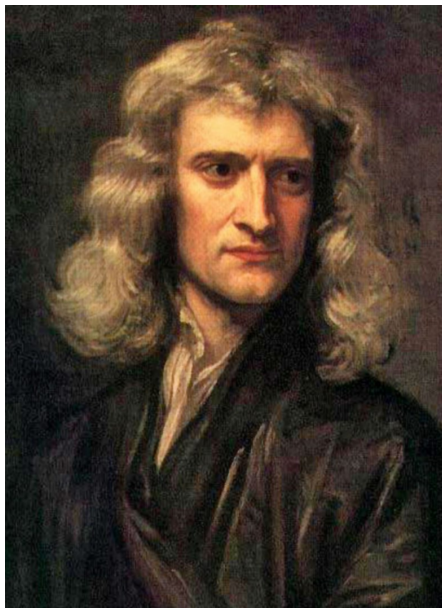


BO1.1. History of Mathematics
Lecture V
Newton's *Principia*

MT23 Week 3

Summary

- ▶ Isaac Newton
(1642–1727)
- ▶ Kepler's laws,
Descartes' theory,
Hooke's conjecture
- ▶ The *Principia*
- ▶ Editions and influence
of the *Principia*



Newton summarised

Alexander Pope, 1730:

*Nature and Nature's Laws lay hid in Night.
God said, Let Newton be! and All was Light.*

Woolsthorpe Manor

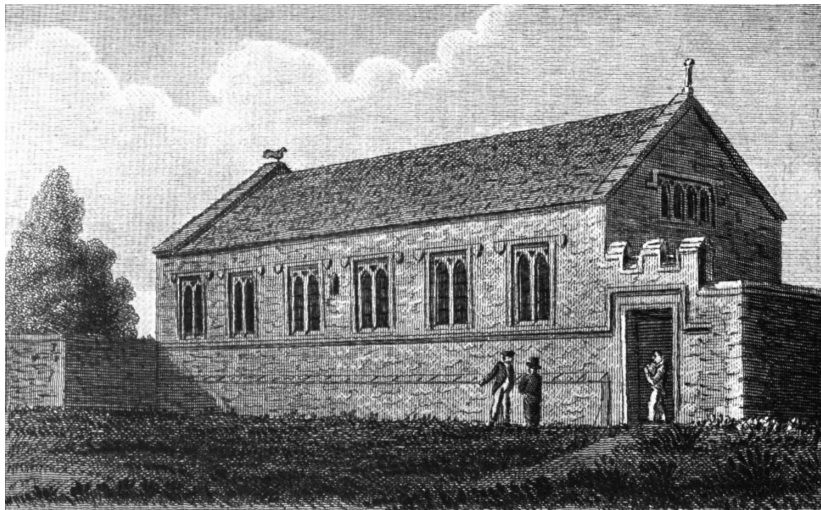


Newton born at Woolsthorpe Manor, 25th December 1642

Woolsthorpe Manor

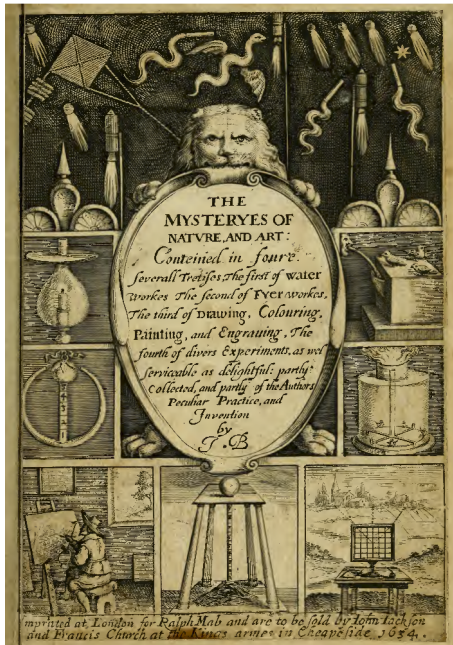


Grantham Grammar School



Now The King's School, Grantham

John Bate: *The mysteries of nature and art* (1634)



John Bate: *The mysteries of nature and art* (1634)

16

The first Booke



little from it on the top.
Having thus prepared the barrell, fit a good thick board unto it, so that it may slip easily up and down from the top of the barrell unto the bottom, mayle a lether about the edges of it, and another upon the top of it : on the underfide of it let there be fastned a good stiffe, but flexible spring of Steele, which may thrust the board from the bottom to the top of the barrell : let the foot of this spring rest upon a barre fastned across the bottom of the barrell: let this board also have tied at the middle a little rope of length sufficient. When you use it, bore a little hole in the table that you set it on, to put the rope thorow, and pull the rope down, which will contract the spring, and with it draw down the board : then poure in water at the basin untill the vessell be full : Note then, as you let slack the rope, the water will spirt out of the pipe, in the middle, and as you pull it straight, the water will run into the vessell againe. You may make birds, or divers images at the top of the pipe, out of which the water may break.

Another

of Water-workes.

17

Another manner of forcing water, whereby the water of any spring may be forced unto the top of a hill.

Let there be two hollow posts, with a succor at the bottom of each, also a succor nigh the top of each : let there be fastned unto both these posts a strong peece of



timber, having, as it were, a beame or scale pinned in it, and having two handles, at each end one. In the tops of

John Bate: *The mysteries of nature and art* (1634)

THE SECOND BOOKE,

Teaching most plainly, and withall
most exactly, the composing of all
manner of Fire-works for Triumph
and Recreation.

By J. B.



LONDON,

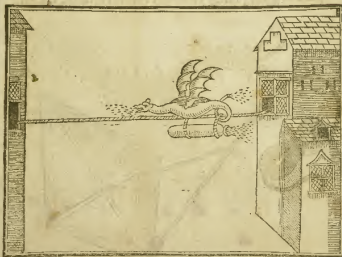
Printed by Thomas Harper for Ralph Mab. 1634.

of Fire-works.

79

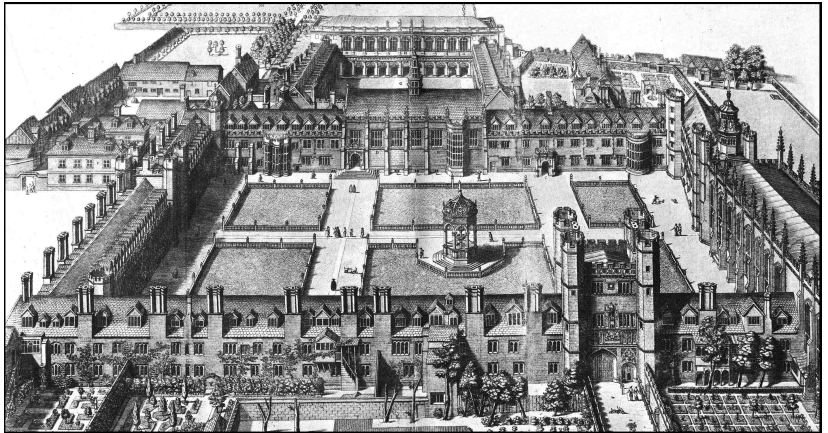
How to make flying Dragons.

The flying Dragon is somewhat troublesome to compose; it must be made either of dry and light wood, or crooked-lane plates, or of thin whalebones covered with Muscovie glasse, and painted over. In the body thereof, there must be a voyde cane to passe the rope through; unto the bottome of this cane must be bound one or two large Rockets, according as the bignesse and



weight of the Dragon shall require; the body must be filled with divers petras, that may consume it, and a sparkling receipt must be so disposed upon it, that being fired, it may burne both at the mouth and at the tail thereof;

Trinity College, Cambridge



Isaac Newton (1642–1727)

Newton's major interests:

1650s: model-building

1660s: optics; (pure) mathematics

1670s: alchemy, theology

1684+: mathematics

1696–1727: Warden of the Mint

Johannes Kepler (1571–1630)

Engaged to sift through the astronomical data gathered by the Danish astronomer Tycho Brahe (1546–1601)

Major works:

Astronomia nova (1609)

Harmonices mundi (1619)



Kepler: *Astronomia nova* (1609)

ASTRONOMIA NOVA ΑΙΤΙΟΛΟΓΗΤΟΣ,

SEV

PHYSICA COELESTIS,

tradita commentariis

DE MOTIBVS STELLE

MARTIS,

Ex observationibus G. V.

TYCHONIS BRAHE:

Iussu & sumptibus

RVDOLPHI II.

ROMANORVM

IMPERATORIS &c:

Plurium annorum pertinaci studio
elaborata Pragæ.

A. S. C. M. S. Mathematico

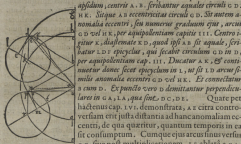
IOANNE KEPLERO.

Consensu C. R. S. privilegio speciali
ANNO MDCXII Dionysianæ cl. Id. c. ix.



DE MOTIBVS STELLE MARTIS

summi mirari idem per hanc etiam verissimam hypothesin tentant, falsa methodo, rursum de terum fumina trepidare cepi. In linea



prodidit distantiam A B iustam: exhibi indicia persuadebat: terminum ipsius A B alterum, quaerendum esse non in o c linea, quod verissimum, tamen erat, sed in o b linea puncto t u e centro a, distans a e, ducentem arcum t u, qui facit o b in r. Esse igitur a t secundum hanc persuasorem iusta distantia, sita & longitudine & t a c anomalia vere coequant. Manifestum est autem, quod s i r arcus facit o c lineam, loco superiorum scilicet in r. itaque anguli t a c & r a o differant quantitate t a r.

Erravi igitur, usurpata linea a i pro a f. Errorem primum experientia deprehendi. Nam cum explorassem quantum arcus a r a g tam per distantias omnes quas per arcum d a i, postea hinc arcus d a c in tempus convertere accommodassem angulum t a c non a g; tunc in superiori hemisphericali parte collegi per s t plus, in inferiori per 4 minus, quam dabam vicinis, latæ certæ. Itaque distantibus æquationibus a vero, cepti rursum accurate verissimas has distantias a t, & librationem Planetæ t a, de crimine, cuius facta mea methodo, quæ i pro r spectabat, erat rea. Quid malis! Ipsa veritas & rerum Natura repudiata, & exulare iussa, per posticum se furim rursum recepit intro, & sub habitu alieno a me recepta fuit. Missis inquam librationibus diametri t a, cepti revocare ellipses, omnino existimatis, me sic longe diversificationis, a librationibus, sequi hypothesin, cum plane coinciderent cap. sequente demonstrabatur nisi quod loque peccaveram prius in methodo, hac ratione fuerunt emendata, & r pro t, ita ut debeat, usurpatum.

Argumentatio mea talis fuit, qualis cap. x l i x. l. & l v i. Circulus cap. x l i i i. peccat excessu, ellipsis capitis x i v peccat defectu. Et sunt excessus ille & hic defectus æquales, Inter circulum vero & ellipsin nihil

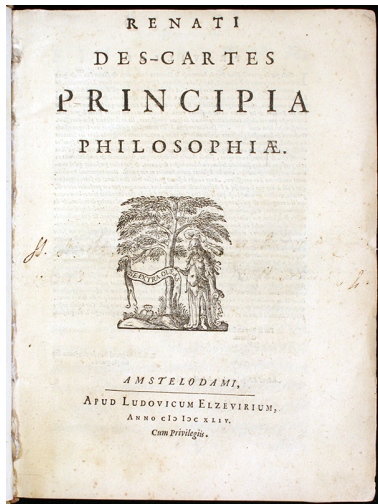
Kepler's laws

Kepler's laws of planetary motion (1609, 1619):

1. Planets move in elliptical orbits with the sun as focus
2. Planets sweep out equal areas in equal times
3. T^2 is proportional to R^3 (where T is time of one revolution, R is mean distance to sun)

All from empirical evidence

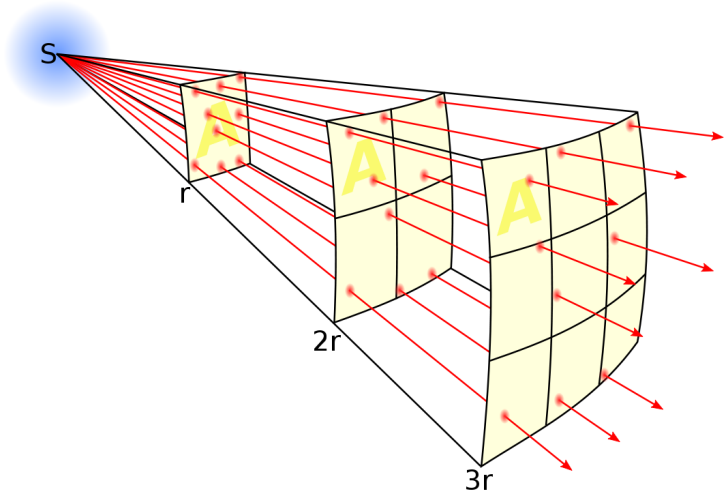
Descartes' theory



Descartes' views of planetary motion in *Principia philosophiæ* (1644):

- ▶ the sun is one star among many
- ▶ asserted that planets are carried round their suns by vortices of the surrounding 'ether'
- ▶ claimed that theory could also explain magnetism and static electricity

An inverse square law?

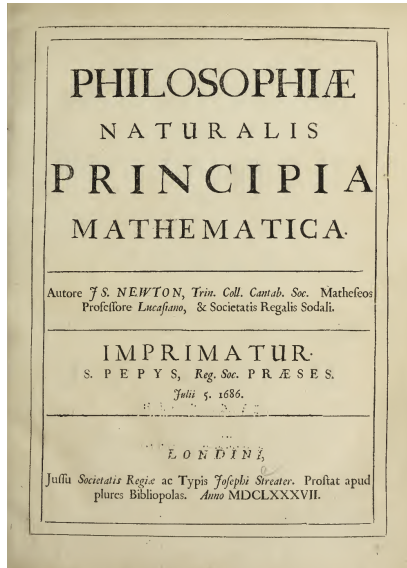


An inverse square law?

Speculations and calculations on an inverse square law of gravity:

- 1645 Ismaël Bullialdus refutes a claim of Kepler that 'gravity' drops off linearly with distance, instead suggesting an inverse square law
- c. 1679 Hooke corresponds with Newton and suggests that an inverse square law might lead to elliptical orbits
- 1684 Halley visits Newton and asks whether this might be the case; Newton sends a short treatise on motion to Halley
- 1687 Publication of Newton's *Principia* at Halley's expense

Isaac Newton: *The mathematical principles of natural philosophy* (London, 1687)



Contents of the *Principia*

- ▶ Eight definitions — of matter, motion, innate force, impressed force, acceleration, ...
- ▶ Three axioms or Laws of Motion (as taught in school), together with six corollaries
- ▶ Book I: The motion of bodies
- ▶ Book II: The motion of bodies in resisting media
- ▶ Book III: The system of the world

The laws of motion

[12]

AXIOMATA SIVE LEGES MOTUS

Lex. I.

Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.

Projectilia perseverant in motibus suis nisi quatenus a resistentia aeris retardantur & vi gravitatis impelluntur deorsum.

Trochus, cujus partes cohaerendo perpetuo retrahunt sese a motibus rectilinis, non cessat rotari nisi quatenus ab aere retardatur. Majora autem Planetarum & Cometarum corpora motus suos & progressivos & circulares in spatiis minus resistentibus factos conservant diutius.

Lex. II.

Mutationem motus proportionalem esse vi motrici impressae, & fieri secundum lineam rectam qua vis illa imprimatur.

Si vis aliqua motum quemvis generet, dupla duplum, tripla tripulum generabit, sive simul & semel, sive gradatim & successive impressa fuerit. Et hic motus quoniam in eandem semper plagam cum vi generatrice determinatur, si corpus antea movebatur, motui ejus vel conspiranti additur, vel contrario subducitur, vel oblique oblique adjicitur, & cum eo secundum utriusq; determinationem componitur.

Lex. III.

[13]

Lex. III.

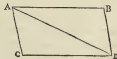
Actioni contrariam semper & aequalem esse reactionem: sive corporum duorum actiones in se mutuo semper esse aequales & in partes contrarias dirig.

Quicquid premit vel trahit alterum, tantundem ab eo premitur vel trahitur. Siquis lapidem digito premit, premitur & hujus digitus a lapide. Si equus lapidem funi allegatum trahit, retrahetur etiam & equus aequaliter in lapidem: nam funis utrinq; distentus eodem relaxandi se conatu urgebit Equum versus lapidem, ac lapidem versus equum, tantumq; impedit progressum unius quantum promovet progressum alterius. Si corpus aliquod in corpus aliud impingens, motum ejus vi sua quomodocumq; mutaverit, idem quoque vicissim in motu proprio eandem mutationem in partem contrariam vi alterius (ob aequalitatem pressiois mutuae) subibit. His actionibus aequales sunt mutationes non velocitatum sed motuum, (scilicet in corporibus non aliunde impeditis:) Mutationes enim velocitatum, in contrarias itidem partes factae, quia motus aequaliter mutantur, sunt corporibus reciproce proportionales.

Corol. I.

Corpus viribus conjunctis diagonalem parallelogrammi eodem tempore describere, quo latera separatis.

Si corpus dato tempore, vi sola *M*, ferretur ab *A* ad *B*, & vi sola *N*, ab *A* ad *C*, compleatur parallelogrammum *ABDC*, & vi utraq; ferretur id eodem tempore ab *A* ad *D*. Nam quoniam vis *N* agit secundum lineam *AC* ipsi *BD* parallelam, haec vis nihil mutabit velocitatem accedendi ad lineam illam *BD* a vi altera genitam. Accedet igitur corpus eodem tempore ad lineam *BD* sive vi *N* imprimatur, sive non, atq; adeo in fine illius temporis reperietur alicubi in linea illa



Book I: Motion of bodies

Book I, Section I: On the method of first and last ratios

Lemma I: Quantities, and ratios of quantities, which [...] approach nearer to each other than by any given difference, become ultimately equal.

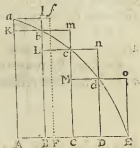
For suppose they are ultimately unequal, and their ultimate difference is D . Then they cannot approach nearer to equality than by that difference.

Book I, Lemma II

[27]

Lemma II.

Si in figura quavis $Aa cE$ rectis Aa , AE , & curva AcE comprehensa, inscribantur parallelogramma quotcumq; Ab , Bc , Cd , &c. sub basibus AB , BC , CD , &c. equalibus, & lateribus Bb , Cc , Dd , &c. figure lateri Aa parallelis contenta; & compleantur parallelogramma $aKbl$, $bLcm$, $cMdn$, &c. Dein horum parallelogrammorum latitudo minuitur, & numerus augeatur in infinitum: dico quod ultime rationes, quas habent ad se invicem figura inscripta $AKbLcMdD$, circumscripta $Aalbmcndoe$, & curvilinea $AabcdE$, sunt rationes equalitatis.



Nam figura inscripte & circumscripte differentia est summa parallelogrammorum $Kl + Lm + Mn + Do$, hoc est (ob æquales omnium bases) rectangulum sub unius basi Kb & altitudinum summa Aa , id est rectangulum $ABla$. Sed hoc rectangulum, eo quod latitudo ejus AB in infinitum minuitur, fit minus quovis dato. Ergo, per Lemma I, figura inscripta & circumscripta & multo magis figura curvilinea intermedia fiunt ultimo æquales. *Q. E. D.*

Lemma III.

Eadem rationes ultime sunt etiam equalitatis, ubi parallelogrammorum latitudines AB , BC , CD , &c. sunt inæquales, & omnes minuantur in infinitum.

Sit enim AF equalis latitudini maxima, & compleatur parallelogrammum $FAaf$. Hoc erit majus quam differentia figure inscripte & si use circumscripte, at latitudine sua AF

Lemma II: Ultimate equality of inscribed figure, circumscribed figure, and curved area

Motion under centripetal forces

[37]

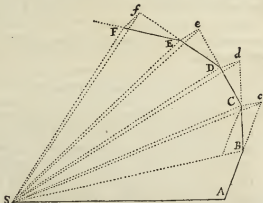
S E C T. II.

De Inventione Virium Centripetarum.

Prop. I. Theorema. I.

Areas quas corpora in gyros acta radiis ad immobile centrum virium ductis describunt, & in planis immobilibus consistere, & esse temporibus proportionales.

Dividatur tempus in partes æquales, & prima temporis parte describat corpus vi insita rectam AB . Idem secunda temporis parte, si nil impediret, recta pergeret ad c , (per Leg. I) describens lineam Bc æqualem ipsi AB , adeo ut radii AS , BS , cS ad centrum actis, confectæ forent æquales arcæ ASB , BSc . Verum ubi corpus venit ad B , agat vis centripeta impulsu unico sed magno, faciatq; corpus a recta Bc deflectere & pergere in recta BC . Ipsi BS parallela agatur cC occurrens BC in C , & completa secunda temporis parte, corpus (per Legum Corol. 1) reperietur in C , in eodem plano cum triangulo ASB . Junge SC , & triangulum SBC , ob parallelas SB , Cc , æquale erit triangulo SBe , atq; adeo etiam triangulo SAB . Simili argumento si



vis

Book I, Section II: Motion under centripetal forces.

Proposition I: Bodies constrained by a central force to orbit a fixed point move in a plane and sweep out equal areas in equal times.

(Kepler's second law)

NB. independent of the 'law of force' involved.

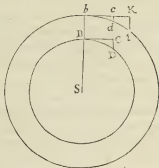
Book I, Section II: Circular motion

[41]

Prop. IV. Theor. IV.

Corporum que diversos circulos æquali motu describunt, vires centripetas ad centra eorundem circularum tendere, & esse inter se ut arcuum simul descriptorum quadrata applicata ad circulorum radios.

Corpora B, b in circumferentiis circulorum BD, bd gyrantia, simul describant arcus BD, bd . Quoniam sola vi insita describerent tangentibus BC, bc his arcibus æquales, manifestum est quod vires centripetæ sunt quæ perpetuo retrahunt corpora de tangentibus ad circumferentias circulorum, atq; adeo hæc sunt ad invicem in ratione prima spatorum nascentium CD, cd : tendunt vero ad centra circulorum per Theor. II, propterea quod aræ radiis descriptæ ponuntur temporibus proportionales. Fiat figura tkb figuræ DCB similis, & per Lemma V, lineola CD erit ad lineolam kt ut



arcus BD ad arcum bt : nec non, per Lemma XI, lineola nascentis tk ad lineolam nascentem dc ut bt quad. ad bd quad. & ex æquo lineola nascentis DC ad lineolam nascentem dc ut $BD \times bt$ ad bd quad. seu quod perinde est, ut $\frac{BD \times bt}{Sb}$ ad $\frac{bd}{Sb}$ quad., adeoq; (ob æquales rationes $\frac{bt}{Sb}$ & $\frac{BD}{Sb}$) ut $\frac{BD}{Sb}$ quad. ad $\frac{bd}{Sb}$ quad.

Q. E. D.

Corol. 1. Hinc vires centripetæ sunt ut velocitatum quadrata applicata ad radios circulorum.

Corol. 2. Et reciproce ut quadrata temporum periodicorum appli-

G

pli-

Book I, Sect. II, Prop. IV: Motion under centripetal forces: motion in a circle.

Corollary 1: For motion in a circle centripetal force is proportional to $\frac{v^2}{r}$.

Corollary 6: For motion in a circle Kepler's third law implies an inverse square law of force.

Book I, Section III: orbits that are conic sections

[50]

S E C T. III.

De motu Corporum in Conicis Sectionibus eccentricis.

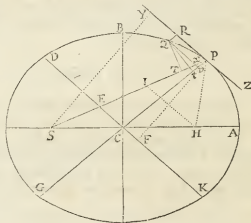
Prop. XI. Prob. VI.

Revolvatur corpus in Ellipsi: Requiritur lex vis centripetæ tendentis ad umbilicum Ellipseos.

Est Ellipseos superioris umbilicus S. Agatur SP secans Ellipseos tum diametrum DK in E, tum ordinatim applicatam Qv in x, & compleatur parallelogrammum QxPR. Patet EP æ-

qualem esse semi-axi majori AC, eo quod acta ab altero Ellipseos umbilico H linea HI ipsi EC parallela, (ob æquales CS, CH) æquantur ES, EI, adeo ut EP sensu summa sit ipsarum PS, PI, id est (ob parallelas HI, PR & angulos æquales IP R, HPZ) ipsorum PS, PH, quæ

conjuncta axem rotum 2 AC adæquant. Ad SP demittatur perpendicularis QT, & Ellipseos latere recto principali (seu $\frac{2 BC \text{ quad.}}{AC}$) disce L, erit $LxQR$ ad $LxPv$ ut QR ad Pv ; id est ut PE (seu AC) ad PC: & $LxPv$ ad GvP ut L ad G ; &

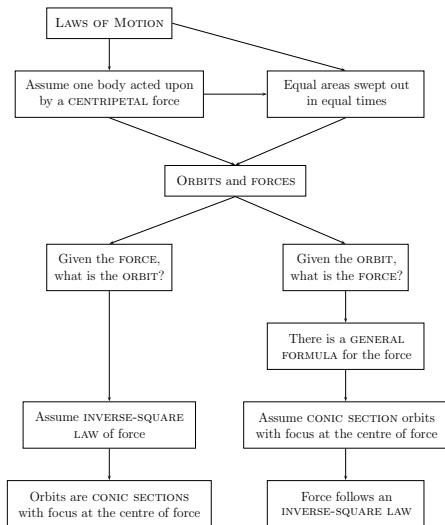


Proposition XI: Motion under centripetal forces: Kepler's First Law (orbit is an ellipse with sun at focus) implies an inverse square law of force.

Proposition XII: Motion under centripetal forces: hyperbolic orbit implies an inverse square law of force.

Proposition XIII: Motion under centripetal forces: parabolic orbit implies an inverse square law of force.

Book I, Sections II and III summarised



Book I, later sections

More mechanics of motion:

- ▶ converses: an inverse square law of force implies that orbits are conic sections;
- ▶ trajectories;
- ▶ much more besides.

All treated geometrically

Books II and III

Book II: Motion of bodies in resisting media:

Conclusion: "... it is manifest that the planets are not carried round in corporeal vortices ..." (Scholium to Proposition LIII)

Book III: The system of the world:

- ▶ Reconciliation of observation and theory
- ▶ Shape of the earth (correct?)
- ▶ Motion of the moon (wrong)
- ▶ Prediction of tides
- ▶ Comets

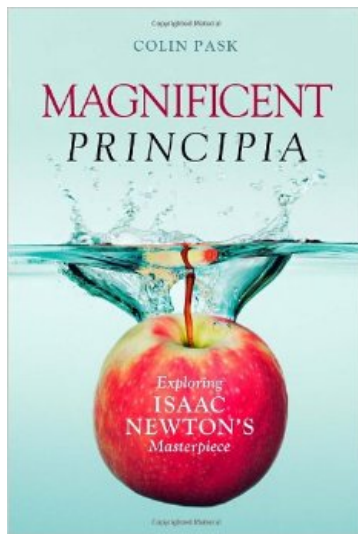
Influence of the *Principia*

Principia showed how mathematical methods could be used to study physical, especially but not exclusively, cosmological phenomena.

New ways of thinking: for example the method of ultimate ratios, though expressed geometrically, came close to a modern concept of limits.

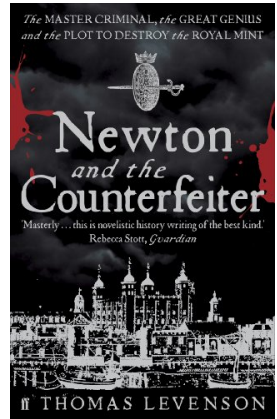
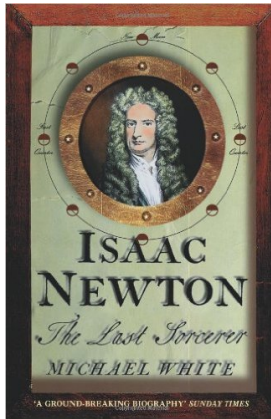
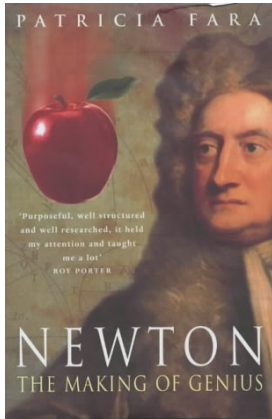
Predictions could be verified by observation and experiment — verified (after some controversy) in the case of the shape of the earth, contradicted in the case of the motion of the moon.

For more on the *Principia*...



(Colin Pask, *Magnificent Principia*, Prometheus Books, 2013)

Three (very different) books among many...



And a lecture given at Gresham College:

<https://www.gresham.ac.uk/lectures-and-events/isaac-newtons-world>