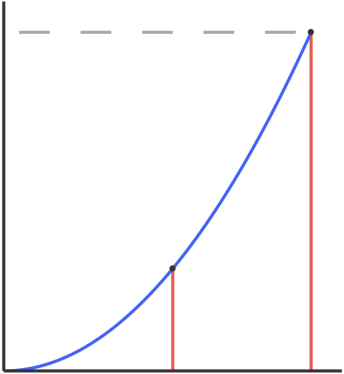


Wallis investigated the cases of small n

For $n = 1$ (one red line),

$$R = \frac{0^2 + 1^2}{1^2 + 1^2} = \frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$

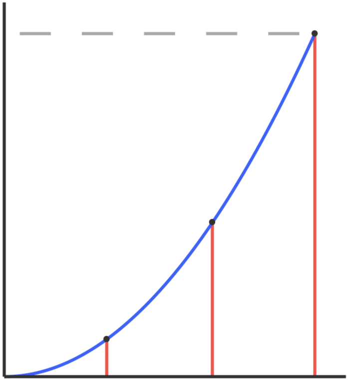
Wallis and the quadrature of a parabola



For $n = 2$ (two red lines),

$$R = \frac{0^2 + 1^2 + 2^2}{2^2 + 2^2 + 2^2} = \frac{5}{12} = \frac{1}{3} + \frac{1}{12}$$

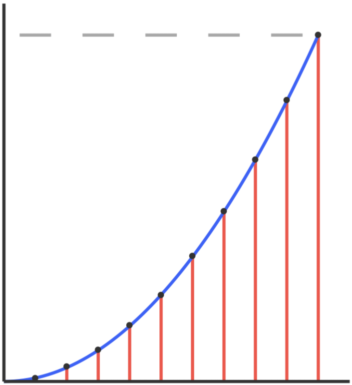
Wallis and the quadrature of a parabola



For $n = 3$ (three red lines),

$$R = \frac{0^2 + 1^2 + 2^2 + 3^2}{3^2 + 3^2 + 3^2 + 3^2} = \frac{14}{36} = \frac{1}{3} + \frac{1}{18}$$

Wallis and the quadrature of a parabola



So as n increases $\frac{A}{x_0 y_0}$ approaches $\frac{1}{3}$, hence $A = \frac{1}{3}x_0^3$, as we'd expect

Wallis called this method of spotting and extending a pattern 'induction' — it was criticised at the time (for example, by Pascal)

Enter Newton...

In his own words:

> *In the winter between the years 1664 and 1665 upon reading Dr Wallis's Arithmetica infinitorum and trying to interpole his progressions for squaring the circle, I found out first an infinite series for squaring the circle and then another infinite series for squaring the Hyperbola ...*

Newton extended Wallis' method of interpolation...

Newton's integration of $(1+x)^{-1}$

	$(1+x)^{-1}$	$(1+x)^{0}$	$(1+x)^{1}$	$(1+x)^{2}$	$(1+x)^{3}$	$(1+x)^{4}$	$\cdots$
x	?	1	1	1	1	1	$\cdots$
$\frac{x^2}{2}$	?	0	1	2	3	4	$\cdots$
$\frac{x^3}{3}$	?	0	0	1	3	6	$\cdots$
$\frac{x^4}{4}$	?	0	0	0	1	4	$\cdots$
$\frac{x^5}{5}$	?	0	0	0	0	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The entry in the row labelled $\frac{x^m}{m}$ and the column labelled $(1+x)^n$ is the coefficient of $\frac{x^m}{m}$ in $\int(1+x)^n dx$. (NB. Newton did not use the notation $\int(1+x)^n dx$.)

Newton's integration of $(1+x)^{-1}$

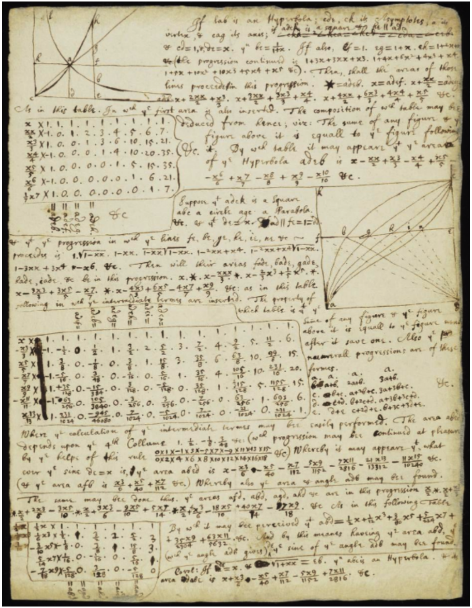
	$(1+x)^{-1}$	$(1+x)^{0}$	$(1+x)^{1}$	$(1+x)^{2}$	$(1+x)^{3}$	$(1+x)^{4}$	$\cdots$
x	1	1	1	1	1	1	$\cdots$
$\frac{x^2}{2}$	-1	0	1	2	3	4	$\cdots$
$\frac{x^3}{3}$	1	0	0	1	3	6	$\cdots$
$\frac{x^4}{4}$	-1	0	0	0	1	4	$\cdots$
$\frac{x^5}{5}$	1	0	0	0	0	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The entry in the row labelled $\frac{x^m}{m}$ and the column labelled $(1+x)^n$ is the coefficient of $\frac{x^m}{m}$ in $\int (1+x)^n dx$. (NB. Newton did not use the notation $\int (1+x)^n dx$.)

The general binomial theorem

CUL Add. MS 3958.3, f. 72

See *Mathematics emerging*, §8.1.1



Newton's calculus: 1664–5

- rules for quadrature (influenced by Wallis's ideas of interpolation)
- rules for tangents (influenced by Descartes' double root method)
- recognition that these are inverse processes

Newton's vocabulary and notation

Newton's calculus 1664–5:

- fluents $x,\ y$, ... (quantities that vary with time t)
- fluxions $\dot{x},\ \dot{y}$, ... (rate of change of those quantities)
- moments o (infinitesimal time in which x increases by $\dot{x}o$)

Newton's calculus in action (*The method of fluxions*, 1736)

12. Ex. 5. As if the Equation $zz + axz - y^4 = 0$ were propos'd to expreſs the Relation between x and y, as alſo $\sqrt{ax-xx} = BD$, for determining a Curve, which therefore will be a Circle. The Equation $zz + axz - y^4 = 0$, as before, will give $2\dot{z}z + a\dot{z}x + a\dot{x}z - 4\dot{y}y^3 = 0$, for the Relation of the Celerities $\dot{x}$, $\dot{y}$, and $\dot{z}$. And therefore ſince it is $\dot{z} = \dot{x} \times BD$ or $= \dot{x}\sqrt{ax-xx}$, ſubſtitute this Value inſtead of it, and there will ariſe the Equation $2\dot{x}z + a\dot{x}x\sqrt{ax-xx} + a\dot{x}z - 4\dot{y}y^3 = 0$, which determines the Relation of the Celerities $\dot{x}$ and $\dot{y}$.

DEMONSTRATION *of the Solution.*

13. The Moments of flowing Quantities, (that is, their indefinitely ſmall Parts, by the acceſſion of which, in indefinitely ſmall portions of Time, they are continually increaſed,) are as the Velocities of their Flowing or Increaſing.

14. Wherefore if the Moment of any one, as x, be repreſented by the Product of its Celerity $\dot{x}$ into an indefinitely ſmall Quantity o (that is, by $\dot{x}o$,) the Moments of the others v, y, z, will be repreſented by $\dot{v}o$, $\dot{y}o$, $\dot{z}o$; becauſe $\dot{v}o$, $\dot{x}o$, $\dot{y}o$, and $\dot{z}o$, are to each other as $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$.

See the Analize des infiniment petites of the M. de L'Hospital.

15. Now ſince the Moments, as $\dot{x}o$ and $\dot{y}o$, are the indefinitely little acceſſions of the flowing Quantities x and y, by which thoſe Quantities are increaſed through the ſeveral indefinitely little intervals of Time; it follows, that thoſe Quantities x and y, after any indefinitely ſmall interval of Time, become $x + \dot{x}o$ and $y + \dot{y}o$. And therefore the Equation, which at all times indifferently expreſſes the Relation of the flowing Quantities, will as well expreſs the Relation between $x + \dot{x}o$ and $y + \dot{y}o$, as between x and y: So that $x + \dot{x}o$ and $y + \dot{y}o$ may be ſubſtituted in the ſame Equation for thoſe Quantities, inſtead of x and y.

16. Therefore let any Equation $x^3 - ax^2 + axy - y^3 = 0$ be given, and ſubſtitute $x + \dot{x}o$ for x, and $y + \dot{y}o$ for y, and there will ariſe

$$\left.\begin{array}{llll} x^3 & + 3\dot{x}ox^2 & + 3\dot{x}^2oox & + \dot{x}^3o^3 \\ - ax^2 & - 2a\dot{x}ox & - a\dot{x}^2oo & \\ + axy & + a\dot{x}oy & + a\dot{y}ox & + a\dot{x}\dot{y}oo \\ - y^3 & - 3\dot{y}oy^2 & - 3\dot{y}^2ooy & - \dot{y}^3o^3 \end{array}\right\} = 0.$$

17.

17. Now by Suppoſition $x^3 - ax^2 + axy - y^3 = 0$, which therefore being expunged, and the remaining Terms being divided by o, there will remain $3\dot{x}x^2 + 3\dot{x}^2ox + \dot{x}^3oo - 2a\dot{x}x - a\dot{x}^2o + a\dot{x}y + a\dot{y}x + a\dot{x}\dot{y}o - 3\dot{y}y^2 - 3\dot{y}^2oy - \dot{y}^3oo = 0$. But whereas o is ſuppoſed to be infinitely little, that it may repreſent the Moments of Quantities; the Terms that are multiply'd by it will be nothing in reſpect of the reſt. Therefore I reject them, and there remains $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + a\dot{y}x - 3\dot{y}y^2 = 0$, as above in Examp. 1.

18. Here we may obſerve, that the Terms that are not multiply'd by o will always vaniſh, as alſo thoſe Terms that are multiply'd by o of more than one Dimenſion. And that the reſt of the Terms being divided by o, will always acquire the form that they ought to have by the foregoing Rule: Which was the thing to be proved.

19. And this being now ſhewn, the other things included in the Rule will eaſily follow. As that in the propos'd Equation ſeveral flowing Quantities may be involved; and that the Terms may be multiply'd, not only by the Number of the Dimenſions of the flowing Quantities, but alſo by any other Arithmetical Progreſſions; ſo that in the Operation there may be the ſame difference of the Terms according to any of the flowing Quantities, and the Progreſſion be diſpos'd according to the ſame order of the Dimenſions of each of them. And theſe things being allow'd, what is taught beſides in Examp. 3, 4, and 5, will be plain enough of itſelf.

PROB. II.

An Equation being propoſed, including the Fluxions of Quantities, to find the Relations of thoſe Quantities to one another.

A PARTICULAR SOLUTION.

1. As this Problem is the Converſe of the foregoing, it muſt be ſolved by proceeding in a contrary manner. That is, the Terms multiply'd by $\dot{x}$ being diſpoſed according to the Dimenſions of x; they muſt be divided by $\frac{\dot{x}}{x}$, and then by the number of their Dimenſions, or perhaps by ſome other Arithmetical Progreſſion. Then the ſame work muſt be repeated with the Terms multiply'd by $\dot{v}$, $\dot{y}$,

or

Newton's calculus in action (*The method of fluxions*, 1736)

12. Ex. 5. As if the Equation $zz + axz - y^4 = 0$ were propos'd to expreſs the Relation between x and y, as alſo $\sqrt{ax - xx} = BD$, for determining a Curve, which therefore will be a Circle. The Equation $zz + axz - y^4 = 0$, as before, will give $2\dot{z}z + a\dot{z}x + a\dot{x}z - 4\dot{y}y^3 = 0$, for the Relation of the Celerities $\dot{x}$, $\dot{y}$, and $\dot{z}$. And therefore ſince it is $\dot{z} = \dot{x} \times BD$ or $= \dot{x}\sqrt{ax - xx}$, ſubſtitute this Value inſtead of it, and there will ariſe the Equation $\overline{2\dot{x}z + a\dot{x}x}\sqrt{ax - xx} + a\dot{x}z - 4\dot{y}y^3 = 0$, which determines the Relation of the Celerities $\dot{x}$ and $\dot{y}$.

Demonstration *of the Solution.*

13. The Moments of flowing Quantities, (that is, their indefinitely ſmall Parts, by the acceſſion of which, in indefinitely ſmall portions of Time, they are continually increaſed,) are as the Velocities of their Flowing or Increaſing.

14. Wherefore if the Moment of any one, as x, be repreſented by the Product of its Celerity $\dot{x}$ into an indefinitely ſmall Quantity o (that is, by $\dot{x}o$,) the Moments of the others v, y, z, will be repreſented by $\dot{v}o$, $\dot{y}o$, $\dot{z}o$; becauſe $\dot{v}o$, $\dot{x}o$, $\dot{y}o$, and $\dot{z}o$, are to each other as $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$.

15. Now ſince the Moments, as $\dot{x}o$ and $\dot{y}o$, are the indefinitely little acceſſions of the flowing Quantities x and y, by which thoſe Quantities are increaſed through the ſeveral indefinitely little intervals of Time; it follows, that thoſe Quantities x and y, after any indefinitely ſmall interval of Time, become $x + \dot{x}o$ and $y + \dot{y}o$. And therefore the Equation, which at all times indifferently expreſſes the Relation of the flowing Quantities, will as well expreſs the Relation between $x + \dot{x}o$ and $y + \dot{y}o$, as between x and y: So that $x + \dot{x}o$ and $y + \dot{y}o$ may be ſubſtituted in the ſame Equation for thoſe Quantities, inſtead of x and y.

16. Therefore let any Equation $x^3 - ax^2 + axy - y^3 = 0$ be given, and ſubſtitute $x + \dot{x}o$ for x, and $y + \dot{y}o$ for y, and there will ariſe

$$\left.\begin{array}{llll} x^3 & + 3\dot{x}ox^2 & + 3\dot{x}^2oox & + \dot{x}^3o^3 \\ - ax^2 & - 2a\dot{x}ox & - a\dot{x}^2oo & \\ + axy & + a\dot{x}oy & + a\dot{y}ox & + a\dot{x}\dot{y}oo \\ - y^3 & - 3\dot{y}oy^2 & - 3\dot{y}^2ooy & - \dot{y}^3o^3 \end{array}\right\} = 0.$$

17. Now by Suppoſition $x^3 - ax^2 + axy - y^3 = 0$, which therefore being expunged, and the remaining Terms being divided by o, there will remain $3\dot{x}x^2 + 3\dot{x}^2ox + \dot{x}^3oo - 2a\dot{x}x - a\dot{x}^2o + a\dot{x}y + a\dot{y}x + a\dot{x}\dot{y}o - 3\dot{y}y^2 - 3\dot{y}^2oy - \dot{y}^3oo = 0$. But whereas o is ſuppoſed to be infinitely little, that it may repreſent the Moments of Quantities; the Terms that are multiply'd by it will be nothing in reſpect of the reſt. Therefore I reject them, and there remains $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + a\dot{y}x - 3\dot{y}y^2 = 0$, as above in Examp. 1.

18. Here we may obſerve, that the Terms that are not multiply'd by o will always vaniſh, as alſo thoſe Terms that are multiply'd by o of more than one Dimenſion. And that the reſt of the Terms being divided by o, will always acquire the form that they ought to have by the foregoing Rule: Which was the thing to be proved.

19. And this being now ſhewn, the other things included in the Rule will eaſily follow. As that in the propos'd Equation ſeveral flowing Quantities may be involved; and that the Terms may be multiply'd, not only by the Number of the Dimenſions of the flowing Quantities, but alſo by any other Arithmetical Progreſſions; ſo that in the Operation there may be the ſame difference of the Terms according to any of the flowing Quantities, and the Progreſſion be diſpos'd according to the ſame order of the Dimenſions of each of them. And theſe things being allow'd, what is taught beſides in Examp. 3, 4, and 5, will be plain enough of itſelf.

PROB. II.

An Equation being propoſed, including the Fluxions of Quantities, to find the Relations of thoſe Quantities to one another.

A Particular Solution.

1. As this Problem is the Converſe of the foregoing, it muſt be ſolved by proceeding in a contrary manner. That is, the Terms multiply'd by $\dot{x}$ being diſpoſed according to the Dimenſions of x; they muſt be divided by $\frac{\dot{x}}{x}$, and then by the number of their Dimenſions, or perhaps by ſome other Arithmetical Progreſſion. Then the ſame work muſt be repeated with the Terms multiply'd by $\dot{v}$, $\dot{y}$,

E or

Leibniz's calculus

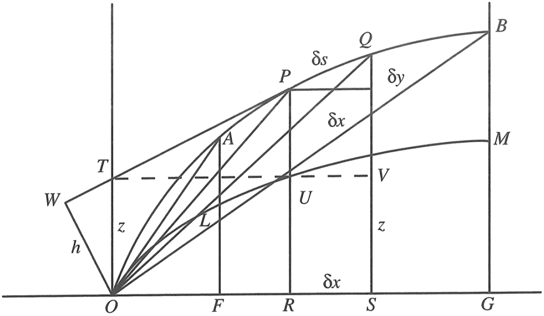
Independently, ten years later than Newton...

Leibniz's calculus, 1673–76:

- rules for quadrature — especially the transformation theorem (a.k.a. the transmutation theorem)
- rules for tangents — by characteristic (or differential) triangles
- recognition that these are inverse processes

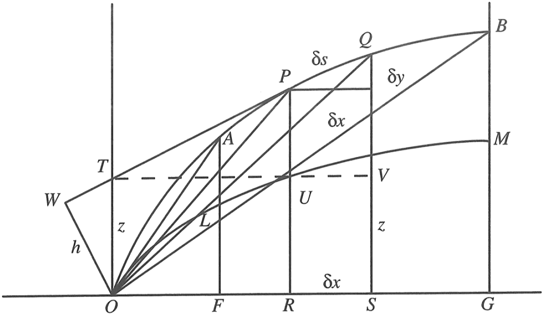
Differentials: du, dv;
integrals: omn. l, later between $S\,\mathsf{l}$ and $\int \mathsf{l}$

Leibniz's Transmutation Theorem



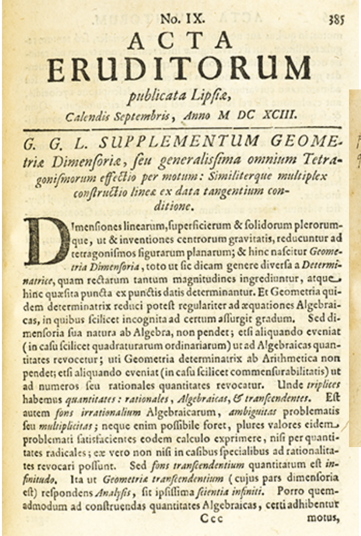
$$OABG = \sum RPQS = \frac{1}{2} OG \cdot GB + \sum OPQ$$

Leibniz's Transmutation Theorem



$$OABG = \sum RPQS = \frac{1}{2} OG \cdot GB + \sum OPQ$$
$$= \frac{1}{2} OG \cdot GB + \frac{1}{2} OLMG$$

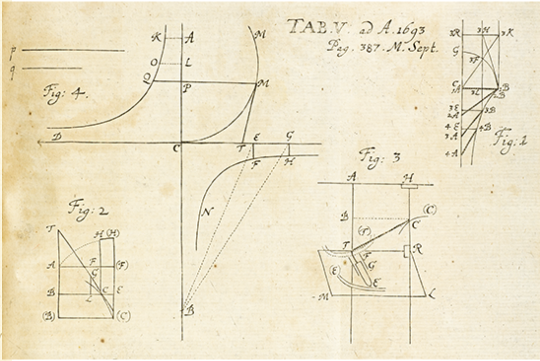
Supplementum Geometriae Dimensoriae . . . (1693)



ACTA
ERUDITORUM
publicata Lipſiæ,
Calendis Septembris, Anno M DC XCIII.

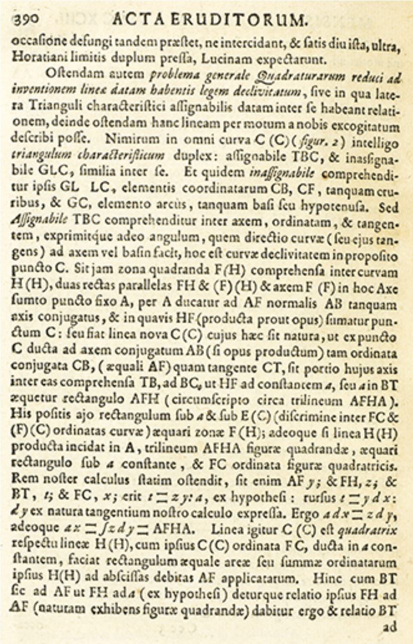
G. G. L. SUPPLEMENTUM GEOMEtriæ Dimenſoriæ, ſeu generaliſſima omnium Tetragoniſmorum effectio per motum: Similiterque multiplex conſtructio lineæ ex data tangentium conditione.

Dimenſiones linearum, ſuperficierum & ſolidorum plerorumque, ut & inventiones centrorum gravitatis, reducuntur ad tetragoniſmos figurarum planarum; & hinc naſcitur *Geometria Dimenſoria*, toto ut ſic dicam genere diverſa a *Determinatrice*, quam rectarum tantum magnitudines ingrediuntur, atque hinc quæſita puncta ex punctis datis determinantur. Et Geometria quidem determinatrix reduci poteſt regulariter ad æquationes Algebraicas, in quibus ſcilicet incognita ad certum aſſurgit gradum. Sed dimenſoria ſua natura ab Algebra, non pendet; etſi aliquando eveniat (in caſu ſcilicet quadraturarum ordinariarum) ut ad Algebraicas quantitates revocetur; uti Geometria determinatrix ab Arithmetica non pendet; etſi aliquando eveniat (in caſu ſcilicet commenſurabilitatis) ut ad numeros ſeu rationales quantitates revocatur. Unde *triplices* habemus *quantitates: rationales, Algebraicas, & tranſcendentes.* Eſt autem *fons irrationalium* Algebraicarum, *ambiguitas* problematis ſeu *multiplicitas*; neque enim poſſibile foret, plures valores eidem problemati ſatisfacientes eodem calculo exprimere, niſi per quantitates radicales; eæ vero non niſi in caſibus ſpecialibus ad rationalitates revocari poſſunt. Sed *fons tranſcendentium* quantitatum eſt *infinitudo*. Ita ut *Geometriæ tranſcendentium* (cujus pars dimenſoria eſt) reſpondens *Analyſis*, ſit ipſiſſima *ſcientia infiniti*. Porro quemadmodum ad conſtruendas quantitates Algebraicas, certi adhibentur



A Supplement to the Geometry of Measurements, or the Most General of all Quadratures to be Effected by a Motion: and likewise the various constructions of a curve from a given condition of the tangent (*Acta eruditorum*, 1693)

Supplementum Geometriae Dimensoriae . . . (1693)



occaſione defungi tandem præſtet, ne intercidant, & ſatis diu iſta, ultra, Horatiani limitis duplum preſſa, Lucinam expectarunt.

Oſtendam autem *problema generale Quadraturarum reduci ad inventionem lineæ datam habentis legem declivitatum*, ſive in qua latera Trianguli characteriſtici aſſignabilis datam inter ſe habeant relationem, deinde oſtendam hanc lineam per motum a nobis excogitatum deſcribi poſſe. Nimirum in omni curva C (C) (*figur. 2*) intelligo *triangulum characteriſticum* duplex: aſſignabile TBC, & inaſſignabile GLC, ſimilia inter ſe. Et quidem *inaſſignabile* comprehenditur ipſis GL LC, elementis coordinatarum CB, CF, tanquam cruribus, & GC, elemento arcûs, tanquam baſi ſeu hypotenuſa. Sed *Aſſignabile* TBC comprehenditur inter axem, ordinatam, & tangentem, exprimitque adeo angulum, quem directio curvæ (ſeu ejus tangens) ad axem vel baſin facit, hoc eſt curvæ declivitatem in propoſito puncto C. Sit jam zona quadranda F(H) comprehenſa inter curvam H (H), duas rectas parallelas FH & (F) (H) & axem F (F) in hoc Axe ſumto puncto fixo A, per A ducatur ad AF normalis AB tanquam axis conjugatus, & in quavis HF (producta prout opus) ſumatur punctum C: ſeu fiat linea nova C (C) cujus hæc ſit natura, ut ex puncto C ducta ad axem conjugatum AB (ſi opus productum) tam ordinata conjugata CB, (æquali AF) quam tangente CT, ſit portio hujus axis inter eas comprehenſa TB, ad BC, ut HF ad conſtantem *a*, ſeu *a* in BT æquetur rectangulo AFH (circumſcripto circa trilineum AFHA). His poſitis ajo rectangulum ſub *a* & ſub E (C) (diſcrimine inter FC & (F) (C) ordinatas curvæ) æquari zonæ F (H); adeoque ſi linea H (H) producta incidat in A, trilineum AFHA figuræ quadrandæ, æquari rectangulo ſub *a* conſtante, & FC ordinata figuræ quadratricis. Rem noſter calculus ſtatim oſtendit, ſit enim AF *y*; & FH, *z*; & BT, *t*; & FC, *x*; erit $t = zy : a$, ex hypotheſi: rurſus $t = y\,dx : dy$ ex natura tangentium noſtro calculo expreſſa. Ergo $a\,dx = z\,dy$, adeoque $ax = \int z\,dy =$ AFHA. Linea igitur C (C) eſt *quadratrix* reſpectu lineæ H (H), cum ipſius C (C) ordinata FC, ducta in *a* conſtantem, faciat rectangulum æquale areæ ſeu ſummæ ordinatarum ipſius H(H) ad abſciſſas debitas AF applicatarum. Hinc cum BT ſit ad AF ut FH ad *a* (ex hypotheſi) deturque relatio ipſius FH ad AF (naturam exhibens figuræ quadrandæ) dabitur ergo & relatio BT ad

"I shall now show *the general problem of quadratures to be reduced to the invention of a line having a given law of declivity*"

i.e., integration is reduced to the finding of a curve with a particular tangent — in modern terms, the antiderivative

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See also: Niccolò Guicciardini, Newton's method and Leibniz's calculus, *A history of analysis* (ed. Hans Niels Jahnke), AMS/LMS, 2003, pp. 73–103

Newton's calculus and Leibniz's calculus compared

Newton (1664–65):	Leibniz (1673–76):
rules for quadrature	rules for quadrature
rules for tangents	rules for tangents
'fundamental theorem'	'fundamental theorem'
dot notation	'modern' notation
physical intuition:	algebraic intuition
rates of change	rules and procedures
PROBLEM:	PROBLEM:
vanishing quantities o	vanishing quantities du, dv, ...

An elementary introduction to the development of calculus