BO1.1. History of Mathematics
Lecture XII
19th-century rigour in real analysis, part 2: real numbers and sets

MT23 Week 6

Summary

- Proofs of the Intermediate Value Theorem revisited
- Convergence and completeness
- Dedekind and the continuum
- Cantor and numbers and sets
- Where and when did sets emerge?
- Early set theory
- Set theory as a language

The Intermediate Value Theorem (1)

Bolzano's criticisms (1817) of existing proofs:

> *The most common kind of proof depends on a truth borrowed from geometry ... But it is clear that it is an intolerable offense against correct method to derive truths of pure (or general) mathematics (i.e., arithmetic, algebra, analysis) from considerations which belong to a merely applied (or special) part, namely, geometry.*

His own proof includes something close to a proof that Cauchy sequences converge:

> *... the true value of X [the limit] therefore ... can be determined as accurately as required ... There is, therefore, a real quantity which the terms of the series, if it is continued far enough, approach as closely as desired.*

But Bolzano assumed the existence of the limit.

The Intermediate Value Theorem (2)

Cauchy's 1st proof (*Cours d'analyse*, 1821, p. 44) is geometric (though he didn't provide a diagram):

> *The function $f(x)$ being continuous between the limits $x = x_0$, $x = X$, the curve which has for equation $y = f(x)$ passes first through the point corresponding to the coordinates x_0, $f(x_0)$, second through the point corresponding to the coordinates X, $f(X)$, will be continuous between these two points: and, since the constant ordinate b of the line which has for equation $y = b$ is to be found between the ordinates $f(x_0)$, $f(X)$ of the two points under consideration, the line will necessarily pass between these two points, which it cannot do without meeting the curve mentioned above in the interval.*

Cauchy's 2nd proof in a different context (p. 460): a numerical method for finding roots of equations — tacitly assumes that bounded monotone sequences of real numbers converge [see Lecture VIII].

The need for a deeper understanding (1)

Emergence of rigour in Analysis:

- Bolzano, *Rein analytischer Beweis ...*, 1817;
- Cauchy, *Cours d'analyse*, 1821, etc.

  By 1821, therefore, attempts to prove the intermediate value theorem had brought three important propositions into play:

  1. *Cauchy sequences are convergent (with an unsuccessful proof by Bolzano in 1817; accepted without proof by Cauchy in 1821).*
  2. *A* [*non-empty*] *set of numbers bounded below has a greatest lower bound (proved by Bolzano in 1817 on the basis of (1)).*
  3. *A monotonic bounded sequence converges to a limit (taken for granted by Cauchy in 1821).*

(*Mathematics emerging*, §16.3.1.)

The need for a deeper understanding (2)

What Bolzano and Cauchy missed: completeness

Completeness of the real number system $\mathbb{R}$ in modern teaching:

- non-empty bounded sets of real numbers have least upper bounds
- monotonic bounded sequences converge
- Cauchy sequences converge
- ...

All equivalent

Equivalence of formulations of completeness

Bolzano–Weierstrass Theorem: A bounded sequence of real numbers has a convergent subsequence.

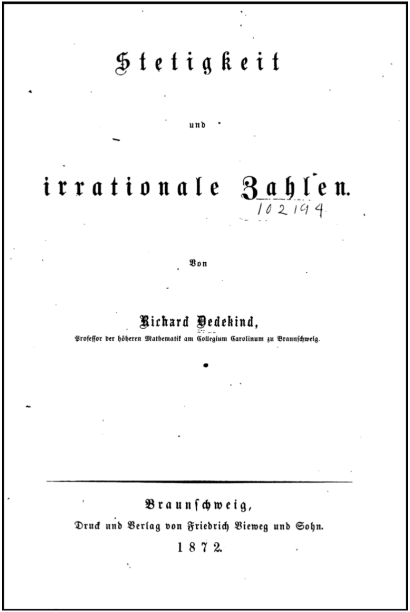
Implicit in Bolzano (1817); explicit in lectures by Karl Weierstrass (1815–1897) in Berlin 1859/60, 1863/64: a step in proofs from other definitions of completeness that Cauchy sequences converge.

Modern proofs often use the lemma that every infinite sequence of real numbers has an infinite monotonic subsequence.

How to incorporate these ideas into analysis in a rigorous way?

All of the above relies upon an intuitive notion of real number — so perhaps provide a formal definition of these? One that includes the idea of completeness?

Richard Dedekind (1831–1916)



Stetigkeit

und

irrationale Zahlen.

Von

Richard Dedekind,

Professor der höheren Mathematik am Collegium Carolinum zu Braunschweig.

Braunschweig,

Druck und Verlag von Friedrich Vieweg und Sohn.

1872.

Dedekind and the foundations of analysis

Teaching calculus in the Zürich Polytechnic (1858), later (from 1862) teaching Fourier series in the Braunschweig Polytechnic, found himself dissatisfied with:

- geometry as a foundation for analysis;
- tacit assumptions about convergence (for monotonic bounded sequences, for example).

Response eventually published in *Stetigkeit und irrationale Zahlen* (1872) [translated as *Continuity and irrational numbers* by Wooster Woodruff Beman, 1901]

Dedekind and continuity (1)

Intuition suggests that numbers (an arithmetical concept) should have the same completeness and continuity properties as a line (a geometrical concept). But we must define these concepts for numbers without appeal to geometrical intuition.

Geometrically, every point separates a line into two parts.

> *I find the essence of continuity in the converse, i.e., in the following principle:*
>
> *"If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions."*

Dedekind and continuity (2)

But Dedekind couldn't *prove* this property, so he had to take it as an axiom:

> *The assumption of this property for the line is nothing but an Axiom, through which alone we attribute continuity to the line, through which we understand continuity in the line.*

(See *Mathematics emerging*, §16.3.2.)

Dedekind and continuity (3)

Next adapt this idea to the arithmetical context:

- every number x separates all other numbers into two classes — those greater than x, and those less than x;
- conversely, every such separation of numbers defines a number.

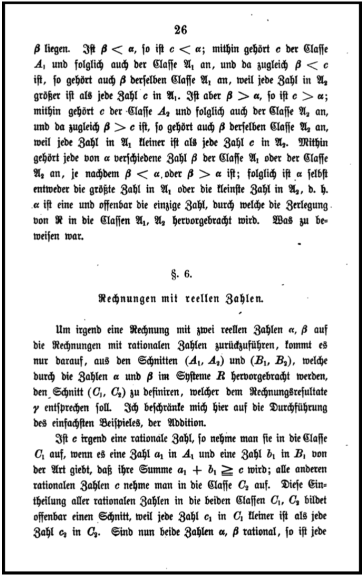
Hence Dedekind cuts (or sections, from the original German Schnitt).

Dedekind cuts (1)

- Start from the system of rational numbers R (assumed known)
- Separate R into two classes A_1 and A_2 such that
  - for any a_1 in A_1, $a_1 < a_2$ for every a_2 in A_2
  - for any a_2 in A_2, $a_2 > a_1$ for every a_1 in A_1
- The cut denoted by (A_1, A_2) defines a number
- Important observation: (A_1, A_2) need not be rational

Whenever, then, we have to do with a cut produced by no rational number, we create a new irrational number, which we regard as completely defined by this cut ...

Dedekind cuts (2)



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β liegen. Ist $\beta < \alpha$, so ist $c < \alpha$; mithin gehört c der Classe A_1 und folglich auch der Classe $\mathfrak{A}_1$ an, und da zugleich $\beta < c$ ist, so gehört auch β derselben Classe $\mathfrak{A}_1$ an, weil jede Zahl in $\mathfrak{A}_2$ größer ist als jede Zahl c in $\mathfrak{A}_1$. Ist aber $\beta > \alpha$, so ist $c > \alpha$; mithin gehört c der Classe A_2 und folglich auch der Classe $\mathfrak{A}_2$ an, und da zugleich $\beta > c$ ist, so gehört auch β derselben Classe $\mathfrak{A}_2$ an, weil jede Zahl in $\mathfrak{A}_1$ kleiner ist als jede Zahl c in $\mathfrak{A}_2$. Mithin gehört jede von α verschiedene Zahl β der Classe $\mathfrak{A}_1$ oder der Classe $\mathfrak{A}_2$ an, je nachdem $\beta < \alpha$ oder $\beta > \alpha$ ist; folglich ist α selbst entweder die größte Zahl in $\mathfrak{A}_1$ oder die kleinste Zahl in $\mathfrak{A}_2$, d. h. α ist eine und offenbar die einzige Zahl, durch welche die Zerlegung von $\mathfrak{R}$ in die Classen $\mathfrak{A}_1$, $\mathfrak{A}_2$ hervorgebracht wird. Was zu beweisen war.

§. 6.

Rechnungen mit reellen Zahlen.

Um irgend eine Rechnung mit zwei reellen Zahlen α, β auf die Rechnungen mit rationalen Zahlen zurückzuführen, kommt es nur darauf, aus den Schnitten (A_1, A_2) und (B_1, B_2), welche durch die Zahlen α und β im Systeme R hervorgebracht werden, den Schnitt (C_1, C_2) zu definiren, welcher dem Rechnungsresultate γ entsprechen soll. Ich beschränke mich hier auf die Durchführung des einfachsten Beispieles, der Addition.

Ist c irgend eine rationale Zahl, so nehme man sie in die Classe C_1 auf, wenn es eine Zahl a_1 in A_1 und eine Zahl b_1 in B_1 von der Art giebt, daß ihre Summe $a_1 + b_1 \geqq c$ wird; alle anderen rationalen Zahlen c nehme man in die Classe C_2 auf. Diese Eintheilung aller rationalen Zahlen in die beiden Classen C_1, C_2 bildet offenbar einen Schnitt, weil jede Zahl c_1 in C_1 kleiner ist als jede Zahl c_2 in C_2. Sind nun beide Zahlen α, β rational, so ist jede

Dedekind showed how to add two cuts, and how to use them in limiting arguments — but did little else with them.

Significance: a major step towards

- understanding completeness, and
- giving a rigorous definition of an irrational number, hence
- setting the foundations of analysis onto a sound logical basis.

Dissemination of Dedekind's ideas

Stetigkeit und irrationale Zahlen reprinted many times, often in conjunction with the later essay *Was sind und was sollen die Zahlen?* (1888) [see below].

Translated into English as *Essays on the theory of numbers* by Wooster Woodruff Beman (1901).

Popularised and organised for teaching, starting from Peano axioms for natural numbers, by Edmund Landau in *Grundlagen der Analysis* [*Foundations of analysis*] (1930), a book that contains very few words.

A good modern (historically sensitive) account can be found in: Leo Corry, *A brief history of numbers*, OUP, 2015, §10.6.

Other approaches

Georg Cantor (1872) and Eduard Heine (1872) created real numbers as equivalence classes of Cauchy sequences of rational numbers. (Also: Charles Méray in 1869.)

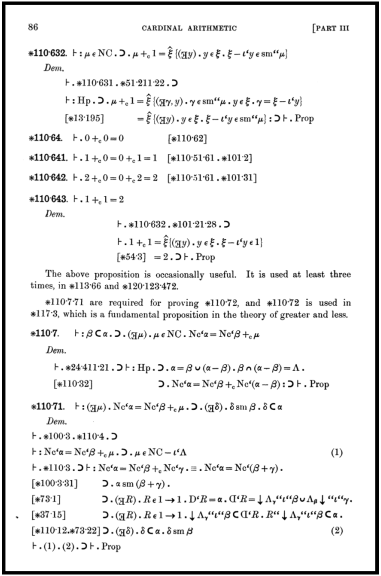
(On Cantor's approach, see *Mathematics emerging*, §16.3.3.)

Heine acknowledged a debt to Cantor and a debt to the lectures of Weierstrass.

Later constructions by many mathematicians and philosophers, often as part of a broader effort to lay down logical foundations for mathematics as a whole — for example:

- Carl Johannes Thomae, 1880, 1890
- Giuseppe Peano, 1889, 1891
- Gottlob Frege, 1884, 1893, 1903
- Otto Hölder, 1901

Extreme formalism



86 CARDINAL ARITHMETIC [PART III

$*110{\cdot}632.\ \vdash : \mu \in \mathrm{NC} . \supset . \mu +_c 1 = \hat{\xi}\{(\exists y) . y \in \xi . \xi - \iota\text{‘}y \in \mathrm{sm}\text{“}\mu\}$

Dem.

$\vdash . *110{\cdot}631 . *51{\cdot}211{\cdot}22 . \supset$

$\vdash : \mathrm{Hp} . \supset . \mu +_c 1 = \hat{\xi}\{(\exists \gamma, y) . \gamma \in \mathrm{sm}\text{“}\mu . y \in \xi . \gamma = \xi - \iota\text{‘}y\}$

$[*13{\cdot}195] \quad = \hat{\xi}\{(\exists y) . y \in \xi . \xi - \iota\text{‘}y \in \mathrm{sm}\text{“}\mu\} : \supset \vdash . \mathrm{Prop}$

$*110{\cdot}64.\ \vdash . 0 +_c 0 = 0 \quad [*110{\cdot}62]$

$*110{\cdot}641.\ \vdash . 1 +_c 0 = 0 +_c 1 = 1 \quad [*110{\cdot}51{\cdot}61 . *101{\cdot}2]$

$*110{\cdot}642.\ \vdash . 2 +_c 0 = 0 +_c 2 = 2 \quad [*110{\cdot}51{\cdot}61 . *101{\cdot}31]$

$*110{\cdot}643.\ \vdash . 1 +_c 1 = 2$

Dem.

$\vdash . *110{\cdot}632 . *101{\cdot}21{\cdot}28 . \supset$

$\vdash . 1 +_c 1 = \hat{\xi}\{(\exists y) . y \in \xi . \xi - \iota\text{‘}y \in 1\}$

$[*54{\cdot}3] \quad = 2 . \supset \vdash . \mathrm{Prop}$

The above proposition is occasionally useful. It is used at least three times, in *113·66 and *120·123·472.

*110·71 are required for proving *110·72, and *110·72 is used in *117·3, which is a fundamental proposition in the theory of greater and less.

$*110{\cdot}7.\ \vdash : \beta \subset \alpha . \supset . (\exists \mu) . \mu \in \mathrm{NC} . \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta +_c \mu$

Dem.

$\vdash . *24{\cdot}411{\cdot}21 . \supset \vdash : \mathrm{Hp} . \supset . \alpha = \beta \cup (\alpha - \beta) . \beta \cap (\alpha - \beta) = \Lambda .$

$[*110{\cdot}32] \quad \supset . \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta +_c \mathrm{Nc}\text{‘}(\alpha - \beta) : \supset \vdash . \mathrm{Prop}$

$*110{\cdot}71.\ \vdash : (\exists \mu) . \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta +_c \mu . \supset . (\exists \delta) . \delta \,\mathrm{sm}\, \beta . \delta \subset \alpha$

Dem.

$\vdash . *100{\cdot}3 . *110{\cdot}4 . \supset$

$\vdash : \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta +_c \mu . \supset . \mu \in \mathrm{NC} - \iota\text{‘}\Lambda \quad (1)$

$\vdash . *110{\cdot}3 . \supset \vdash : \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}\beta +_c \mathrm{Nc}\text{‘}\gamma . \equiv . \mathrm{Nc}\text{‘}\alpha = \mathrm{Nc}\text{‘}(\beta + \gamma) .$

$[*100{\cdot}3{\cdot}31] \quad \supset . \alpha \,\mathrm{sm}\, (\beta + \gamma) .$

$[*73{\cdot}1] \quad \supset . (\exists R) . R \in 1 \to 1 . \mathrm{D}\text{‘}R = \alpha . \mathrm{Cl}\text{‘}R = \downarrow \Lambda_\gamma\text{“}\iota\text{“}\beta \cup \Lambda_\beta \downarrow \text{“}\iota\text{“}\gamma .$

$[*37{\cdot}15] \quad \supset . (\exists R) . R \in 1 \to 1 . \downarrow \Lambda_\gamma\text{“}\iota\text{“}\beta \subset \mathrm{Cl}\text{‘}R . R\text{“}\downarrow \Lambda_\gamma\text{“}\iota\text{“}\beta \subset \alpha .$

$[*110{\cdot}12 . *73{\cdot}22] \supset . (\exists \delta) . \delta \subset \alpha . \delta \,\mathrm{sm}\, \beta \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

Alfred North Whitehead and Bertrand Russell, *Principia mathematica*, 3 vols., Cambridge University Press, 1910, 1912, 1913

Vol. II, p. 86: $1 + 1 = 2$

"The above proposition is occasionally useful."

NB. This is not the source of our axioms for the reals.

See: *Logicomix: An epic search for truth* (2009)

New ideas

An idea that emerged as central to Dedekind's work: that of a set

In fact, naive notions of sets had already appeared all but unremarked earlier in the nineteenth century

- as Gauss' classes, orders, genera (of binary quadratic forms with integer coefficients) [see Lecture XIV];
- as Galois' groupes (of permutations and of substitutions);
- as Cauchy's systèmes (of substitutions);
- as Dedekind's Zahlkörpern (of algebraic numbers).

This is by no means an exhaustive list of examples; see *Mathematics emerging*, §18.2 for others.

Formalisation of the concept of a set

Georg Cantor: series of articles in *Mathematische Annalen*, 1879–1883

Final one also published separately as *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* [*Foundations of a general theory of aggregates*], Teubner, Leipzig, 1883:

> *By an "aggregate" (Menge) we are to understand any collection into a whole (Zusammenfassung zu einem Ganzen) M of definite and separate objects m of our intuition or our thought.*

Cantor and the continuum

Cantor's major interest: the continuum (i.e., the set of real numbers).

How to characterise this set within the collection of all sets? — A question that Cantor never satisfactorily answered.

Cantor's first great insight regarding sets (1873): infinite sets can have different sizes.

Cantor's first proof that the continuum is uncountable

Proposition: Given any sequence of real numbers $\omega_1, \omega_2, \omega_3, \ldots$ and any interval $[\alpha, \beta]$, there is a real number in $[\alpha, \beta]$ that is not contained in the given sequence.

Proof proceeds by construction of a sequence of nested intervals $[\alpha, \beta] \supseteq [\alpha_1, \beta_1] \supseteq [\alpha_2, \beta_2] \supseteq [\alpha_3, \beta_3] \supseteq \cdots$. Cantor considered the different cases where the sequence terminates or does not, but in all instances he constructed a real number in the interval that does not lie in the original sequence.

Next suppose that the continuum is countable, i.e., that the real numbers may be listed $\omega_1, \omega_2, \omega_3, \ldots$. But then there is a real number in any interval $[\alpha, \beta]$ that does not belong to this list — a contradiction.

The more famous diagonal argument came later (1891).

One-to-one correspondences

Also in the 1874 paper:

The algebraic $\mathbb{A}$ numbers are countable — therefore transcendental numbers exist.

> **NB:** In 1851 Joseph Liouville had already produced a constructive proof of the existence of transcendental numbers.
>
> Charles Hermite proved in 1873 that e is transcendental.
>
> Proof of the transcendence of π was finally accomplished by Carl Louis Lindemann in 1882.

Cantor to Dedekind (1877): there is a one-to-one correspondence between a line and the plane — “Je le vois, mais je ne le crois pas!” (“I see it, but I don't believe it!”)

Cantor's *Mengenlehre*

Developed at the end of the nineteenth century (1878–1897): a general theory of sets and of transfinite numbers — infinite cardinals (e.g., $\#\mathbb{N} = \aleph_0$, $\#\mathbb{R} = c$), transfinite ordinals, ...

Mixed terminology: Inbegriff, System, Mannigfaltigkeit, Menge

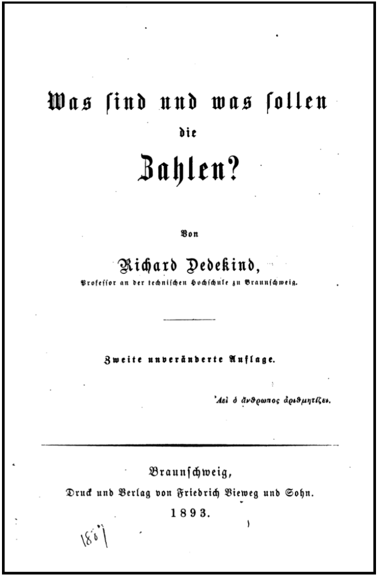
Continuum hypothesis (1878): there is no infinite cardinal strictly between $\aleph_0$ and c

Power set construction given in 1890: $\mathscr{P}(S)$ — the set of all subsets of a set S

Cantor's Theorem: $\#\mathscr{P}(S) > \#S$

Further: $\#\mathscr{P}(\mathbb{N}) = \#\mathbb{R}$, or $2^{\aleph_0} = c$

Was sind und was sollen die Zahlen?



Was sind und was sollen
die
Zahlen?

Von

Richard Dedekind,
Professor an der technischen Hochschule zu Braunschweig.

Zweite unveränderte Auflage.

Ἀεὶ ὁ ἄνθρωπος ἀριθμητίζει.

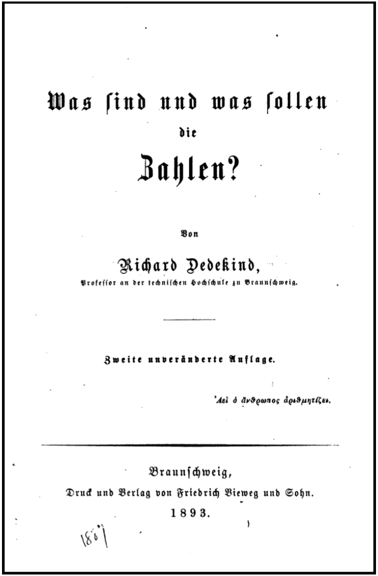
Braunschweig,
Druck und Verlag von Friedrich Vieweg und Sohn.
1893.

Richard Dedekind, *Was sind und was sollen die Zahlen?* Braunschweig, 1893

Contains, amongst other things:

- a definition of infinite sets;
- an axiomatisation of the natural numbers (soon simplified by Peano).

Was sind und was sollen die Zahlen?



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Also includes a definition of a function as a mapping between sets (p. 6):

"By a mapping of a system S we understand a law according to which every determinate element s of S is associated with a determinate thing which is called the *image* of s and is denoted by $\phi(s)$ ..."

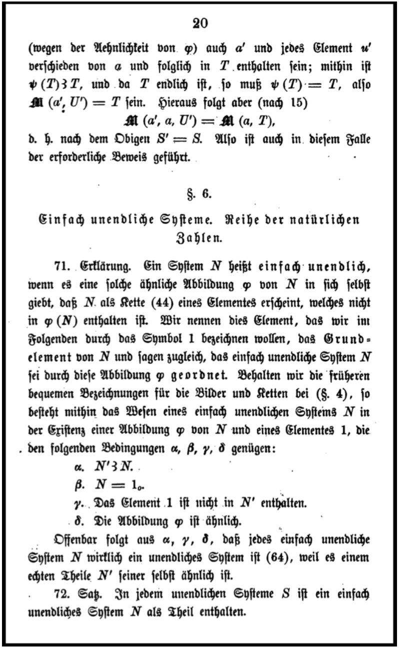
Was sind und was sollen die Zahlen?

Extract from William Ewald, *From Kant to Hilbert: a source book in the foundations of mathematics*, OUP, 1996, vol. II, p. 790:

> *The title of Dedekind's paper is subtle: rigidly translated it asks 'What are, and what ought to be, the numbers?' But sollen here carries several senses—among them, 'What is the best way to regard the numbers?'; 'What is the function of numbers?; 'What are numbers supposed to be?'. But perhaps Dedekind's title is famous enough to be left in the original.*

W. W. Beman translated the essay under the title *The nature and meaning of numbers* (1901).

Was sind und was sollen die Zahlen?



(wegen der Aehnlichkeit von φ) auch a' und jedes Element u' verſchieden von a und folglich in T enthalten ſein; mithin iſt $\psi(T)\,3\,T$, und da T endlich iſt, ſo muß $\psi(T) = T$, alſo $\mathfrak{M}(a', U') = T$ ſein. Hieraus folgt aber (nach 15)

$$\mathfrak{M}(a', a, U') = \mathfrak{M}(a, T),$$

d. h. nach dem Obigen $S' = S$. Alſo iſt auch in dieſem Falle der erforderliche Beweis geführt.

§. 6.

Einfach unendliche Syſteme. Reihe der natürlichen Zahlen.

71. Erklärung. Ein Syſtem N heißt einfach unendlich, wenn es eine ſolche ähnliche Abbildung φ von N in ſich ſelbſt giebt, daß N als Kette (44) eines Elementes erſcheint, welches nicht in $\varphi(N)$ enthalten iſt. Wir nennen dies Element, das wir im Folgenden durch das Symbol 1 bezeichnen wollen, das Grundelement von N und ſagen zugleich, das einfach unendliche Syſtem N ſei durch dieſe Abbildung φ geordnet. Behalten wir die früheren bequemen Bezeichnungen für die Bilder und Ketten bei (§. 4), ſo beſteht mithin das Weſen eines einfach unendlichen Syſtems N in der Exiſtenz einer Abbildung φ von N und eines Elementes 1, die den folgenden Bedingungen $\alpha, \beta, \gamma, \delta$ genügen:

α. $N'\,3\,N$.

β. $N = 1_0$.

γ. Das Element 1 iſt nicht in N' enthalten.

δ. Die Abbildung φ iſt ähnlich.

Offenbar folgt aus α, γ, δ, daß jedes einfach unendliche Syſtem N wirklich ein unendliches Syſtem iſt (64), weil es einem echten Theile N' ſeiner ſelbſt ähnlich iſt.

72. Satz. In jedem unendlichen Syſteme S iſt ein einfach unendliches Syſtem N als Theil enthalten.

Written in an explicitly set-theoretic language

(But with slightly different notation from ours.)

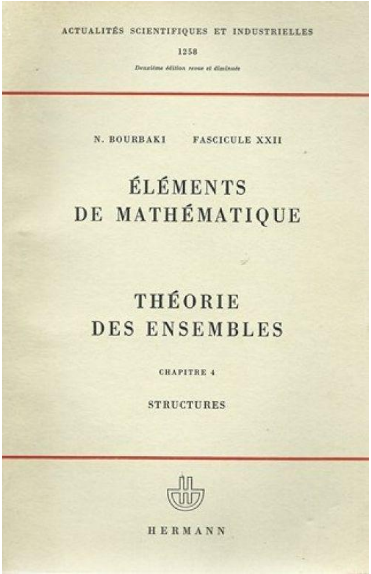
For a summary, see: Kathryn Edwards, ‘Richard Dedekind (1831–1916)’, *Mathematics Today* 52(1) (Feb 2016) 212–215

Set theory in our lives

Set theory as an effective language for mathematics:

- Set-builder notation
- Unification of ideas concerning functions and relations

Nicolas Bourbaki (1934–????)



Collective of French mathematicians who set out to reformulate mathematics on extremely formal, abstract, structural lines — the language of sets has a significant role to play.

Association des collaborateurs de Nicolas Bourbaki

SMP/New Math

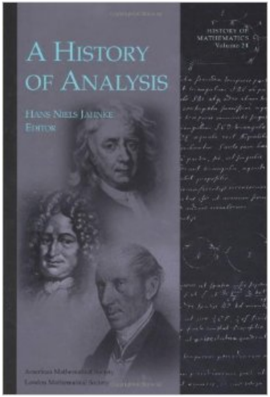
School Mathematics Project (UK)/New Mathematics (USA):

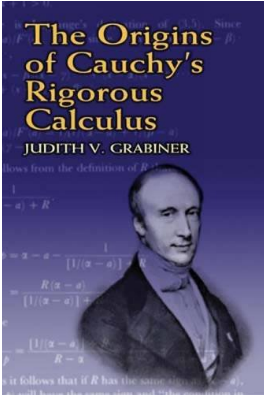
- Response to the launch of Sputnik I in 1957
- Traditional school arithmetic and geometry replaced by abstract algebra, matrices, symbolic logic, different bases, . . . — in short, mathematical topics based on set theory
- Much debate — now usually regarded as a passing fad
- Tom Lehrer — 'New Math'

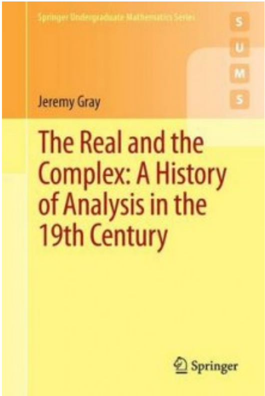
Conclusions

- Our modern perception of real numbers took well over 2000 years to crystallise, with geometric, arithmetic, set-theoretic intuitions to the fore at different times.
- The concept of set emerged at about the same time as the modern concept of real number, 1870–1890.
- This coincidence is no coincidence.

Further reading on the development of analysis...







...and on set theory and foundations