

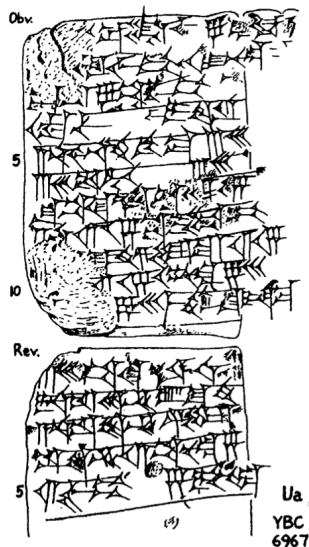
BO1.1. History of Mathematics  
Lecture IX  
Classical algebra: equation solving  
1800 BC – AD 1800

MT23 Week 5

# Summary

- ▶ Early quadratic equations
- ▶ Cubic and quartic equations
- ▶ Further 16th-century developments
- ▶ 17th century ideas
- ▶ 18th century ideas
- ▶ Looking back

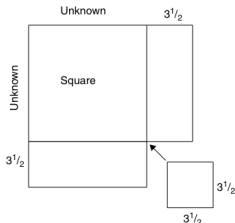
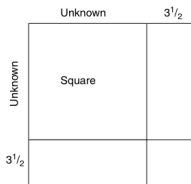
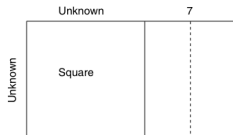
## Completing the square, c. 1800 BC



A Babylonian scribe, clay tablet BM 13901, c. 1800 BC:

A reciprocal exceeds its reciprocal by 7. What are the reciprocal and its reciprocal? Break in half the 7 by which the reciprocal exceeds its reciprocal, and  $3\frac{1}{2}$  will come up. Multiply  $3\frac{1}{2}$  by  $3\frac{1}{2}$  and  $12\frac{1}{4}$  will come up. Append 60, the area, to the  $12\frac{1}{4}$  which came up for you and  $72\frac{1}{4}$  will come up. What is the square-side of  $72\frac{1}{4}$ ?  $8\frac{1}{2}$ . Put down  $8\frac{1}{2}$  and  $8\frac{1}{2}$  and subtract  $3\frac{1}{2}$  from one of them; append  $3\frac{1}{2}$  to one of them. One is 12, the other is 5. The reciprocal is 12, its reciprocal 5.

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## Points to note

- ▶ We have used the word 'equation' without writing down anything in symbols
- ▶ Solution recipe derived from geometrical insight
- ▶ **Not** (explicitly) a general solution — but reader ought to be able to adapt the method
- ▶ Is this algebra? Geometrical algebra?

# Diophantus of Alexandria (3rd century AD)

## Arithmeticon Liber I.

47

Ad positiones erit primus  $\frac{1}{2}$ .  
secundus  $\frac{1}{3}$ . tertius  $\frac{1}{4}$ . quartus  
 $\frac{1}{5}$ . Abiciatur denominator par-  
tium. Erit itaque primus  $\frac{1}{2}$ . se-  
cundus  $\frac{1}{3}$ . tertius  $\frac{1}{4}$ . quartus  
 $\frac{1}{5}$ . & satisfaciunt questioni.

*Ἄρα πρῶτος ὁ μὲν πρῶτος μὲν πρῶτος ὁ δὲ πρῶτος  
πρῶτος ὁ δὲ πρῶτος μὲν πρῶτος ὁ δὲ πρῶτος  
πρῶτος ὁ δὲ πρῶτος μὲν πρῶτος ὁ δὲ πρῶτος*

### IN QUESTIONEM XXVI.

**F**AD ratio est huius questionis, quæ & precedentis. Quo fit infinitas reci-  
pit solutiones, & si determinanda sit ad vicinam, præscribendus est numerus in  
quo fieri debet æqualitas, tuncque operabimur vt in precedente traditum est. Quod  
autem denominatores abici iubet Diophantus, vt solutio in integris habeatur, id fit  
quia si inueni semel numeri questionis satisfaciens, per eundem multiplicetur vel  
diuidatur, producta eisdem & quotientes quælibet inueniuntur, cuius rei ratio est  
quam attingit Xilander, quia scilicet quærit numeri, partes proportionales vicissim  
dant & accipiunt, quæ autem partium cognominum eadem totorum inter se, ac vi-  
cissim est ratio. Vnde etiam colligi potest alius modus soluendi huiusmodi quæstio-  
nes, cum numero præscribitur in quo fiat æqualitas. Nam si quæritur prius soluat  
per operationem Diophanti, & numerus in quo fit æqualitas diuidatur per eum qui  
præscribitur, & per quotientem diuidatur item inuenti numeri per operationem  
Diophanti, habebuntur quærit numeri. Verbi gratia, si quærantur quatuor nume-  
ridanes & accipientes eisdem partes quæ requirit Diophantus, ita vt facta contri-  
butione quilibet repetatur 19, solus prius quæritur cum Diophanto, & inue-  
niuntur numeri 10, 21, 120, 114. Et numerus in quo fit æqualitas erit 119. Hunc ergo  
si diuidas per numerum præscriptum 19, t erit quotiens 6, per quem si diuidas il-  
ligatam inuentos numeros, sient 75, 46, 60, 57, quærit numeri. Posset etiam tam  
hæc quam præcedens paulo aliter proponi, requirendo scilicet facta noua contri-  
butione fiant numeri diuersi non æquales. Verbi gratia, si inueniendi quatuor nu-  
meri, vt prius dando sui trientem & accipiendo sextantem quarti fiat 6. Secundus  
dando sui quadrantem, & accipiendo trientem primi fiat 7. Tertius dando sui quin-  
tantem, & accipiendo quadrantem secundi fiat 12. Quartus dando sui sextantem, &  
recipiendo quintantem terti, fiat 13. Et tunc imitabimur artificium operationis quæ ad  
precedentem tradita est, hoc modo. Ponatur primus 3. N. cum ergo multatus suo trien-  
te & additus sextante quarti faciat 6. erit 6 - 2 N. sextans quartæ, & ipse quatuor 16 - 12  
N. vnde ablato sextante, manent 30. - 10 N. quæ cum triente terti debent  
facere 31. Igitur quintans terti est 10 N. - 7. Ideoque ipse tertius est 10 N. - 11,  
qui multatus quintante manet 40 N. - 18. debetque tunc cum quadrante secundi  
facere 14. Quare 42 - 40 N. est quadrans secundi, & ipse secundus 168 - 160 N.  
vnde ablato quadrante manent 126 - 110 N. quæ cum triente primi debent facere  
7, sed faciunt 126 - 119 N. hoc ergo æquatur 7, & fit 1 N. 1. Ad positiones pri-  
mi est 3, secundus 8, tertius 15, quartus 14.

### QUESTIO XXVII.

**I**NVENIRE tres numeros vt  
quilibet à reliquis duobus  
coniunctis partem imperatam  
accipiat, & fiant æquales. Sta-  
tutum sit primum à reliquis

*ΕΥΡΕΙΝ τρεῖς ἀριθμοὺς ὅπως  
ἕκαστος παρὰ τὴν λοιπὴν δύο ὡς  
ἐκὸς λαβὼν μέρους τὸν ἡμισυχόμενον  
καὶ συνάσται ἴσους. Ὁρισθέντος δὲ  
τοῦ πρώτου ἀπὸ τῶν λοιπῶν*

Problem I.27: Find two numbers  
such that their sum and product  
are given numbers

# Muḥammad ibn Mūsā al-Khwārizmī (c. 780–c. 850)

Noted six cases of equations:



1. Squares are equal to roots  
( $ax^2 = bx$ )
2. Squares are equal to numbers  
( $ax^2 = c$ )
3. Roots are equal to numbers  
( $bx = c$ )
4. Squares and roots are equal to numbers  
( $ax^2 + bx = c$ )
5. Squares and numbers are equal to roots  
( $ax^2 + c = bx$ )
6. Roots and numbers are equal to squares  
( $bx + c = ax^2$ )

# Muḥammad ibn Mūsā al-Khwārizmī (c. 780–c. 850)

السطح الاعظم وهو سطح دة وقد علمنا ان ذلك  
كله اربعة وستون واحد اضلاعه حليرة وهو  
ثمانية فاذا نقصنا من الثمانية مثل ربع العشرة مرتين  
من طرفي ضلع السطح الاعظم الذي هو سطح دة فهو  
خمسه بقي من ضلعه ثلثه وهو جذ ذلك للال  
وانما نصفنا العشرة الاجزاء وضربناها في مثاتها ووزنا  
ها على العدد الذي هو تسعة وثلاثون ليتم لنا بناء  
السطح الاعظم بما نقص من زوايا الاربعة لان  
كل عدد يضرب ربعه في مثله ثم في اربعة يكون  
مثل ضرب نصفه في مثله فاستغنينا بضرب  
نصف الاجزاء في مثاتها عن الربع في مثله ثم في اربعة  
وهذا صورته

	ب	
ج	ب	ا
	د	

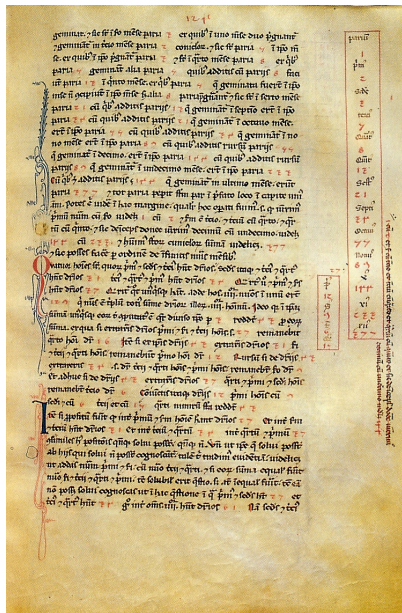
وله ايضا صورة اخرى تؤدي الى هذا وهي سطح  
اب وهو المال فاردنا ان نزيد عليه مثل عشرة

An algorithm for case (4) on  
the previous slide

# Leonardo of Pisa (Fibonacci) (c. 1175–c. 1240/50)

*Liber abaci* (or *Liber abbaci*),  
Pisa, 1202:

- ▶ included al-Khwārizmī's recipes
- ▶ geometrical demonstrations and lots of examples
- ▶ didn't go very far beyond predecessors, **but** began transmission of Islamic ideas to Europe



# Cubic equations (1)

Italy, early 16th century:

solutions to cubics of the form  $x^3 + px = q$

- ▶ found by Scipione del Ferro (or Ferro) (c. 1520)
- ▶ taught to Antonio Maria Fiore (pupil)
- ▶ and Annibale della Nave (son-in-law)
- ▶ rediscovered by Niccolò Tartaglia (1535)
- ▶ passed in rhyme to Girolamo Cardano (1539)

## Cubic equations (2)

$$x^3 + px = q$$

*When the cube with the things next after  
Together equal some number apart  
Find two others that by this differ  
And this you will keep as a rule  
That their product will always be equal  
To a third cubed of the number of things  
The difference then in general between  
The sides of the cubes subtracted well  
Will be your principal thing.*

(Tartaglia, 1546; see: *Mathematics emerging*, §12.1.1)

## Cubic equations (3)

Interpretation of Tartaglia's rhyme:

Find  $u, v$  such that

$$x^3 + px = q$$

$$u - v = q, \quad uv = \left(\frac{p}{3}\right)^3.$$

*When the cube with the things next after  
Together equal some number apart  
Find two others that by this differ  
And this you will keep as a rule  
That their product will always be equal  
To a third cubed of the number of things  
The difference then in general between  
The sides of the cubes subtracted well  
Will be your principal thing.*

Then

$$x = \sqrt[3]{u} - \sqrt[3]{v}$$

**NB:** In an equation

$y^3 + ay^2 + by + c = 0$  we can put  
 $y = x - \frac{a}{3}$  to remove the square  
term, so this solution is general.



## Cubic equations (4)

In modern terms, one of the solutions of the equation  $ax^3 + bx^2 + cx + d = 0$  has the form

$$x = \sqrt[3]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)} + \sqrt{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3} \\ + \sqrt[3]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)} - \sqrt{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3} - \frac{b}{3a}$$

with similar expressions (in **radicals**) for the remaining two roots

# Cardano's *Ars magna, sive de regulis algebraicis* (1545)

**HIERONYMI CAR  
DANI, PRÆSTANTISSIMI MATHE-  
MATICI, PHILOSOPHI, AC MEDICI,  
ARTIS MAGNÆ,  
SIVE DE REGVLIS ALGEBRAICIS,  
Lib. unus. Qui & totius operis de Arithmetica, quod  
OPVS PERFECTVM  
inscriptus, est in ordine Decimus.**



**H**Abes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cos-  
ta vocant) nouis adinventum omnibus, ac demonstrationibus ab Authore ita  
locupletatas, ut pro pauculis ante uulgo trinitis, tam sepe uagina existerint. Neq-  
ue solum, ubi tunc numerus alter, aut duo uni, uerum etiam, ubi duo duobus,  
aut tres uni equales fuerint, nodum explicant. Hunc autem librum ideo seors-  
im edere placuit, ut hoc altissimum, & plane inexcussibile totius Arithmeti-  
ce thesaurum in lucem eruo, & quasi in theatro quodam omnibus ad spectan-  
dum exposito, Lectori innotaret, ut reliquos Operis Perfecti libros, qui per  
Tomo eduntur, tanto audius amplectantur, ac minore fastidio perdiscant.

**HIERONYMI CARDANI**

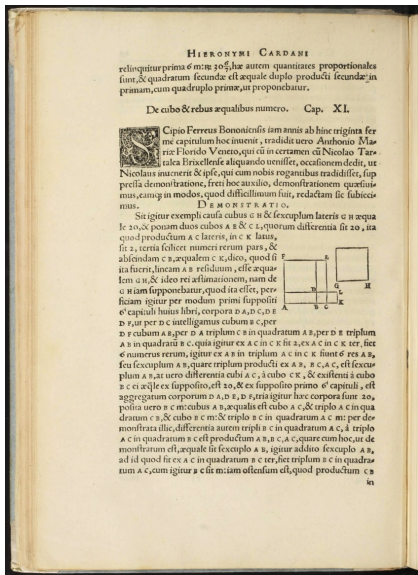
**3** Numerus & positio, & ignota quantitas.  
**4** Numerus & quadrati positionis, & ignota quantitas, seu numerus  
& quadrati positionis, seu numerus cu quadrato positionis  
quantitatis ignota, seu numerus & productum ex positione in quan-  
tatem ignotam, cum altera earum, uel cum quadrato unius earum.

**Capitula primitiua.**

**Capitula deriuatiua.**

- |  |  |
|--|--|
| <b>1</b> Numerus aequalis qd & rebus.                  | <b>1</b> Numerus aequalis qd qd & qd.      |
| <b>2</b> Numerus & res aequalia quadrato.              | <b>2</b> Numerus aequalis cu qd & cub.     |
| <b>3</b> Numerus & qd equalia rebus.                   | <b>3</b> Numerus & qd aequalis qd qd.      |
| <b>4</b> Numerus aequalis cubo & rebus.                | <b>4</b> Numerus & cub aequalis cub qd.    |
| <b>5</b> Numerus & res aequalia cubis.                 | <b>5</b> Numerus aequalis qd & cub qd.     |
| <b>6</b> Numerus & cubus aequalis rebus aequo prima.   | <b>6</b> Numerus aequalis cub & cubo cubi. |
| <b>7</b> Numerus & cub equalia rebus qd secunda.       | <b>7</b> Numerus & qd equalia cub qd.      |
| <b>8</b> Numerus equalis quadrato & cubo.              | <b>8</b> Numerus & cub equalia cub cubi.   |
| <b>9</b> Numerus & quadratum equalia cubo.             | <b>9</b> Numerus & cub qd equalia cub qd.  |
| <b>10</b> Numerus & cub equalia quadrato qd prima.     | <b>10</b> Numerus & cub qd equalia cub qd. |
| <b>11</b> Numerus & cub equalia quadrato qd secunda.   | <b>11</b> Numerus & cub qd equalia cub qd. |
| <b>12</b> Numerus equalis rebus quadrato & cubo.       | <b>12</b> Numerus & cub qd equalia cub qd. |
| <b>13</b> Numerus & res equalia quadrato qd prima.     | <b>13</b> Numerus & cub qd equalia cub qd. |
| <b>14</b> Numerus & res & qd equalia cubo.             | <b>14</b> Numerus & cub qd equalia cub qd. |
| <b>15</b> Numerus & qd equalia rebus & cub qd prima.   | <b>15</b> Numerus & cub qd equalia cub qd. |
| <b>16</b> Numerus & qd equalia rebus & cub qd secunda. | <b>16</b> Numerus & cub qd equalia cub qd. |
| <b>17</b> Numerus & qd equalia rebus & cub qd prima.   | <b>17</b> Numerus & cub qd equalia cub qd. |
| <b>18</b> Numerus & qd equalia rebus & cub qd secunda. | <b>18</b> Numerus & cub qd equalia cub qd. |
| <b>19</b> Numerus & qd equalia rebus & cub qd prima.   | <b>19</b> Numerus & cub qd equalia cub qd. |
| <b>20</b> Numerus & qd equalia rebus & cub qd secunda. | <b>20</b> Numerus & cub qd equalia cub qd. |
| <b>21</b> Numerus & qd equalia rebus & cub qd prima.   | <b>21</b> Numerus & cub qd equalia cub qd. |
| <b>22</b> Numerus & qd equalia rebus & cub qd secunda. | <b>22</b> Numerus & cub qd equalia cub qd. |
| <b>23</b> Numerus & qd equalia rebus & cub qd prima.   | <b>23</b> Numerus & cub qd equalia cub qd. |
| <b>24</b> Numerus & qd equalia rebus & cub qd secunda. | <b>24</b> Numerus & cub qd equalia cub qd. |
| <b>25</b> Numerus & qd equalia rebus & cub qd prima.   | <b>25</b> Numerus & cub qd equalia cub qd. |
| <b>26</b> Numerus & qd equalia rebus & cub qd secunda. | <b>26</b> Numerus & cub qd equalia cub qd. |
| <b>27</b> Numerus & qd equalia rebus & cub qd prima.   | <b>27</b> Numerus & cub qd equalia cub qd. |
| <b>28</b> Numerus & qd equalia rebus & cub qd secunda. | <b>28</b> Numerus & cub qd equalia cub qd. |
| <b>29</b> Numerus & qd equalia rebus & cub qd prima.   | <b>29</b> Numerus & cub qd equalia cub qd. |
| <b>30</b> Numerus & qd equalia rebus & cub qd secunda. | <b>30</b> Numerus & cub qd equalia cub qd. |
| <b>31</b> Numerus & qd equalia rebus & cub qd prima.   | <b>31</b> Numerus & cub qd equalia cub qd. |
| <b>32</b> Numerus & qd equalia rebus & cub qd secunda. | <b>32</b> Numerus & cub qd equalia cub qd. |

# Cardano on the cubic



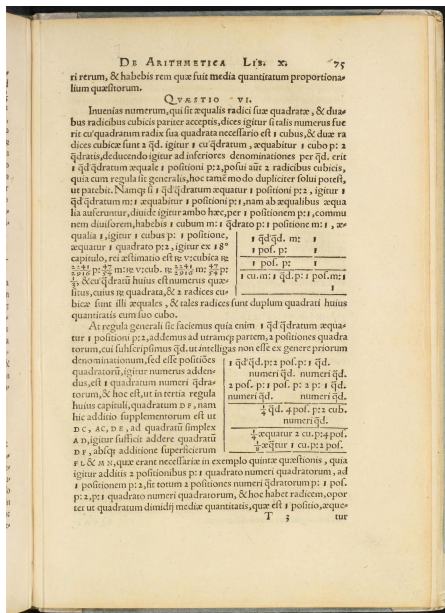
► Geometrical justification remains

► General solution (to particular case), rather than example to be followed

► Make substitution  $x = y - \frac{a}{3}$  in  $y^3 + ax^2 + bx + c = d$  to suppress square term and obtain equation of the form  $x^3 + px = q$  — manipulation of equations prior to solution

# Quartic equations (1)

General solution discovered  
(again on a case-by-case  
basis) by Lodovico Ferrari  
(c. 1540) and published by  
Cardano, in the form of  
worked examples, alongside  
solution of cubic



## Quartic equations (2)

In modern terms, suppose that

$$x^4 = px^2 + qx + r.$$

Add  $2yx^2 + y^2$  to each side to give

$$(x^2 + y)^2 = (p + 2y)x^2 + qx + (r + y^2).$$

Now we seek  $y$  such that the right hand side is a perfect square:

$$8y^3 + 4py^2 + 8ry + (4pr - q^2) = 0.$$

So the problem is reduced to solving a cubic equation and then a quadratic.

**NB:** In an equation  $y^4 + ay^3 + by^2 + cy + d = 0$  we can put  $y = x - \frac{a}{4}$  to remove the cube term, so this solution is general.

## Quartic equations (3)

Formulae for the solutions of the general quartic equation, in all their unedifying glory, may be found at:

<http://planetmath.org/QuarticFormula>

Cardano's *Ars Magna* may also be found online [here](#)

## Further 16th-century developments



Rafael Bombelli, *L'algebra* (1572):

- ▶ heavily influenced by Cardano
- ▶ equation solving, new notation
- ▶ exploration of complex numbers  
[to be dealt with in a later lecture]

## Further 16th-century developments

### L'ARITHMETIQUE DE SIMON STEVIN DE BRUGES:

Contenant les computations des nombres  
Arithmetiques ou vulgaires :

*Aussi l'Algebre, avec les equations de cinq quantitez.*

Ensemble les quatre premiers liures d'Algebre  
de Diophante d'Alexandrie, maintenant pre-  
mierement traduits en François.

*Encore vn liure particulier de La Pratique d'Arithmetique,  
contenant entre autres, Les Tables d'Interest, La Difme;  
Et vn traité des Incommensurables grandeurs :  
Avec l'Explication du Dixiesme Liure d'Euclide.*



A LEYDE,  
De l'Imprimerie de Christophle Plantin.  
c15. 15. LXXXV.

Simon Stevin, *L'arithmetique ... aussi  
l'algebre* (1585):

- ▶ heavily influenced by Cardano through Bombelli
- ▶ appended his treatise on decimal notation



## Further 16th-century developments

François Viète (1590s):

- ▶ links between algebra and geometry
- ▶ (algebra as 'analysis' or 'analytic art')
- ▶ notation [recall Lecture III]
- ▶ numerical methods for solving equations



## Thomas Harriot (c. 1600)

$a + b = aaaa + baaa$   
 $a + c = aaaa + caaa + baaa$   
 $a + d = aaaa + daaa + baaa$   
 $a - f = aaaa - faaa + cdaa + bdaa$   
 $\quad \quad \quad - bfaa - bfaa$   
 $\quad \quad \quad - bfaa - bfaa$   
 $\quad \quad \quad - bfaa - cdaa - bdaa = 0000.$

Ergo:  $bcd \equiv bda + baaa$   
 $\quad \quad \quad - bfaa + baaa$   
 $\quad \quad \quad - bfaa + cdaa + baaa$   
 by the definition of the first sequence,  $a \equiv f.$   
 $\quad \quad \quad - cdaa - bfaa + aaaa$   
 $\quad \quad \quad - bfaa - faaa + aaaa.$

Si:  $b + c + d \equiv f.$  Tollitur tertius gradus, nemp (aaa).  
 Et per reductionem,  
 $f + bcd \equiv -bcaa$   
 $+ bcd \equiv -bdaa - bbaa$   
 $+ bcd \equiv -bcaa - ccaa$   
 $\quad \quad \quad - cdaa - ddaa$   
 $\quad \quad \quad - bdaa - bcaa$   
 $\quad \quad \quad - cdaa - bdaa$   
 $\quad \quad \quad - 2bdaa - bdaa + aaaa.$

Si:  $b + cd + ed \equiv b + c + d + f.$  Tollitur secundus gradus, nemp (aa).  
 Tu:  $\frac{b + cd + ed}{b + c + d} \equiv f.$  Et per reductionem,  
 $f + bcd \equiv -bcaa + bbaa$   
 $+ bcd \equiv -bdaa + ccaaa$   
 $+ bcd \equiv -bdaa + ddaa$   
 $\quad \quad \quad - bdaa + bbaa$   
 $\quad \quad \quad - bdaa + bdaa$   
 $\quad \quad \quad - bdaa + cdaa + aaaa.$   
 $\quad \quad \quad \frac{b + cd}{b + c + d}$

Si:  $bcd \equiv bcd + bcd + bcd.$  Tollitur primus gradus, nemp (a).  
 Tu:  $\frac{bcd}{bcd + bcd} \equiv f.$  Et per reductionem,  
 $f + bcd \equiv -bbaaa + bbaaa$   
 $+ bcd \equiv -bdaaa + bdaaaa$   
 $+ bcd \equiv -bdaaa + bbaaaa$   
 $\quad \quad \quad - bdaaa + cdaaaa$   
 $\quad \quad \quad + bdaaa + bdaaaa$   
 $\quad \quad \quad \frac{bcd + bcd}{bcd + bcd} + 2bdaaaa + aaaa$

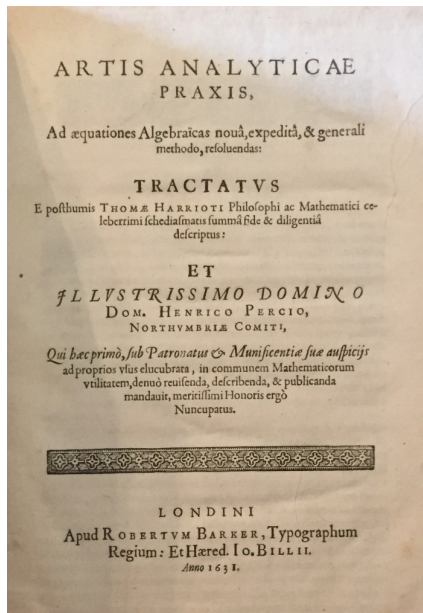
$aaaaa + 8aaaa + 18aaa \equiv 27 \equiv + 18aaa - 8aaaa + aaaa$   
 $a \equiv 1. \quad \frac{(bcd)}{(bcd + ccd + gggg)}$   
 $a \equiv 3. b.$   
 $a \equiv 3. c.$   
 $a \equiv 3. d.$

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Note:

- ▶ notation [see lecture III]:
- ▶ appearance of polynomials as products of linear factors.

# Thomas Harriot (1631)

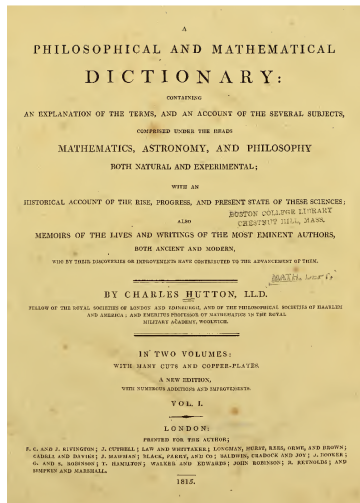


Some of Harriot's ideas found their way into his *Artis analyticae praxis* (*The practice of the analytic art*), published posthumously in 1631

**But** editors did not permit negative or imaginary roots [to be discussed further in a later lecture]

See *Mathematics emerging*, §12.2.1.

# Commentary on Harriot



Charles Hutton, *A mathematical and philosophical dictionary*, London, 1795, vol. 1, p. 91 (p. 96 of revised edition, 1815):

*He shewed the universal generation of all the compound or affected equations, by the continual multiplication of so many simple ones; thereby plainly exhibiting to the eye the whole circumstances of the nature, mystery and number of the roots of equations; with the composition and relations of the coefficients of the terms; . . .*

# Algebra in the 17th century

From 1600 onwards, 'algebra' as a set of recipes and techniques began to diverge in two (linked) directions:

- ▶ 'algebra' as a tool or a language (a.k.a. 'analysis' or the 'analytic art')
- ▶ 'algebra' as an object of study in its own right (the 'theory of equations')

# Descartes on algebra

Polynomials feature in Descartes' *La géométrie* (1637), e.g.:

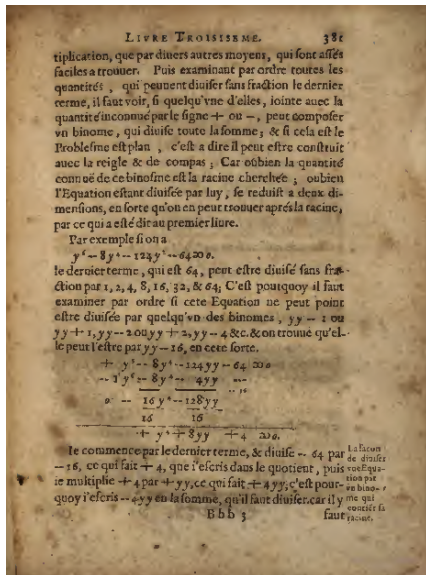
- ▶ one example to show that polynomials can be constructed from their roots (influenced by Harriot?);
- ▶ 'rule of signs': the number of positive ('true') roots of a polynomial is at most the number of times that the sign changes as we read term-by-term; the number of negative ('false') roots is at most the number of successions of the same sign; for example,

$$x^4 - 4x^3 - 19x^2 + 106x - 120 = 0$$

has at most 3 positive roots and at most one negative;

- ▶ can always make a transformation to remove the second-highest term.

# Descartes on cubics



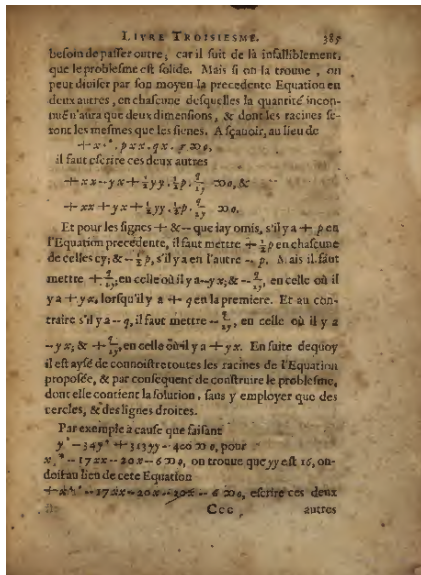
Search for roots of a cubic by examining the factors of the constant term:

if  $\alpha$  is such a factor, test whether  $x - \alpha$  divides the polynomial.

Examines the example

$$y^6 - 8y^4 - 124y^2 - 64 = 0$$

# Descartes on quartics



To solve  $+x^4 \star .pxx.qx.r = 0$   
(Descartes' notation), that is,

$$x^4 \pm pxx \pm qx \pm r = 0,$$

he sought to write the quartic as a product of two quadratics. This led him to

$$y^6 \pm 2py^4 + (pp \pm 4r)yy - qq = 0$$

As in Ferrari's/Cardano's method: a quartic is reduced to a cubic



## Summary and a glance ahead

By 1600, general solutions were available for quadratic, cubic and quartic equations — specifically, general solutions **in radicals**, i.e., solutions constructed from the coefficients of a given polynomial equation via  $+$ ,  $-$ ,  $\times$ ,  $\div$ ,  $\sqrt{\phantom{x}}$ ,  $\sqrt[3]{\phantom{x}}$ ,  $\sqrt[4]{\phantom{x}}$ ,  $\dots$

**NB:** A solution in radicals may be constructed by ruler and compass.

**Spoiler:** the general quintic equation is **not** solvable in radicals.

By the 1770s, mathematicians (notably Lagrange) had come to suspect this, but it was not **proved** until the 1820s.

So did anything interesting happen in algebra during the 17th and 18th centuries?

## A typical 20th-century view

Luboš Nový, *Origins of modern algebra* (1973), p. 23:

*From the propagation of Descartes' algebraic knowledge up to the publication of the important works of Lagrange [and others] in the years 1770–1, the evolution of algebra was, at first glance, hardly dramatic and one would seek in vain for great and significant works of science and substantial changes.*

Fair point? Or not?

# Some 17th-century developments: Hudde's rule (1657)

Published 1659 as an addendum to van Schooten's Latin translation of Descartes' *La géométrie*:

$$\text{take } x^3 - 4xx + 5x - 2 = 0$$

multiply the terms of the equation by numbers in arithmetic progression:

$$3x^3 - 8xx + 5x = 0$$

$$\gcd(x^3 - 4xx + 5x - 2, 3x^3 - 8xx + 5x) = x - 1$$

so the original polynomial can be divided by  $x - 1$  twice — hence it has a double root at  $x = 1$

See *Mathematics emerging*, §12.2.2.

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quæro, per Methodum superius explicatam, maximum earum communem diviforem; atque hujus ope æquationem Propositam toties divido, quoties id fieri potest.

Exempli gratiâ, proponatur hæc æquatio  $x^3 - 4xx + 5x - 2 = 0$ , in qua duæ sunt æquales radices. Multiplico ergo ipsam per Arithmeticam Progressionem qualemcunque, hoc est, cujus incrementum vel decrementum sit vel 1, vel 2, vel 3, vel alius quilibet numerus; & cujus primus terminus sit vel 0, vel +, vel - quam 0: Ita ut semper ejus ope talis terminus æquationis tolli possit, qualem quis voluerit, collocando tantum sub eo 0.

Ut si, exempli causâ, ultimum ejus terminum auferre velim, multiplicatio fieri potest ipsius  $x^3 - 4xx + 5x - 2 = 0$  per hanc progressionem

$$\begin{array}{r} 3. \quad 2. \quad 1. \quad 0. \\ 3x^3 - 8xx + 5x - 2 = 0. \end{array}$$

Maxima autem communis divisor hujus & Propositæ æquationis est  $x - 1 = 0$ , per quam Propositâ bis dividi potest; ita ut ejusdem radices sint 1, 1, & 2.

Sic si cupiam  $1^{\text{um}}$  æquationis terminum auferre, multiplicatio institui potest ipsius  $x^3 - 4xx + 5x - 2 = 0$  per hanc progressionem

$$\begin{array}{r} 0. \quad 1. \quad 2. \quad 3. \\ 3x^3 - 4xx + 10x - 6 = 0. \end{array}$$

Cujus quidem ac Propositæ æquationis maximus communis divisor, ut antea, est  $x - 1 = 0$ .

Similiter si  $2^{\text{um}}$  terminum tollere lubeat, multiplicatio fieri potest, hoc pacto:

$$\begin{array}{r} +1. \quad 0. \quad -1. \quad -1. \\ 3x^3 - 4xx + 5x - 2 = 0 \end{array}$$

& prodibit  $x^3 - 5x + 4 = 0$ .

Cujus item & Propositæ maximus communis divisor est  $x - 1 = 0$ .

Ubi notandum, non necessarium esse, semper uti Progressione cujus excessus sit 1, quanquam ea communiter sit optima.

# Some 17th-century developments: Tschirnhaus transformations (1683)

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## ACTA ERUDITORUM METHODUS AUFERENDI OMNES TER- minos intermedios ex data æquatione, per D. T.

EX Geometria Dn. Des Cartes notum est, qua ratione semper secun-  
dus terminus ex data æquatione possit auferri; quoad plures termi-  
nos intermedios auferendos, hæcenus nihil inventum vidi in Arte Ana-  
lytica, imo non paucos offendi, qui crediderunt, id nulla arte perfici  
posse. Quapropter hic quædam circa hoc negotium aperire conatus  
verum saltem pro iis, qui Artis Analyticæ apprime gnari, cum aliis  
tam brevi explicatione vix satisfieri possit: reliqua, quæ hic desiderari  
possent, alii tempori refervans.

Primo itaque loco, ad hoc attendendum; sit data aliqua æquatio  
cubica  $x^3 - px^2 + qx - r = 0$ , in qua  $x$  radices hujus æquationis defi-  
gnat;  $p, q, r$ , cognitæ quantitates representant: ad auferendum jam  
secundum terminum supponatur  $x = y + a$ ; jam ope harum duarum æ-  
quationum inveniat tertiam, ubi quantitas  $x$  abest, & erit  
 $y^3 + 3ay^2 + 3a^2y + a^3 = 0$  Ponatur nunc secundus terminus æqua-  
tionis nihilominus (quia hunc auferre possumus in-  
tentio) eritque  $3ay^2 - py - p^2a = 0$  Unde  
 $-py - p^2a = 0$   $3ay^2 = p^2a$   $y^2 = \frac{p^2}{3}$  id quod indicat, ad auferendum  
secundum terminum in æquatione Cubica, supponendum esse loco  
 $x = y + a$  (prout modo fecimus)  $x = y + \frac{p}{3}$ . Hæc jam vulgata admo-  
dum sunt, nec hic referuntur aliam ob causam, quam quia sequentia  
admodum illustrent, dum hæc bene intellectis, eo facilius, quæ modo  
proponam, capiuntur.

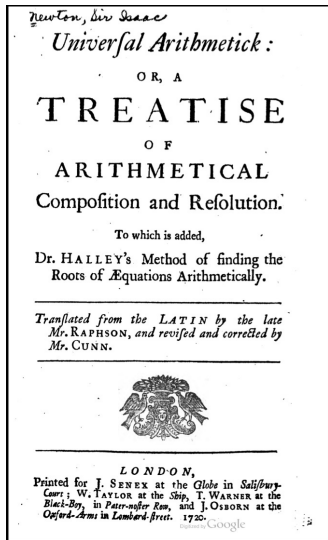
Sint jam secundo in æquatione data auferendi duo termini:  
dico, quod supponendum sit,  $xx = bx + y + a$ ; si tres,  $x^3 = cx^2 + bx + y + a$ ; si quatuor,  $x^4 = dx^3 + cx^2 + bx + y + a$ , atque sic in  
infinitum. Vocabo autem has æquationes assumptas, ut eas distin-  
guam ab æquatione, quæ ut data consideratur. Ratio autem ho-  
rum est: quod eadem ratione, prout ope æquationis  $x = y + a$  saltem  
unicus terminus poterat auferri, quia nimirum unica saltem inde-  
terminata hic existit  $a$ , sic eadem ratione ope hujus  $xx = bx + y + a$ ,  
non nisi duo termini possunt auferri, quia duæ indeterminatæ  $a$  &  $b$   
adiunguntur.

For an equation  $x^3 - px^2 + qx - r = 0$

- ▶ to remove one term put  $x = y + a$   
(where  $a = p/3$ )
- ▶ can we remove both the middle  
terms?
- ▶ to remove two terms put  
 $x^2 = bx + y + a$

See *Mathematics emerging*, §12.2.3.

# An 18th-century development: Newton's *Arithmetica universalis* (1707)



Rules for sums of powers of roots of

$$x^n - px^{n-1} - qx^{n-2} - rx^{n-3} - sx^{n-4} - \dots = 0$$

sum of roots	=	$a$	=	$p$
sum of roots <sup>2</sup>	=	$b$	=	$pa + 2q$
sum of roots <sup>3</sup>	=	$c$	=	$pb + qa + 3r$
sum of roots <sup>4</sup>	=	$d$	=	$pc + qb + ra + 4s$

# Developments of the 17th and 18th centuries

- ▶ Symbolic notation
- ▶ Understanding of the structure of polynomials
- ▶ ... of the number and nature of their roots
- ▶ ... of the relationship between roots and coefficients
- ▶ ... of how to manipulate them
- ▶ ... of how to solve them numerically
- ▶ The leaving behind of geometric intuition?

# Some 18th-century theory of equations

Recall:

- ▶ cubic equations can be solved by means of quadratics
- ▶ quartic equations can be solved by means of cubics

# Some 18th-century theory of equations

Recall:

- ▶ quadratic equations can be solved by means of linear equations
- ▶ cubic equations can be solved by means of quadratics
- ▶ quartic equations can be solved by means of cubics

The 'reduced' or 'resolvent' equation:

- ▶ for cubics, the reduced equation is of degree 2
- ▶ for quartics, the reduced equation is of degree 3
- ▶ for quintics, the reduced equation is of degree ?



## Some 18th-century hypotheses

Euler's hypothesis (1733):

- ▶ for an equation of degree  $n$  the degree of the reduced equation will be  $n - 1$

Bézout's hypothesis (1764):

- ▶ for an equation of degree  $n$  the degree of the reduced equation will in general be  $n!$
- ▶ though always reducible to  $(n - 1)!$
- ▶ possibly further reducible to  $(n - 2)!$

# Lagrange's 'Réflexions' 1770/71

J.-L. Lagrange, 'Réflexions sur la résolution algébrique des équations', Berlin (1770/1):

Examined all known methods of solving

- ▶ quadratics: the well-known solution
- ▶ cubics: methods of Cardano, Tschirnhaus, Euler, Bézout
- ▶ quartics: methods of Cardano, Descartes, Tschirnhaus, Euler, Bézout

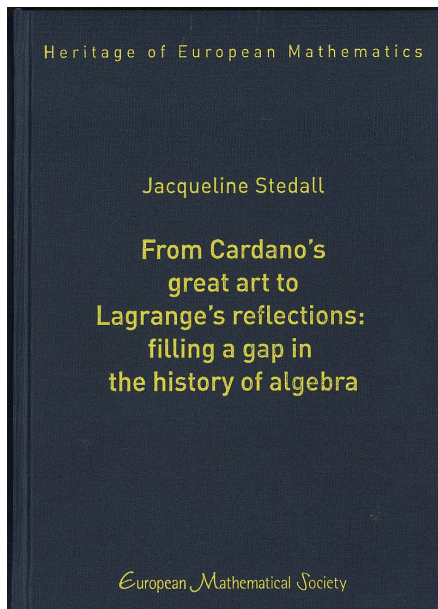
seeking to identify a uniform method that could be extended to higher degree

## A typical 20th-century view revisited

Luboš Nový, *Origins of modern algebra* (1973), p. 23:

*From the propagation of Descartes' algebraic knowledge up to the publication of the important works of Lagrange [and others] in the years 1770–1, the evolution of algebra was, at first glance, hardly dramatic and one would seek in vain for great and significant works of science and substantial changes.*

# Filling a gap in the history of algebra (2011)



*The hitherto untold story  
of the slow and halting  
journey from Cardano's  
solution recipes to  
Lagrange's sophisticated  
considerations of  
permutations and  
functions of the roots of  
equations . . . [Preface]*

## From Stedall's preface:

*This assertion . . . from Nový quoted above, betrays yet another fundamental shortcoming of several earlier accounts, a view that mathematics somehow progresses only by means of 'great and significant works' and 'substantial changes'. Fortunately, the truth is far more subtle and far more interesting: mathematics is the result of a cumulative endeavour to which many people have contributed, and not only through their successes but through half-formed thoughts, tentative proposals, partially worked solutions, and even outright failure. No part of mathematics came to birth in the form that it now appears in a modern textbook: mathematical creativity can be slow, sometimes messy, often frustrating.*