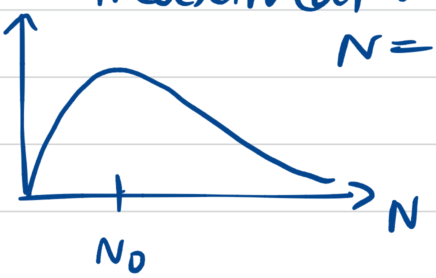
PS1 Q1

$$\frac{dN}{dt} = \underbrace{PN(t-t_0)e^{-N(t-t_0)/N_0}} - \underbrace{\delta N(t)}$$

(a) birth rate, including maturation time, maximal at $N=N_0$ — death



(b) Let $N = [N]n$, $t = [T]\tau$ $\Rightarrow$ $\frac{d}{dt} = \frac{1}{[T]}\frac{d}{d\tau}$

$$\frac{[N]}{[T]}\frac{dn}{d\tau} = P[N]n(\tau-\tau_0)e^{-[N]n(\tau-\tau_0)/N_0} - \delta[N]n$$

Take $[N] = N_0$, $[T] = \frac{1}{\delta}$, $\tau_0 = t_0\delta$ to get

$$\frac{dn}{d\tau} = p\,n(\tau-\tau_0)e^{-n(\tau-\tau_0)} - n(\tau)$$

$$= \frac{P}{\delta}$$

NB units of N = density, t = time, N_0 = density, t_0 = time, P = time^{-1}, δ = time^{-1} $\Rightarrow$ units of $n = 1$ ✓, $\tau = 1$ ✓, $\tau_0 = 1$ ✓, $p = 1$ ✓

(c) Steady states satisfy $0 = pn_* e^{-n_*} - n_*$

$\therefore \quad n_* = 0$ or $n_* = \ln p$ (feasible as $p > 1$)

(d) Let $n(t) = n_* + \varepsilon n_1(\tau) + \dots$

$$\varepsilon \frac{dn_1}{d\tau} = p[n_* + \varepsilon n_1(\tau - \tau_0) + \dots]\exp[-n_* - \varepsilon n_1(\tau - \tau_0) - \dots] - [1 + \varepsilon n_1(\tau) + \dots]$$

$$= pe^{-n_*}[n_* + \varepsilon n_1(\tau - \tau_0) + \dots][1 - \varepsilon n_1(\tau - \tau_0) + \dots] - [1 - \varepsilon n_1(\tau) + \dots]$$

$\frac{1}{p} =$

$$\frac{dn_1}{d\tau} = pe^{-n_*}[-n_* n_1(\tau - \tau_0) + n_1(\tau - \tau_0)] - n_1(\tau)$$

$$= (1 - \ln p)n_1(\tau - \tau_0) - n_1(\tau)$$

(e) Seek solutions $n_1(\tau) = n_1(0)e^{\lambda t}$

$$\lambda n_1(0)e^{\lambda\tau} = (1 - \ln p)n_1(0)e^{\lambda(\tau - \tau_0)} - n_1(0)e^{\lambda\tau}$$

$$\therefore \quad \lambda = -1 - (\ln p - 1)e^{-\lambda\tau_0} \qquad (4)$$

Suppose $\lambda \in \mathbb{R}$. If $\tau_0 = 0$, $\lambda = -\ln p > 0$ and hence the steady state is stable.

Now suppose that $\tau_0 > 0$. For $\lambda < 0$ we need to show that we can choose τ_0 such that
$$-1 - (\ln p - 1)e^{-\lambda\tau_0} < 0$$

If $\ln p > 1$ then $-1 - \underbrace{(\ln p - 1)}_{>0}\underbrace{e^{-\lambda T_0}}_{>0} = \lambda < 0$

If $0 < \ln p < 1$ then write

$$\lambda = -1 - (\ln p - 1)e^{-\lambda T_0} = \underbrace{(-1 + e^{\lambda T_0})}_{} - \underbrace{\ln p \, e^{-\lambda T_0}}_{>0}$$

Can choose T_0 sufficiently small such that $\lambda < 0$

Hence, the steady state $n_* = \ln p$ is stable for sufficiently small delays, T_0, and it can only become unstable by Eq. (4) having complex solutions with positive real part.

(f) Let $\lambda = \mu + i\omega$

$$\Rightarrow \mu + i\omega = -1 - (\ln p - 1)e^{-(\mu + i\omega)T_0}$$
$$= -1 - (\ln p - 1)e^{-\mu T_0}\left(\cos(\omega T_0) - i\sin(\omega T_0)\right)$$

$$\therefore \quad \mu = -1 - (\ln p - 1)e^{-\mu T_0}\cos(\omega T_0)$$
$$\omega = (\ln p - 1)e^{-\mu T_0}\sin(\omega T_0)$$

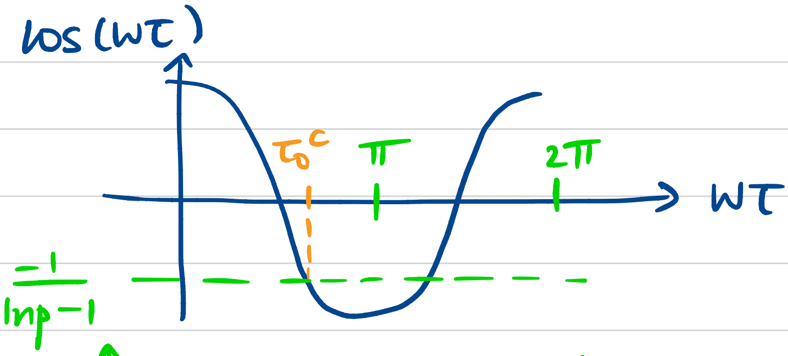
The critical value of the delay occurs when $\mu = 0$ — here let $T_0 = T_0^c$ and $\omega = \omega^c$.

$$0 = -1 - (\ln p - 1)\cos(\omega^c T_0^c)$$
$$\omega^c = (\ln p - 1)\sin(\omega^c T_0^c)$$

Alternatively,

$$\cos(w^c \tau_0^c) = \frac{-1}{lnp - 1} \quad (5)$$

$$\sin(w^c \tau_0^c) = \frac{w^c}{lnp - 1} \quad (6)$$



↑ NB since $lnp > 2$, $lnp - 1 > 1 \Rightarrow$ solutions exist.

Then $\tau_0^c = \frac{1}{w^c}\left[\pi - \cos^{-1}\left(\frac{1}{lnp-1}\right)\right]$

Also, (5), (6) $\Rightarrow (lnp - 1)^2 = 1 + (w^c)^2$

$$\therefore \quad \tau_0^c = \frac{\pi - \cos^{-1}\left(\frac{1}{lnp-1}\right)}{\left((lnp-1)^2 - 1\right)^{\frac{1}{2}}}$$

PS1 Q5

$$n_t + (1+\beta\phi) n_\phi = -\mu n \qquad \begin{aligned} n(\phi, 0) &= f(\phi) \\ n(0,t) &= 2n(1,t) \end{aligned}$$

Seek solution of the form $n(\phi, t) = e^{\gamma t} N(\phi)$

$$\Rightarrow \quad (1+\beta\phi)\frac{dN}{d\phi} = -(\mu+\gamma)N$$

$$\int_{N_0}^{N} \frac{1}{\tilde{N}}\, d\tilde{N} = -(\mu+\gamma)\int_0^{\phi} \frac{1}{1+\beta\phi}\, d\phi$$

$$\ln\left(\frac{N}{N_0}\right) = -\frac{(\mu+\gamma)}{\beta}\ln(1+\beta\phi)$$

$$\Rightarrow \quad N(\phi) = N(0)(1+\beta\phi)^{-(\mu+\gamma)/\beta}$$

Note that $N(0) = 2N(1)$ ∴

$$\cancel{N(0)} = 2\cancel{N(0)}(1+\beta)^{-(\mu+\gamma)/\beta}$$

ie $$\ln 2 = \frac{\mu+\gamma}{\beta}\ln(1+\beta)$$

For time-independent solutions, $\gamma = 0$ hence

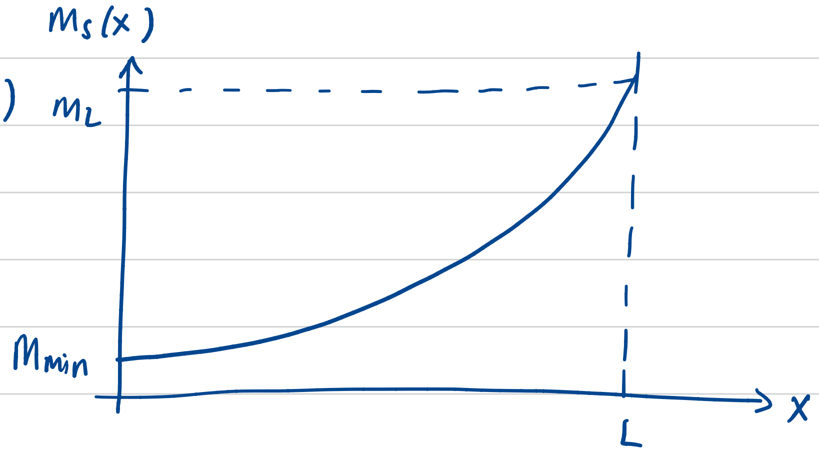
$$\boxed{\mu = \mu^*(\beta) = \frac{\beta \ln 2}{\ln(1+\beta)}}$$

PS2Q1

(a) $m_s(x)$ is a solution of $\begin{cases} 0 = Dm_{xx} - \lambda m & x \in (0, L) \\ m_x(0) = 0, & m(L) = m_L \end{cases}$

$$\therefore \quad m_s(x) = \frac{m_L \cosh(\sqrt{\lambda/D}\, x)}{\cosh(\sqrt{\lambda/D}\, L)}$$

(b)



minimum value at $x = 0$

$$m_{min} = \frac{m_L}{\cosh(\sqrt{\frac{\lambda}{D}}\, L)}$$

Case 1 : $0 < \theta < m_{min} < m_L \Rightarrow$ all cells of type I

Case 2 : $0 < m_{min} < m_L < \theta \Rightarrow$ all cells of type II

Case 3 : $0 < m_{min} < \theta < m_L$

NB critical domain size for cells of type II :

$$\theta = m_{min} = \frac{m_L}{\cosh(\sqrt{\frac{\lambda}{D}}\, L^*)}$$

$$\theta = \frac{m_L \cosh(\sqrt{\lambda/D}\, x_\theta)}{\cosh(\sqrt{\lambda/D}\, L)}$$

$$\Rightarrow \quad x_\theta = \sqrt{\frac{D}{\lambda}} \cosh^{-1}\left(\theta \cdot \frac{\cosh(\sqrt{\lambda/D}\, L)}{m_L} \right)$$

then $m_s(x) > \theta$ for $x_\theta < x < L$ ie type I

$m_s(x) < \theta$ for $0 < x < x_\theta$ ie type II

(c) $x_\theta = \sqrt{\frac{D}{\lambda}} \cosh^{-1}\left(\frac{\theta}{m_L} \cosh\left(\sqrt{\frac{\lambda}{D}} L\right)\right)$

To make progress, note that

① $\cosh x = \frac{1}{2}(e^x + e^{-x}) \approx \frac{1}{2} e^x$ for large x

② $\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1}) \approx \ln(2x)$ for large x

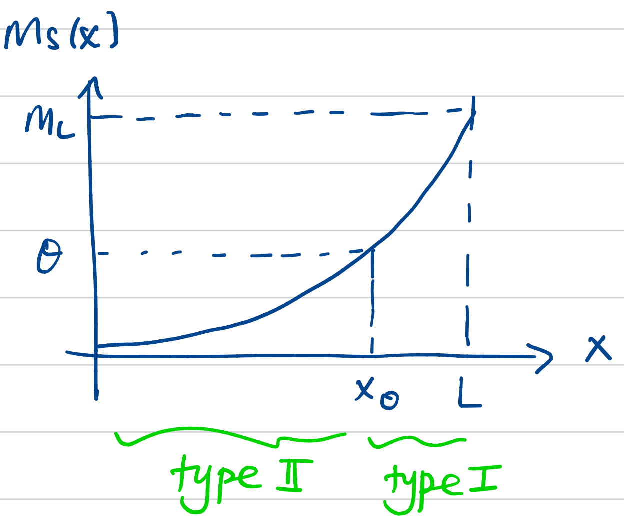
$$\Rightarrow x_\theta \approx \sqrt{\frac{D}{\lambda}} \ln\left(2 \frac{\theta}{m_L} \cdot \frac{1}{2} e^{\sqrt{\lambda/D} L}\right) \quad \text{as } L \to \infty$$

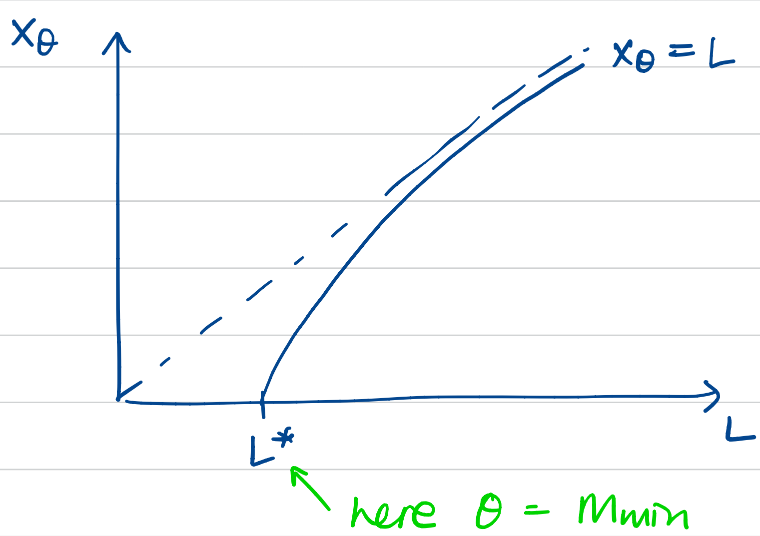
$$\therefore e^{\sqrt{\lambda/D}\, x_\theta} \approx \frac{\theta}{m_L} e^{\sqrt{\lambda/D} L}$$

$$\Rightarrow x_\theta \approx L + \sqrt{\frac{D}{\lambda}} \ln\left(\frac{\theta}{m_L}\right) \quad \text{as } L \to \infty$$

Assuming $m_{min} < x_\theta < m_L$ then, as $L \to \infty$, the width of the tissue with type I cells tends to a constant value of

$$L - x_\theta \approx \sqrt{\frac{D}{\lambda}} \ln\left(\frac{m_L}{\theta}\right) \quad //$$





PS2Q2

(a) $$\underbrace{u_t = Du_{xx}}_{\text{diffusion}} + \underbrace{ru\left(1-\frac{u}{K}\right)}_{\text{logistic growth}} \underbrace{-Eu}_{\text{harvesting}}$$

$$\underbrace{u=0 \text{ on } x=H}_{\text{over fishing outside } |x|<H.} \text{ and } \underbrace{\frac{\partial u}{\partial x}=0 \text{ at } x=0}_{\text{fish cannot leave sea and enter dry land}}$$

(b) spatially uniform steady states satisfy

$$ru\left(1-\frac{u}{K}\right)-Eu=0 \Rightarrow \underbrace{u^*=0,}_{\text{NB only this one satisfies the BCs.}} \left(1-\frac{E}{r}\right)K$$

(c) Let $u(x,t) = \varepsilon u_1(x)e^{\lambda t} + O(\varepsilon^2)$

$$\varepsilon\lambda u_1(x)e^{\lambda t} = \varepsilon u_1''(x)e^{\lambda t}D + r\varepsilon u_1(x)e^{\lambda t} - E\varepsilon u_1(x)e^{\lambda t} + O(\varepsilon^2)$$

$$\Rightarrow Du_1'' + \underbrace{(r-E-\lambda)}_{:=\mu}u_1 = 0 \quad \text{with} \quad \begin{aligned} u_1(H)&=0 \\ u_1'(0)&=0 \end{aligned}$$

(d) suppose $\mu<0$ and let $\frac{\mu}{D}=-q^2$ for some $q\in\mathbb{R}$.

$$u_1''-q^2u_1=0 \Rightarrow u_1(x) = Ae^{qx}+Be^{-qx}$$
$$u_1(H)=0=Ae^{qH}+Be^{-qH}$$
$$u_1'(0)=0=q(A-B)$$

Cannot satisfy both BCs unless $A=0=B \Rightarrow u_1\equiv 0$ $\cancel{\times}$

Suppose $\mu = 0 \Rightarrow u_1'' = 0 \Rightarrow u_1(x) = Ax + B$
Again, cannot satisfy BCs unless $A = 0 = B$
$\Rightarrow u_1 \equiv 0$ ※.

Suppose $\mu > 0$, and let $\frac{\mu}{D} = +q^2$ for some $q \in \mathbb{R}$.

Then
$$u_1(x) = A\cos(qx) + B\sin(qx)$$
$$u_1'(0) = 0 = qB \Rightarrow B = 0$$
$$u_1(H) = 0 = A\cos(qH)$$

Hence for a non-trivial solution we require $qH = \frac{(2n+1)\pi}{2}$
$n = 0, 1, \ldots$

$$\Rightarrow H\sqrt{\frac{r - E - \lambda}{D}} = \tfrac{1}{2}(2n+1)\pi$$
$n = 0, 1, 2, \ldots$

$$u_1(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{(2n+1)\pi x}{2H}\right) e^{\lambda_n t}$$

with
$$\lambda_n = r - E - D\left(\frac{(2n+1)\pi}{2H}\right)^2$$

(e) For instability of the extinction state we need $\lambda > 0$ for at least one value of n (i.e. $n = 0$)

$$\Rightarrow \text{Require } r - E - D\left(\frac{\pi}{2H}\right)^2 > 0$$

$$\Leftrightarrow H\sqrt{\frac{r-E}{D}} > \frac{\pi}{2} \quad //$$

(f) Ecological implications – we need sufficiently wide fishing zones near coastal areas in order to prevent species extinction.

PS2Q5

$$I_t = DI_{xx} + rIS - aI$$
$$S_t = -rIS + BS\left(1 - \frac{S}{S_0}\right)$$

(a) Let $I = I_0 u$, $S = S_0 V$, $t = T\tau$ and $x = Xy$

$$\frac{\partial}{\partial t} = \frac{1}{T}\frac{\partial}{\partial \tau} \qquad \frac{\partial}{\partial x} = \frac{1}{X}\frac{\partial}{\partial y}$$

$$\frac{I_0}{T}\frac{\partial u}{\partial \tau} = \frac{DI_0}{X^2}\frac{\partial^2 u}{\partial y^2} + rI_0S_0uv - aI_0u$$

$$\frac{S_0}{T}\frac{\partial v}{\partial \tau} = -rI_0S_0uv + BS_0v\left(1 - \frac{S_0v}{S_0}\right)$$

So that

$$\frac{\cancel{I_0}}{\cancel{T}}\frac{\partial u}{\partial \tau} = T\frac{D\cancel{I_0}}{X^2}\frac{\partial^2 u}{\partial y^2} + Tr\cancel{I_0}S_0uv - Ta\cancel{I_0}u$$

$$\frac{\cancel{S_0}}{\cancel{T}}\frac{\partial v}{\partial \tau} = -TrI_0\cancel{S_0}uv + TB\cancel{S_0}v\left(1 - \frac{\cancel{S_0}v}{\cancel{S_0}}\right)$$

Take $T = \frac{1}{rS_0}$

$\lambda = \frac{a}{rS_0}$

$X = \frac{rS_0}{D}$

$b = \frac{B}{rS_0}$

Take $I_0 = S_0$ to make this coefficient unity.

$$\Rightarrow \boxed{\begin{aligned} u_\tau &= u_{yy} + uv - \lambda u \\ v_\tau &= -uv + bv(1-v) \end{aligned}}$$

(b) Change to TW coordinates $z = y - c\tau$ with $u(y,\tau) = U(z)$, $v(y,\tau) = V(z)$.

$$u'' + cu' + uv - \lambda u = 0$$
$$cv' - uv + bv(1-v) = 0.$$

Far ahead of the wavefront ie as $z \to \infty$: $V \to 1$ (S)

$u \to 0$ (I)

Linearise by setting $u = \hat{u}$

$v = 1 - \tilde{v}$

Substitute into the TW equations:

$$0 = \hat{u}'' + c\hat{u}' + (1-\lambda)\hat{u} \quad \leftarrow \text{equation for } \hat{u} \text{ decouples}$$

$$0 = c\tilde{v}' + \hat{u} - b\tilde{v}$$

Let $\hat{u}' = \hat{w}$

$\hat{w}' = -c\hat{w} - (1-\lambda)\hat{u}$

Jacobian: $J = \begin{pmatrix} 0 & 1 \\ -(1-\lambda) & -c \end{pmatrix}$

Solutions $\Leftrightarrow$ $\det(J - \sigma I) = 0$ ie $\sigma^2 + c\sigma + (1-\lambda) = 0$

$$\Rightarrow \sigma_{\pm} = \tfrac{1}{2}\left[-c \pm \sqrt{c^2 - 4(1-\lambda)}\right]$$

Hence a TW may exist if $0 < \lambda < 1$ and, in this case,

$$c \geq c_{min} = 2\sqrt{1-\lambda} \;//$$

PS3 Q1

$$A_t = D_A A_{xx} + \frac{\rho A^2}{(1+KA^2)H} - \mu A = D_A A_{xx} + f(A,H)$$

$$H_t = D_H H_{xx} + \rho' A^2 - \nu H = D_H H_{xx} + g(A,H)$$

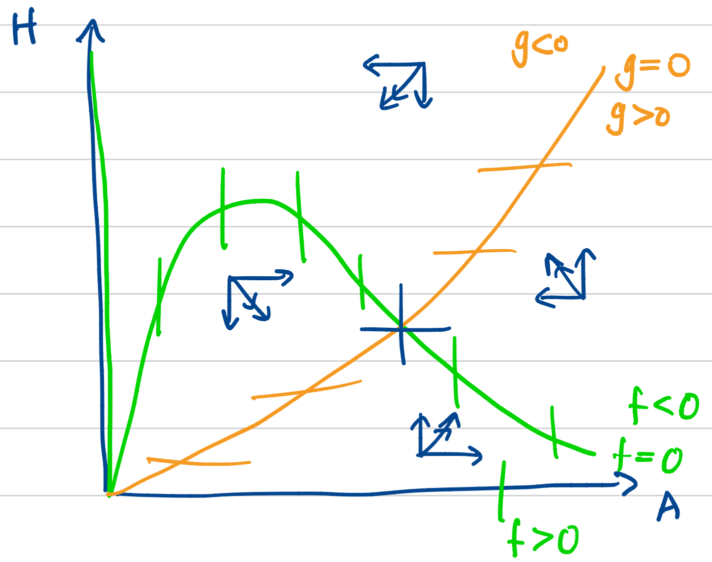
(a) Recall that for a DDI we require

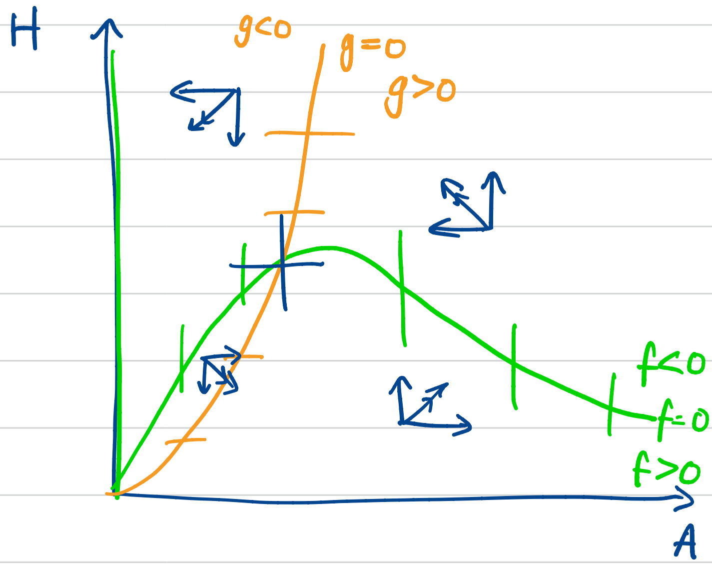
$$J = \left.\begin{pmatrix} f_A & f_H \\ g_A & g_H \end{pmatrix}\right|_{HSS} = \begin{pmatrix} + & - \\ + & - \end{pmatrix} \text{ or } \begin{pmatrix} + & + \\ - & - \end{pmatrix}$$

Null curves: $f(A,H) = 0 \Rightarrow A = 0$ or $H = \dfrac{\rho A}{\mu(1+KA^2)}$

$g(A,H) = 0 \Rightarrow H = \dfrac{\rho'}{\nu} A^2$

Two possible cases:





$$J = \begin{pmatrix} - & - \\ + & - \end{pmatrix}$$

NO DDI possible

$$J = \begin{pmatrix} + & - \\ + & - \end{pmatrix}$$

DDI possible

(b) Conditions for a DDI:

Steady state is $(\bar{A}, \bar{H})$ where $\bar{H} = \frac{\rho' \bar{A}^2}{\nu^2}$ and $\bar{H} = \frac{\rho \bar{A}}{\mu(1+K\bar{A}^2)}$.

$$f_A = \frac{2\rho\bar{A}}{(1+K\bar{A}^2)^2\bar{H}} - \mu \qquad f_H = \frac{-\rho\bar{A}^2}{(1+K\bar{A}^2)\bar{H}^2}$$

$$= \left(\frac{2}{(1+K\bar{A}^2)} - 1\right)\mu \qquad = \frac{\mu\bar{A}}{\bar{H}}$$

$$g_A = 2\rho'\bar{A} \qquad g_H = -\nu$$

(i) $f_A + g_H < 0 \Rightarrow \left(\frac{2}{(1+K\bar{A}^2)} - 1\right)\mu - \nu < 0$ //

(ii) $f_A g_H - f_H g_A > 0 \Rightarrow \mu\nu\left[3 - \frac{2}{1+K\bar{A}^2}\right] > 0$ //

(iii) $D_H f_A + D_A g_H > 2\sqrt{D_A D_H (f_A g_H - f_H g_A)} > 0$

$\Rightarrow D_H\left(\frac{2}{(1+K\bar{A}^2)} - 1\right)\mu - \nu D_A > 2\sqrt{D_A D_H \mu\nu\left(3 - \frac{2}{1+K\bar{A}^2}\right)} > 0$ //

PS3 Q3

$$n_t = \mu n_{xx} - (n c_x)_x$$

$$c_t = D c_{xx} + \frac{n}{(1+n)^2} - c$$

(a) spatially uniform steady state $\Big(n_0, \underbrace{\frac{n_0}{(1+n_0)^2}}_{c_0}\Big).$

(b) seek solutions of the linearised system of the form

$$(n, c) = (n_0, c_0) + e^{ikx + \sigma t}(N, K)$$

$|N|, |K| \ll n_0.$

Then

$$\sigma N = -\mu k^2 N + n_0 k^2 K$$

$$\sigma K = -Dk^2 K + \frac{(1-n_0)}{(1+n_0)^3} N - K$$

For non-trivial solutions we require $N, K \neq 0$

$$\Rightarrow \begin{vmatrix} \sigma + \mu k^2 & -n_0 k^2 \\ -\frac{(1-n_0)}{(1+n_0)^3} & \sigma + Dk^2 + 1 \end{vmatrix} = 0$$

$$\Rightarrow \sigma^2 + \underbrace{(1 + Dk^2 + \mu k^2)}_{A(k^2) > 0}\sigma + \underbrace{k^2\left[\mu + \mu D k^2 - \frac{n_0(1-n_0)}{(1+n_0)^3}\right]}_{B(k^2)} = 0$$

//

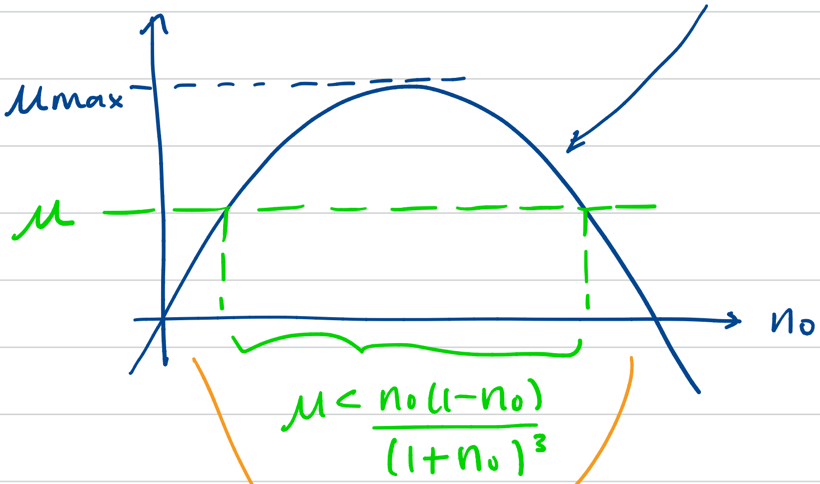
(c) $2\sigma_{\pm} = \underbrace{-A(k^2)}_{>0} \pm \sqrt{A(k^2)^2 - \underbrace{4B(k^2)}}$

hence require $B(k^2) < 0$ for some $k^2 \neq 0$ for instability.

$$B(k^2) = k^2 \left[\mu + \mu D k^2 - \frac{n_0(1-n_0)}{(1+n_0)^3} \right]$$

$B(k^2) < 0$ if $\mu + \mu D k^2 - \dfrac{n_0(1-n_0)}{(1+n_0)^3} < 0$

If $\mu < \dfrac{n_0(1-n_0)}{(1+n_0)^3}$



hence no spatial patterns if n_0 sufficiently large or small since then $B(k^2) > 0$.

PS4 Q1

(a) $$c_t = \frac{D}{r^2}\underbrace{(r^2 c_r)_r}_{\text{diffusion}} - \underbrace{\lambda}_{\text{uptake}} \qquad 0 \le r < R(t)$$ (tumour radially symmetric) (14)

$c_r = 0$ at $r = 0$ — nutrient profile symmetric about tumour centre. (15)

$C(R(t), t) = C^*$ — nutrient concentration at outer bdry. (16)

$$\frac{dV}{dt} = 4\pi \int_0^{R(t)} P(C) r^2 dr$$ change in tumour volume due to cell proliferation, with rate of cell proliferation dependent on nutrient levels. (17)

$R(0) = R_0$ initial radius. (18)

NB $V = \frac{4\pi}{3}R^3 \Rightarrow \frac{dV}{dt} = 4\pi R^2 \frac{dR}{dt}$

in detail: $$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi R^3\right) = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} P(C) r^2 \sin\theta \, dr \, d\theta \, d\phi$$
$$= 2\pi \int_0^{\pi} \sin\theta \, d\theta \int_0^R P(C) r^2 dr$$
$$\Rightarrow R^2 \frac{dR}{dt} = \int_0^R P(C) r^2 dr.$$

(b) Let $r = R_0 \rho \Rightarrow \frac{\partial}{\partial r} = \frac{1}{R_0}\frac{\partial}{\partial \rho}$, $t = \frac{\tau}{P_0} \Rightarrow \frac{\partial}{\partial t} = P_0 \frac{\partial}{\partial \tau}$

$$C = C^* c, \quad P(C) = P_0 p(c), \quad R = R_0 s(\tau)$$

(14) $\Rightarrow$ $$\frac{R_0^2 \cancel{C_0} P_0}{D} c_\tau = \frac{\cancel{D C_0}}{\cancel{R_0^3}}\left(\cancel{R_0^2}\rho^2 \frac{1}{\cancel{R_0}} c_\rho\right)_\rho \underbrace{- \frac{\lambda}{C_0}\frac{R_0^2}{D}}_{\mu}$$

$$\Rightarrow \frac{R_0^2 P_0}{D} c_\tau = (\rho^2 c_\rho)_\rho - \mu$$

Assuming $\frac{R_0^2 P_0}{D} \ll 1 \Rightarrow 0 = (\rho^2 c_\rho)_\rho - \mu$ (19)

$(15) \Rightarrow \quad c_\rho = 0 \text{ on } \rho = 0$ (20)

$(16) \Rightarrow \quad c = 1 \text{ on } \rho = s(\tau)$ (21)

$(17) \Rightarrow \quad \cancel{P_0}\cancel{R_0^3}\, s^2 \frac{ds}{d\tau} = \int_0^s \cancel{P_0}\cancel{R_0^3}\, p(c)\rho^2\, d\rho$

$$s^2 \frac{ds}{d\tau} = \int_0^{s(\tau)} p(c)\rho^2 d\rho \qquad (22)$$

$(18) \Rightarrow \quad s = 1 \text{ at } \tau = 0.$ (23).

(c) Solving (19)–(21) $\Rightarrow \quad c = 1 - \frac{1}{6}\mu(s^2 - \rho^2)$

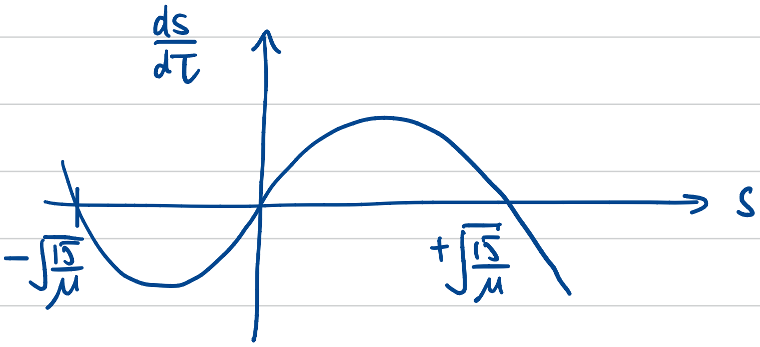
Then (22) gives $\quad s^2 \frac{ds}{d\tau} = \int_0^{s(\tau)} \{1 - \frac{1}{6}\mu(s^2 - \rho^2)\}\rho^2 d\rho$

$$= \left[\frac{1}{3}\rho^3 - \frac{1}{6}\mu s^2 \cdot \frac{1}{3}\rho^3 + \frac{1}{6}\mu \cdot \frac{1}{5}\rho^5\right]_0^s$$

$$= \frac{1}{3}s^3 - \frac{1}{45}\mu s^5$$

$\therefore \quad \frac{ds}{d\tau} = \frac{s}{3}\left(1 - \frac{1}{15}\mu s^2\right)$ with $s(0) = 1$. //

(d) Steady states: $\frac{ds}{d\tau} = 0 \Rightarrow s = 0, \sqrt{\frac{15}{\mu}}$.



$\frac{ds}{d\tau} = f(s)$

$f'(0) > 0 \Rightarrow$ tumour-free steady state unstable //

For $s = \sqrt{\frac{15}{\mu}}$, $c(\rho) = 1 - \frac{1}{6}\mu(s^2 - \rho^2) \Rightarrow c(0) = 1 - \frac{\mu s^2}{6}$

$$= 1 - \frac{5}{3} < 0$$

Hence steady state $s = \sqrt{\frac{15}{\mu}}$ physically unrealistic //

PS4 Q3

$$A \xrightarrow{\lambda} A+A, \quad A \xrightarrow{\mu} \phi, \quad \phi \xrightarrow{\beta} A$$

(a) $$\frac{d}{dt} P_n(t) = \lambda(n-1)P_{n-1} - \lambda n P_n + \mu(n+1)P_{n+1} - \mu n P_n + \beta P_{n-1} - \beta P_n$$

$$P_n(0) = \begin{cases} 1 & n=N_0 \\ 0 & \text{o/w} \end{cases}$$

(b) $$G(s,t) = \sum_{n=0}^{\infty} P_n(t) s^n$$

$$\begin{aligned}
\frac{\partial}{\partial t} G(s,t) &= \lambda\left(\sum_{n=0}^{\infty}(n-1)P_{n-1}s^n - \sum_{n=0}^{\infty} n P_n s^n\right) \\
&\quad + \mu\left(\sum_{n=0}^{\infty}(n+1)P_{n+1}s^n - \sum_{n=0}^{\infty} n P_n s^n\right) \\
&\quad + \beta\left(\sum_{n=0}^{\infty} P_{n-1}s^n - \sum_{n=0}^{\infty} P_n s^n\right) \\
&= \lambda\left(s^2\sum_{n=0}^{\infty}(n-1)P_{n-1}s^{n-2} - s\sum_{n=0}^{\infty} n P_n s^{n-1}\right) \\
&\quad + \mu\left(\sum_{n=0}^{\infty} n P_n s^{n-1} - s\sum_{n=0}^{\infty} n P_n s^{n-1}\right) \\
&\quad + \beta\left(s\sum_{n=0}^{\infty} P_n s^n - \sum_{n=0}^{\infty} P_n s^n\right) \\
&= \lambda s(s-1)\frac{\partial G}{\partial s} + \mu(1-s)\frac{\partial G}{\partial s} + \beta(s-1)G
\end{aligned}$$

with $G(s,0) = s^{N_0}$

(c) For $\lambda \neq \mu$, we differentiate the expression on the sheet to give

$$m(t) = \frac{1}{\mu-\lambda}\left[(N_0\mu - N_0\lambda - \beta)e^{(\lambda-\mu)t} + \beta\right] //$$

Note that for $\lambda = \mu$ then we effectively (at the mean level) have only to consider $\phi \xrightarrow{\beta} A$

$$\Rightarrow m(t) = N_0 + \beta t$$

(d) $\lambda > \mu$: at large times, the exponential term in the numerator dominates, hence

$$m(t) \sim \frac{1}{\lambda - \mu} \left[(\lambda - \mu) N_0 + \beta \right] e^{(\lambda - \mu) t}$$

(ie exponential growth)

$\lambda = \mu$: linear growth, $m(t) \sim \beta t$.

$\lambda < \mu$: $m(t) \longrightarrow \frac{\beta}{\mu - \lambda}$ (ie steady state).

$P_m = P_L + P_R$

PS4 Q5

(a).

$$p(A_n, t+dt) - p(A_n, t)$$

$$\mathbb{P}(\text{left}) = \frac{P_L}{P_L+P_R} = \frac{1}{2}(1-\rho) \qquad \rho = \frac{P_L - P_R}{P_L + P_R}$$

$$\mathbb{P}(\text{right}) = \frac{P_R}{P_L+P_R} = \frac{1}{2}(1+\rho) \qquad = \frac{P_L - P_R}{P_m}$$

$$= \tfrac{1}{2}(1+\rho) P_m dt\, p(A_{n-1}, O_n, t) - \tfrac{1}{2}(1-\rho) P_m dt\, p(O_{n-1}, A_n, t)$$
$$+ \tfrac{1}{2}(1-\rho) P_m dt\, p(O_n, A_{n+1}, t) - \tfrac{1}{2}(1+\rho) P_m dt\, p(A_n, O_{n+1}, t)$$
$$+ \tfrac{1}{2} P_b dt \left[p(A_{n-1}, O_n, t) + p(O_n, A_{n+1}, t) \right] - P_d\, p(A_n, t)$$

Simplify the movement terms using conservation statements of the form

$$p(A_n, A_m, t) + p(A_n, O_m, t) = p(A_n, t)$$

and invoke the mean-field assumption:

$$p(A_n, O_m, t) = p(A_n, t)\, p(O_m, t)$$

$$\frac{1}{dt}\left[p(A_n, t+dt) - p(A_n, t) \right]$$

$$= \tfrac{1}{2} P_m \left[p(A_{n-1}, t) - 2p(A_n, t) + p(A_{n+1}, t) \right]$$
$$+ \tfrac{1}{2} P_m \rho \Big[(1 - p(A_n, t)) \left[p(A_{n-1}, t) - p(A_{n+1}, t) \right]$$
$$+ p(A_n, t) \left[(1 - p(A_{n-1}, t)) - (1 - p(A_{n+1}, t)) \right] \Big]$$
$$+ \tfrac{1}{2} P_b \left[(1 - p(A_n, t)) \left[p(A_{n-1}, t) + p(A_{n+1}, t) \right] \right]$$
$$- P_d\, p(A_n, t)$$

(b)

Then, identify with a continuous lattice site occupancy probability by writing

$$p(A_n, t) = p(n\,dx, t)$$

$$\frac{1}{dt}\left[p(ndx,t+dt)-p(ndx,t)\right] =$$

$$\frac{1}{2}P_m\left[p((n-1)dx,t)-2p(ndx,t)+p((n+1)dx,t)\right]$$

$$+\frac{1}{2}P_m\,p\Big\{[1-p(ndx,t)][p((n-1)dx,t)-p((n+1)dx,t)]$$

$$+p(ndx,t)\left[[1-p((n-1)dx,t)-[1-p((n+1)dx,t)]\right]\Big\}$$

$$+\frac{1}{2}P_b\left[[1-p(ndx,t)]\right]\left[p((n-1)dx,t)+p((n+1)dx,t)\right]$$

$$-P_d\,p(ndx,t)$$

Taylor expand and take the limit as $dx, dt \to 0$:

$$\frac{\partial p}{\partial t}+o(dt)$$

$$=\frac{1}{2}P_m\left[p-dx\frac{\partial p}{\partial x}+\frac{dx^2}{2}\frac{\partial^2 p}{\partial x^2}-2p+p+dx\frac{\partial p}{\partial x}+\frac{dx^2}{2}\frac{\partial^2 p}{\partial x^2}+\dots\right]$$

$$+\frac{1}{2}P_m\,p\Big\{(1-p)\left[p-dx\frac{\partial p}{\partial x}+\frac{dx^2}{2}\frac{\partial^2 p}{\partial x^2}-p-dx\frac{\partial p}{\partial x}-\frac{dx^2}{2}\frac{\partial^2 p}{\partial x^2}+\dots\right]$$

$$+p\Big[\Big(1-p+dx\frac{\partial p}{\partial x}-\frac{dx^2}{2}\frac{\partial^2 p}{\partial x^2}+\dots\Big)$$

$$-\Big(1-p-dx\frac{\partial p}{\partial x}-\frac{dx^2}{2}\frac{\partial^2 p}{\partial x^2}+\dots\Big)\Big]\Big\}$$

$$+\frac{1}{2}P_b\Big\{(1-p)\left[p-dx\frac{\partial p}{\partial x}+\frac{dx^2}{2}\frac{\partial^2 p}{\partial x^2}+\dots+p+dx\frac{\partial p}{\partial x}+\frac{dx^2}{2}\frac{\partial^2 p}{\partial x^2}+\dots\right]\Big\}$$

$$-P_d\,p$$

$$\frac{\partial p}{\partial t} + o(\cancel{dt})$$

$$= \frac{1}{2}P_m\left[\cancel{p} - \cancel{dx\frac{\partial p}{\partial x}} + \boxed{\frac{dx^2}{2}\frac{\partial^2 p}{\partial x^2}} - \cancel{2p} + \cancel{p} + \cancel{dx\frac{\partial p}{\partial x}} + \boxed{\frac{dx^2}{2}\frac{\partial^2 p}{\partial x^2}} + \dots\right]$$

$$+\frac{1}{2}P_m p\Big\{(1-p)\Big[\cancel{p} - \boxed{dx\frac{\partial p}{\partial x}} + \cancel{\frac{dx^2}{2}\frac{\partial^2 p}{\partial x^2}} - \cancel{p} - \boxed{dx\frac{\partial p}{\partial x}} - \cancel{\frac{dx^2}{2}\frac{\partial^2 p}{\partial x^2}} + \dots\Big]$$

$$+p\Big[\Big(\cancel{1} - \cancel{p} + \boxed{dx\frac{\partial p}{\partial x}} - \cancel{\frac{dx^2}{2}\frac{\partial^2 p}{\partial x^2}} + \dots\Big)$$

$$-\Big(\cancel{1} - \cancel{p} - \boxed{dx\frac{\partial p}{\partial x}} - \cancel{\frac{dx^2}{2}\frac{\partial^2 p}{\partial x^2}} + \dots\Big)\Big]\Big\}$$

$$+\frac{1}{2}P_b\Big\{\underbrace{(1-p)\Big[\boxed{p}}_{1} - \cancel{dx\frac{\partial p}{\partial x}} + \underbrace{\boxed{\frac{dx^2}{2}\frac{\partial^2 p}{\partial x^2}}}_{2} + \dots + \underbrace{p}_{1} + \cancel{dx\frac{\partial p}{\partial x}} + \underbrace{\boxed{\frac{dx^2}{2}\frac{\partial^2 p}{\partial x^2}}}_{2} + \dots\Big]\Big\}$$

$$-P_d p$$

$$\Rightarrow \frac{\partial p}{\partial t} = Dp_{xx} - v(p(1-p))_x + P_b p(1-p) - P_d p$$

$$D = \lim_{dx\to 0} \frac{1}{2}P_m dx^2 \qquad V = \lim_{dx\to 0} P_m p\, dx$$

$$= \lim_{dx\to 0} (P_R - P_L)dx$$

Note: $P_m \sim \frac{1}{dx^2}$ $\quad p \sim dx$ or $P_R - P_L \sim \frac{1}{dx}$.

For 1 and 2: 1 $\Rightarrow P_b \sim O(1)$ so $\frac{\partial^2 p}{\partial x^2}$ terms $\to 0$.

In addition, write the birth-death terms as

$$(P_b - P_d)p\left(1 - \frac{P_b}{P_b - P_d}p\right) = rp\left(1 - \frac{p}{K}\right)$$

$r = P_b - P_d$ $\qquad K = \frac{P_b - P_d}{P_b}$ $\qquad$ (Assuming $P_b - P_d > 0$)

$$\therefore \quad p_t = Dp_{xx} - v(p(1-p))_x + rp\left(1 - \frac{p}{K}\right)$$