BO1.1 History of Mathematics
Lecture VI
Successes of and difficulties with the calculus: the 18th-century beginnings of 'rigour'

MT23 Week 3

Summary

- Publication and acceptance of the calculus
- Some successes of the calculus
- Functions
- Problems with the calculus
- Some responses: beginnings of 'rigour' in Analysis

Reminder: a comparison from lecture IV

Newton (1664–65):	Leibniz (1673–76):
rules for quadrature	rules for quadrature
rules for tangents	rules for tangents
‘fundamental theorem’	‘fundamental theorem’
dot notation	‘modern’ notation
physical intuition:	algebraic intuition
rates of change	rules and procedures
PROBLEM:	PROBLEM:
vanishing quantities o	vanishing quantities du, dv, ...

Newton's publication (or not) of his calculus

1669: 'De analysi' shown to Barrow and Collins

1671: 'Treatise on fluxions and infinite series' withdrawn before publication

1676: two long letters to Leibniz, plus a coded message

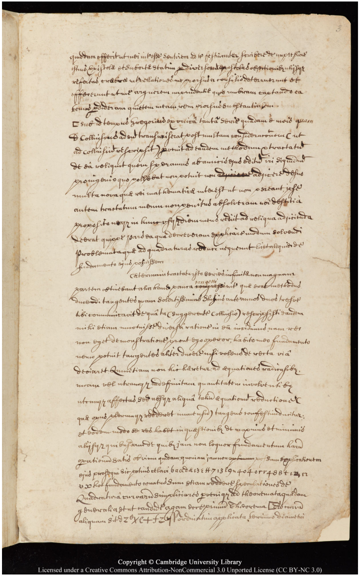
1685: partial publication of the letters to Leibniz by Wallis in his *Treatise of algebra*

1693: further partial publication by Wallis in his *Opera mathematica*

1694: some material written up by David Gregory, but not published

1704: 'Treatise of quadrature' appended to published *Opticks*

Newton's coded message



CUL MS Add.3977 f. 3r

Letter from Isaac Newton to Henry Oldenburg, 24 October 1676 ('Epistola posterior')

"The foundation of these operations is evident enough, in fact; but because I cannot proceed with the explanation of it now, I have preferred to conceal it thus: $6accdae13eff7i3l9n4o4qrr4s8t12vx$."

"Data aequatione quotcunque fluentes quantitates involvente, fluxiones invenire: et vice versa."

= "Given an equation involving any number of fluent quantities, to find the fluxions: and vice versa."

Leibniz's publication of his calculus

1680s: Papers in *Acta eruditorum* (journal founded 1682)

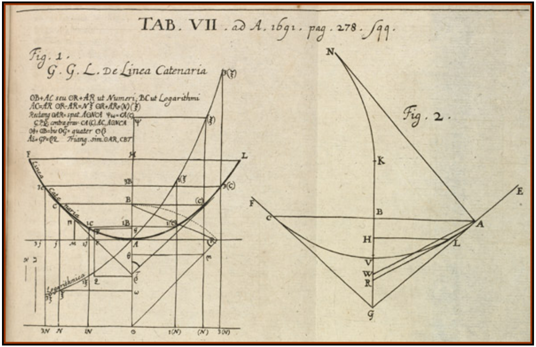
1691: Bernoulli brothers (Johann and Jacob) begin to apply Leibniz' methods

1696: Exposition by L'Hôpital based on teachings of Johann Bernoulli

Challenge problems

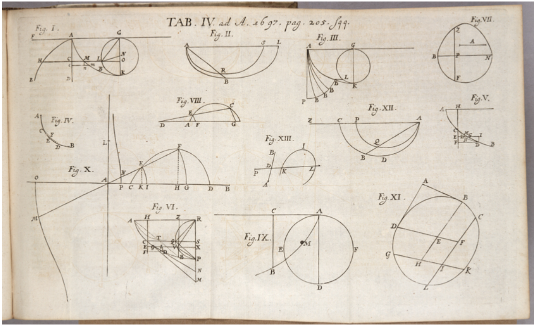
1687: Isochrone — curve of uniform descent
(posed by Leibniz; solved by Jacob Bernoulli)

1691: Catenary — curve of a hanging chain
(posed by Jacob Bernoulli; solved by Johann Bernoulli, Huygens, Leibniz)



Leibniz' and Huygens' solutions, *Acta eruditorum*, 1691.

Challenge problems



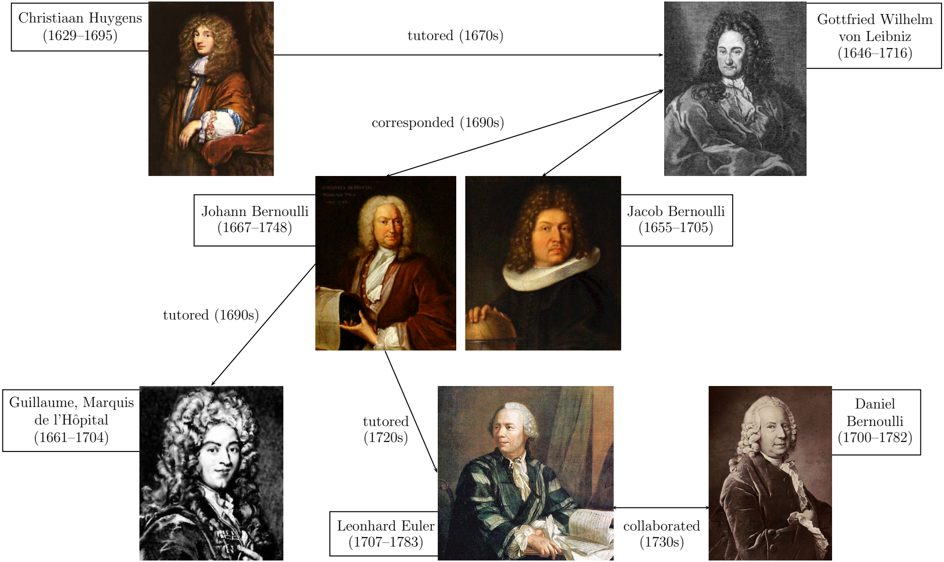
Solutions by Johann & Jacob Bernoulli, l'Hospital, and Newton, *Acta eruditorum*, 1696.

1696: Brachistochrone — curve of fastest descent (posed by Johann Bernoulli; shown to be cycloid by Jacob Bernoulli, Leibniz, Newton, l'Hôpital)

1697: Isoperimeter problems — find curve of given length that maximises a certain integral (classical problem; variant posed by Jacob Bernoulli, solved by him 1701)

And many others

People and connections



Leonhard Euler (1707–1783): a major 18th-century figure

1707: Euler born in Basel

1720: attended University of Basel, taught by Johann Bernoulli

1727: left Basel for Saint Petersburg with Daniel Bernoulli

1741: invited to Frederick the Great's new Academy in Berlin

1766: returned to St Petersburg

1783: died in St Petersburg

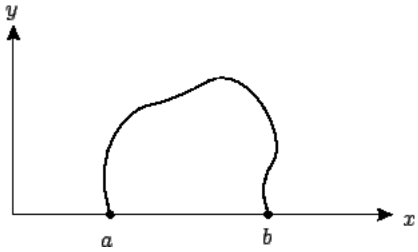
Influence of the challenge problems

These challenge problems and others helped to

- consolidate and validate Leibnizian calculus
- introduce new questions about ‘functions’, ‘differentiability’, ‘continuity’, ...

Functions: isoperimeter problem

Classical Problem (Virgil's *Aeneid*): Find the closed curve of given length L that maximises the area enclosed.

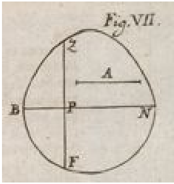


Modern Formulation: Find a function f and corresponding curve $y = f(x)$ between $(a, 0)$ and $(b, 0)$ of given length L (where $L > b - a$) that maximises the area beneath it.

But what is meant by 'function'?

Functions: isoperimeter problem

Isoperimeter problem posed by Jacob Bernoulli to Johann Bernoulli, May 1697, verbally and geometrically (ratio and proportion)



December 1697: problem rephrased by Johann in terms of powers

Solved by Johann in June 1698; published in 1706, with problem phrased in terms of functions (undefined)

In 1718, gave the following definition:

> *Here one calls a function of a variable magnitude, a quantity composed in any manner possible from this variable magnitude and constants.*

(See *Mathematics emerging*, §9.1.1.)

Functions: the wave equation

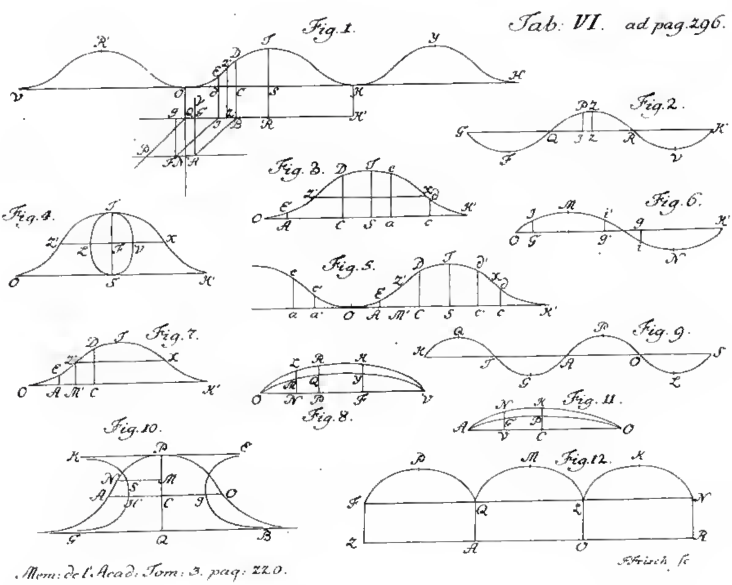
Another success of calculus: the wave equation

$$\frac{\partial^2 y}{\partial s^2} = c^2 \frac{\partial^2 y}{\partial t^2}$$

Solved by d'Alembert (1747) and Euler (1748) with solutions of the form

$$y = \Psi(s + ct) - \Phi(s - ct).$$

Functions: the wave equation



Functions: the wave equation

But which 'functions' are admissible as solutions?

Must they be

- continuous?
- differentiable?
- ... whatever these mean ...

What is a function?

Euler's definition of a function (1748):

> *A function of a variable quantity is an analytic expression composed in any way from that variable quantity and from numbers or constant quantities.*
>
> ...
>
> *Functions are divided into algebraic and transcendental; the former are those composed by algebraic operations alone, but the latter are those in which transcendental operations are involved.*

L. Euler: *Introductio in analysin infinitorum* (1748) [*Introduction to the analysis of the infinite*], available in translation, Springer-Verlag, 1988.

What is a function?

Euler's new definition of a function (1755):

> *Moreover, the quantities that depend in this way on others, so that the latter having changed, they themselves also undergo change, are usually called functions; which name opens up most generally all the ways in which one quantity may be determined from others involved with it.*

L. Euler: *Institutiones calculi differentialis* [*Foundations of differential calculus*] (1755)

What is a function?

In fact, this question took a long time to settle.

Nineteenth-century authors were split between those who preferred Euler's definition of 1748 and that of 1755 (see *Mathematics emerging*, §9.3).

The idea of a function as a mapping began to emerge at the end of the nineteenth century, in, for example, Dedekind's *Was sind und was sollen die Zahlen?* (1888), a text that we will come back to in a later lecture.

[For a list of different definitions of functions, ranging from 1718 to 1939, see: Dieter Rüthing, 'Some definitions of the concept of function from Joh. Bernoulli to N. Bourbaki', *The Mathematical Intelligencer* 6(4) (1984) 72–77]

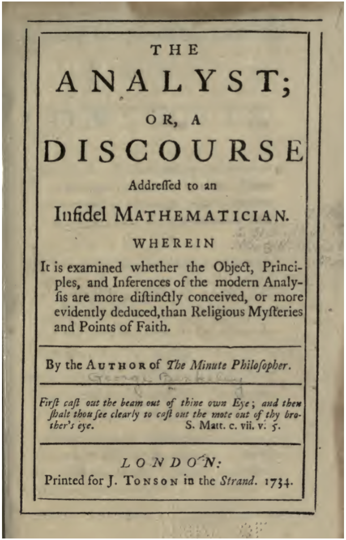
More problems: infinitely small quantities

Thomas Hobbes, *Six lessons to the Professors of Mathematicks* (1656):

> *The least Altitude is Somewhat or Nothing. If Somewhat, then the first character of your Arithmeticall Progression must not be zero;*
>
> ...
>
> *If Nothing, then your whole figure is without Altitude and consequently your Understanding nought.*

Wallis tried to provide further explanation in his *Due correction for Mr. Hobbes* (1656), but wasn't too concerned by the problems

More problems: infinitely small quantities



THE

ANALYST;

OR, A

DISCOURSE

Addreſſed to an

Infidel MATHEMATICIAN.

WHEREIN

It is examined whether the Object, Principles, and Inferences of the modern Analyſis are more diſtinctly conceived, or more evidently deduced, than Religious Myſteries and Points of Faith.

By the AUTHOR of *The Minute Philoſopher.*

Firſt caſt out the beam out of thine own Eye; and then ſhalt thou ſee clearly to caſt out the mote out of thy brother's eye. S. Matt. c. vii. v. 5.

LONDON:

Printed for J. TONSON in the *Strand.* 1734.

George Berkeley (1734)

Qu. 43: Whether an algebraist, fluxionist, geometrician, or demonstrator of any kind can expect indulgence for obscure principles or incorrect reasoning? And whether an algebraical note or species can at the end of a process be interpreted in a sense which could not have been substituted for it at the beginning?

Qu. 45: Whether, although geometry be a science, and algebra allowed to be a science, and the analytical a most excellent method, in the application nevertheless of the analysis to geometry, men may not have admitted false principles and wrong methods of reasoning?

Some responses to the difficulties

Guillaume Marquis de l'Hôpital, *Analyse des infiniment petits* (1696)

Colin Maclaurin, *A treatise of fluxions* (1742)

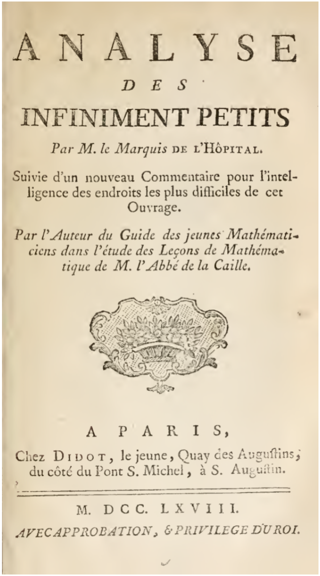
Maria Agnesi, *Instituzioni analitiche ad uso della gioventù italiana* (1748)

Leonhard Euler, *Institutiones calculi differentialis* (1755)

John Landen, *A discourse concerning the residual analysis* (1758)

Joseph-Louis Lagrange, *Théorie des fonctions analytiques* (1797)

Responses to the difficulties: l'Hôpital



ANALYSE
DES
INFINIMENT PETITS

Par M. le Marquis DE L'HÔPITAL.

Suivie d'un nouveau Commentaire pour l'intelligence des endroits les plus difficiles de cet Ouvrage.

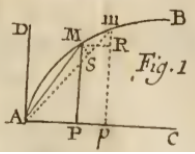
Par l'Auteur du Guide des jeunes Mathématiciens dans l'étude des Leçons de Mathématique de M. l'Abbé de la Caille.

A PARIS,
Chez DIDOT, le jeune, Quay des Auguſtins, du côté du Pont S. Michel, à S. Auguſtin.

M. DCC. LXVIII.
AVEC APPROBATION, & PRIVILEGE DU ROI.

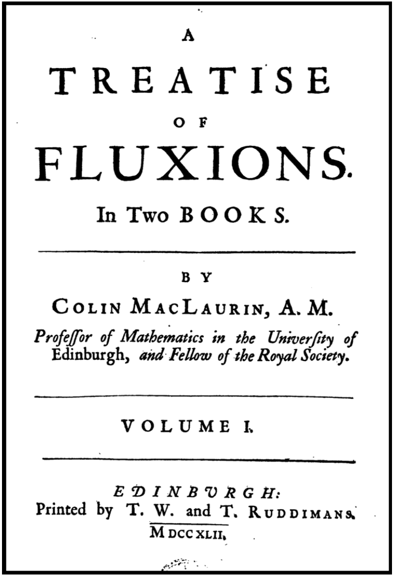
Guillaume, Marquis de l'Hôpital (1696)

Definition. The infinitely small part whereby a variable quantity is continually increased or decreased, is called the differential of that quantity.



Postulate. Grant that two quantities whose difference is an infinitely small quantity may be taken (or used) indifferently for each other: or (which is the same thing) that a quantity which is increased or decreased only by an infinitely small quantity may be considered as remaining the same.

Responses to the difficulties: Maclaurin



A

TREATISE

OF

FLUXIONS.

In Two BOOKS.

BY

COLIN MACLAURIN, A. M.

Professor of Mathematics in the University of Edinburgh, and Fellow of the Royal Society.

VOLUME I.

EDINBURGH:

Printed by T. W. and T. RUDDIMANS.

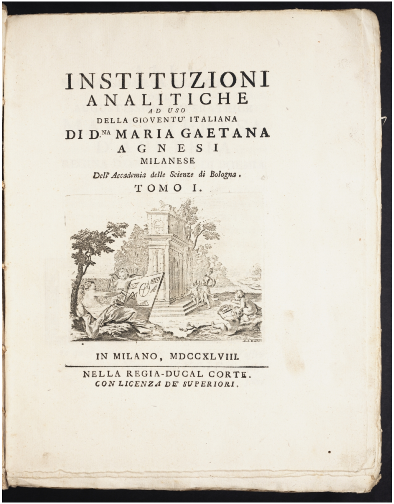
MDCCXLII.

Colin Maclaurin (1742)

- written in direct response to Berkeley
- attempted to prove all propositions of calculus by classical Archimedean methods

  (‘double contradiction’: derive a contradiction from the assumption that $a > b$; derive a contradiction from the assumption that $b > a$; then it must be the case that $a = b$).

Responses to the difficulties: Agnesi

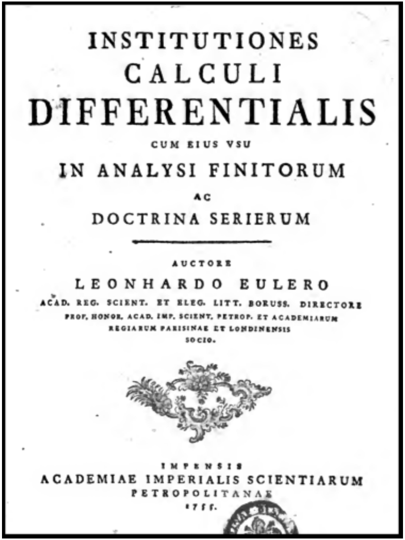

INSTITUZIONI
ANALITICHE
AD USO
DELLA GIOVENTU' ITALIANA
DI D.NA MARIA GAETANA
AGNESI
MILANESE
Dell' Accademia delle Scienze di Bologna.
TOMO I.

IN MILANO, MDCCXLVIII.
NELLA REGIA-DUCAL CORTE.
CON LICENZA DE' SUPERIORI.

Maria Agnesi (1748)

- systematic introduction to calculus
- sought to place calculus on an algebraic basis
- translated into English (1801)

Responses to the difficulties: Euler



INSTITUTIONES
CALCULI
DIFFERENTIALIS
CUM EIUS VSU
IN ANALYSI FINITORUM
AC
DOCTRINA SERIERUM

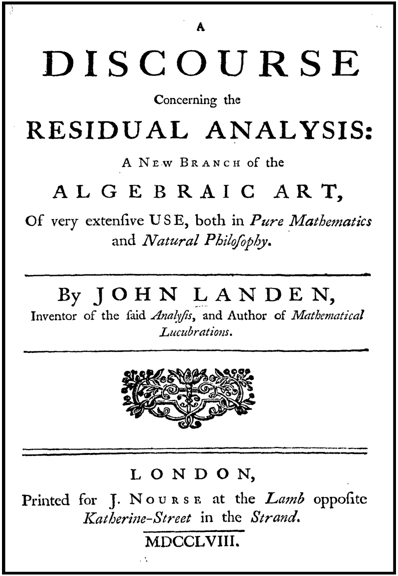
AUCTORE
LEONHARDO EULERO
ACAD. REG. SCIENT. ET ELEG. LITT. BORUSS. DIRECTORE
PROF. HONOR. ACAD. IMP. SCIENT. PETROP. ET ACADEMIARUM
REGIARUM PARISINAE ET LONDINENSIS
SOCIO.

IMPENSIS
ACADEMIAE IMPERIALIS SCIENTIARUM
PETROPOLITANAE
1755.

Leonard Euler (1755):

> *An infinitely small quantity is nothing other than a vanishing quantity, and is therefore really equal to 0.*
>
> . . .
>
> *If there occur different infinitely small quantities dx and dy, although both are equal to 0, nevertheless their ratio is not constant.*

Responses to the difficulties: Landen


A

DISCOURSE

Concerning the

RESIDUAL ANALYSIS:

A NEW BRANCH of the

ALGEBRAIC ART,

Of very extenfive USE, both in *Pure Mathematics* and *Natural Philofophy*.

By JOHN LANDEN,

Inventor of the faid *Analyfis*, and Author of *Mathematical Lucubrations*.

LONDON,

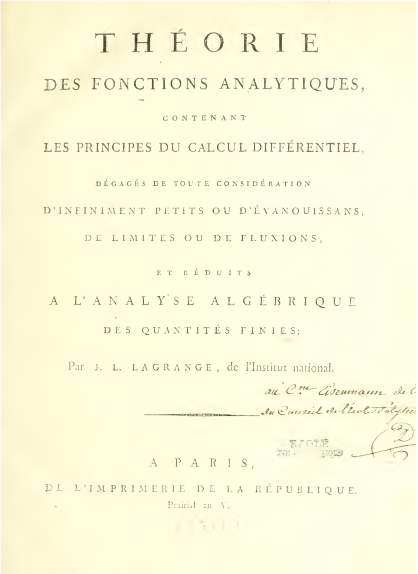
Printed for J. NOURSE at the *Lamb* oppofite *Katherine-Street* in the *Strand*.

MDCCLVIII.

John Landen (1758)

- 'Fluxions are not immediately applicable to algebraic quantities …'
- attempted a purely algebraic development of calculus

Responses to the difficulties: Lagrange



THÉORIE

DES FONCTIONS ANALYTIQUES,

CONTENANT

LES PRINCIPES DU CALCUL DIFFÉRENTIEL,

DÉGAGÉS DE TOUTE CONSIDÉRATION

D'INFINIMENT PETITS OU D'ÉVANOUISSANS,

DE LIMITES OU DE FLUXIONS,

ET RÉDUITS

A L'ANALYSE ALGÉBRIQUE

DES QUANTITÉS FINIES;

Par J. L. LAGRANGE, de l'Institut national.

A PARIS,

DE L'IMPRIMERIE DE LA RÉPUBLIQUE.

Prairial an V.

Joseph-Louis Lagrange (1797)

Another attempt to avoid 'infinitely small quantities'

(by taking functions to be defined by power-series expansions)

‘Rigour’ and ‘professionalisation’

Finally, note the increasing ‘professionalisation’ of mathematics in the eighteenth century:

- more university positions
  - Jacob Bernoulli in Basel;
  - Johann Bernoulli in Groningen, then Basel;
- new Academies in St Petersburg and Berlin provided positions with salaries
  - Euler at St Petersburg and Berlin;
  - d’Alembert in Paris;
  - Lagrange followed Euler to Berlin, later went to Paris;
- each Academy had its own ‘Mémoires’ or ‘Transactions’ enabling wider (and sometimes faster) circulation of new ideas.