Problem Sheet 3: Solutions

1. Let $g_m = (\alpha^2 + m^2\pi^2)$. For $\sigma = \mathrm{i}\Omega$, Ω real and non-zero, we have

$$-A\Omega^2 + \mathrm{i}B\Omega + C = 0$$

and hence (i) $B = 0$, $\Omega^2 = C/A$. Noting $A > 0$ we require (ii) $C > 0$.

Thus at the bifurcation we have

$$g_m^2(1 + \mathrm{Le}) + \alpha^2 (\mathrm{Ra}_s - \mathrm{RaLe}) = 0,$$

and thus

$$\left(\mathrm{Ra} - \frac{\mathrm{Ra}_s}{\mathrm{Le}}\right) = \left(1 + \frac{1}{\mathrm{Le}}\right) \frac{g_m^2}{\alpha^2}.$$

From (ii), for we require

$$g_m^3 + (\mathrm{Ra}_s - \mathrm{Ra})\alpha^2 g_m > 0$$

and thus

$$\mathrm{Ra} - \mathrm{Ra}_s < \frac{g_m^2}{\alpha^2}.$$

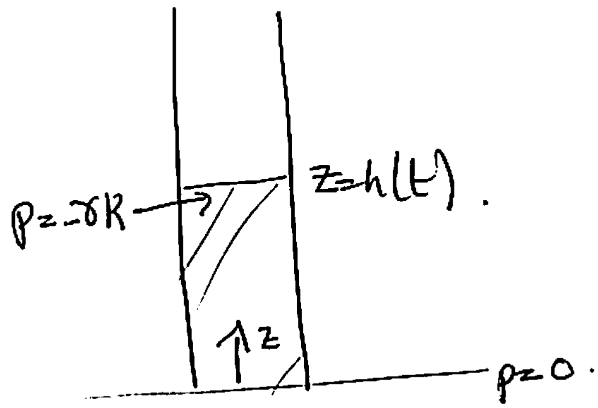
Hence, at the bifurcation we must have

$$\left(\mathrm{Ra} - \frac{\mathrm{Ra}_s}{\mathrm{Le}}\right) > \left(1 + \frac{1}{\mathrm{Le}}\right) (\mathrm{Ra} - \mathrm{Ra}_s)$$

and hence

$$\mathrm{Ra}_s > \frac{\mathrm{Ra}}{\mathrm{Le}}.$$

2. The problem is illustrated schematically below



Darcy's law in the porous medium gives the vertical Darcy velocity

$$w = -\frac{k}{\mu}\left(\frac{\partial p}{\partial z} + \rho g\right).$$

Given that the problem is assumed one-dimensional, incompressibility requires that $\partial w/\partial z = 0$ so that $\partial^2 p/\partial z^2 = 0$. Hence we must have $p = Az + B$ with the constants of integration A and B chosen to satisfy the boundary conditions that $p(0) = 0$ and $p(h) = -\gamma\kappa$, which gives

$$p = -\gamma\kappa\frac{z}{h}.$$

From Darcy's law the *Darcy velocity* within the liquid is

$$w = -\frac{k}{\mu}\left(\rho g - \gamma\kappa/h\right).$$

Now, we would like to use the kinematic condition to match the velocity at which the interface moves with the fluid velocity in the layer. However, the physical fluid velocity at the interface is $\dot{h}$ and so $w = \phi\dot{h}$ where $\phi$is the porosity. We therefore have

$$\dot{h} = -\frac{k}{\mu\phi}\left(\rho g - \gamma\kappa/h\right) = \frac{k\rho g}{\mu\phi}\left(\frac{h_\infty - h}{h}\right)$$

where $h_\infty = \gamma\kappa/\rho g$ is the equilibrium rise height of the liquid.

For $h \ll h_\infty$, we have $h\dot{h} \approx \rho g k h_\infty/\mu\phi$ so that

$$h \approx \left(\frac{2\rho g k h_\infty}{\mu\phi}t\right)^{1/2} = \left(\frac{2\gamma\kappa k}{\mu\phi}t\right)^{1/2}.$$

Expect that as $t \to \infty$, $\dot{h} \to 0$ and so $h \to h_\infty$, as required.

To find the time taken to reach a given height, s, we separate variables and integrate to find

$$\int_0^s \frac{h \, \mathrm{d}h}{h_\infty - h} = \int_0^{t_s} \frac{k\rho g}{\mu\phi} \, \mathrm{d}t = \frac{k\rho g}{\mu\phi} t_s$$

and hence

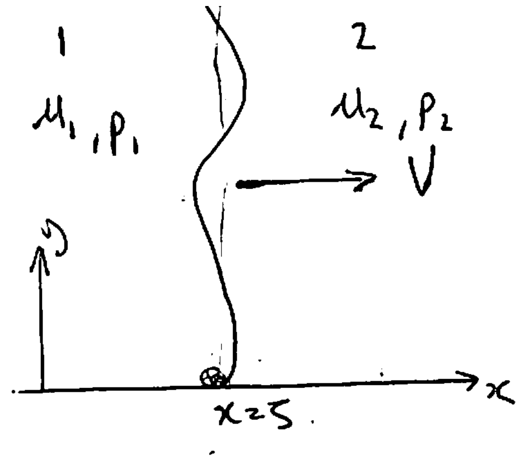
$$-h_\infty \log\left(\frac{h_\infty - s}{h_\infty}\right) - s = \frac{k\rho g}{\mu\phi} t_s.$$

Finally we have

$$t_s = -\frac{\mu\phi}{k\rho g}\left[s + h_\infty \log\left(1 - s/h_\infty\right)\right],$$

as required.

3. We consider the scenario illustrated schematically below.



(a) Since the fluids occupy a porous medium, we have $\mathbf{u}_i = -k/\mu_i \nabla p_i$ for $i = 1, 2$. We also have that both fluids are incompressible, so that $\nabla \cdot \mathbf{u}_i = 0$ and hence $\nabla^2 p_i = 0$.

In the base state, both fluids move with velocity $\mathbf{u}_i = V\mathbf{i}$ so that $p_i^{(0)} = -\mu_i V x/k + A_i$. However, since the interface is flat, the pressure must be continuous across the interface at $x = Vt$ and so we take, wlog, $p_i^{(0)}(Vt) = 0$, i.e.

$$p_i^{(0)}(x) = \frac{\mu_i V}{k}(Vt - x)$$

(b) We now consider the perturbation of the interface to $x = \zeta(y, t) = Vt + \epsilon e^{\sigma t} \sin \kappa y$ where we assume that ϵ is small. Because of the perturbation to the interface, the pressure fields, and hence velocities, will also be perturbed. We let $p_i = p_i^{(0)} + \epsilon p_i^{(1)} + ...$ and $\mathbf{u}_i = \mathbf{u}_i^{(0)} + \epsilon \mathbf{u}_i^{(1)} + ...$

Since we expect the pressure perturbations to inherit the sinusoidal dependence on y of the perturbed interface and they must be harmonic functions, we take as our ansatz

$$p_1^{(1)} = A_1 e^{\sigma t + \kappa x} \sin \kappa y, \quad p_2^{(1)} = A_2 e^{\sigma t - \kappa x} \sin \kappa y$$

where the signs of the exponents in x have been taken to ensure that $p_1^{(1)} \to 0$ as $x \to -\infty$ and similarly for $p_2^{(1)}$.

To determine the coefficients A_i, we must use the kinematic boundary condition, which takes the form

$$\mathbf{u}_i \cdot \mathbf{i} = \frac{\mathrm{D}\zeta}{\mathrm{D}t} = V + \epsilon e^{\sigma t}\left[\sigma \sin \kappa y + \epsilon \kappa \cos \kappa y \ \mathbf{u}_i^{(1)} \cdot \mathbf{j}\right] \approx V + \epsilon e^{\sigma t}\left[\sigma \sin \kappa y\right].$$

The left hand side may be calculated from Darcy's law and the above solution for the pressure field; we find that

$$-\frac{k}{\mu_i}\frac{\partial}{\partial x}\left[\frac{\mu_i V}{k}(Vt - x) + \epsilon A_i e^{\sigma t \pm \kappa x} \sin \kappa y\right]\Bigg|_{x = Vt + \epsilon e^{\sigma t} \sin \kappa y} \approx V + \epsilon e^{\sigma t}\left[\sigma \sin \kappa y\right]$$

and hence

$$A_i = \mp \sigma \frac{\mu_i}{k\kappa} e^{\mp \kappa V t}$$

with the $-$ taken for $i = 1$ and the $+$ for $i = 2$.

In summary, we have that

$$p_1^{(1)} = -\sigma \frac{\mu_1}{k\kappa} e^{\sigma t + \kappa(x - Vt)} \sin \kappa y$$

and

$$p_2^{(1)} = \sigma \frac{\mu_2}{k\kappa} e^{\sigma t - \kappa(x - Vt)} \sin \kappa y.$$

(c) To determine the growth rate, we use the pressure jump due to surface tension, which reads

$$p_2 - p_1|_{x=\zeta(y,t)} = \gamma \left(\frac{\partial^2 \zeta}{\partial y^2} \right) = -\gamma \kappa^2 \epsilon e^{\sigma t} \sin \kappa y.$$

Being careful to include the perturbation to $p_i^{(0)}$ from the perturbed interface, we have

$$\frac{(\mu_1 - \mu_2)V}{k} + \sigma \frac{\mu_1 + \mu_2}{k\kappa} = -\gamma \kappa^2.$$

from which we have

$$\sigma(\mu_1 + \mu_2) = -\gamma k \kappa^3 - (\mu_1 - \mu_2) V \kappa, \qquad (1)$$

as required.

Now, if $(\mu_1 - \mu_2)V > 0$ then $\sigma < 0$ for all $k > 0$ and so the interface is stable. (This corresponds to the more viscous liquid invading the less viscous.)

If $(\mu_1 - \mu_2)V < 0$ then $\sigma > 0$ (i.e. the interface is unstable) for $0 < \kappa < \kappa_0$ where

$$0 = -\gamma k \kappa_0^3 - (\mu_1 - \mu_2) V \kappa_0$$

i.e. $\kappa_0 = [(\mu_2 - \mu_1)V/\gamma k]^{1/2}$. Thus the interface is unstable provided that the wavelength is too large for surface tension to suppress it.

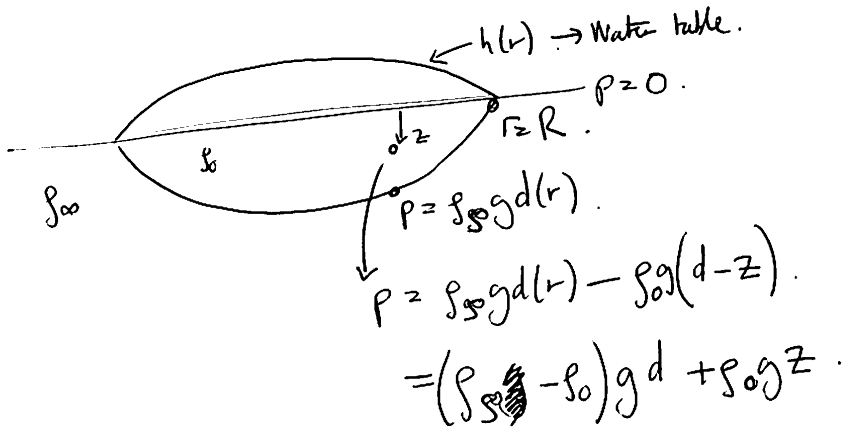
(d) Experimentally, we expect to observe the mode that grows most quickly. The fastest growing mode, $\kappa = \kappa_m$, has $\sigma'(\kappa_m) = 0$ so that

$$0 = -3\gamma k \kappa_m^2 + (\mu_2 - \mu_1)V$$

so that $\kappa_m = 3^{-1/2}\kappa_0$. The corresponding wavelength is

$$\lambda = \frac{2\pi}{\kappa_m} = 2\sqrt{3}\pi k^{1/2} \mathrm{Ca}^{-1/2}.$$

(e) This is not likely to be a realistic model of the Saffman-Taylor instability in a porous medium because the surface tension force was related to the macroscopic curvature of the interface. In the physical problem the surface tension jump is more likely to be determined by the microscopic curvature, and hence be related to the scale of the pores themselves.



4. The problem is illustrated schematically below. To determine the pressure *within* the lens, begin at sea level at some distance from the island. Going vertically down, we see that the pressure at the base of the lens must be $p_{base} = \rho_s g d(r)$ where ρ_s is the density of the (salty) sea.

   Now, if the lens is long and thin, then the scaling arguments from lectures show that the pressure within the lens must be approximately hydrostatic. Moving from the base vertically upwards, and assuming that the pressure is hydrostatic, then we have that at a depth z below sea level

   $$p[z(r)] = p_{base} - \rho_0 g[d(r) - z] = (\rho_s - \rho_0) g d(r) + \rho_0 g z. \tag{2}$$

   From Darcy's law, the horizontal (Darcy) velocity within the lens is given by

   $$u = -\frac{k}{\mu}\frac{\partial p}{\partial r} = -\frac{k(\rho_s - \rho_0)g}{\mu} d'(r).$$

   Conservation of mass across a ring at a radius r yields that $2\pi r \times u \times d(r) = \pi r^2 w_r$ (we assume that the problem is in steady state and so the total flux of fluid in must balance that across the ring). We therefore have that

   $$-2\frac{k(\rho_s - \rho_0)g}{\mu} d d' = r w_r,$$

   from which we find that

   $$d^2 = \frac{w_r \mu}{2(\rho_s - \rho_0) g k}(R^2 - r^2)$$

   where the radius R is that at which $d = 0$ and hence where the lens meets sea level. The result for $d(r)$ then follows.

   [*Note that we have not said anything about what R is. In reality, this must be determined by the length scale over which mixing between the salty and fresh water can happen to ensure a steady state. The word 'steady' is not used in the question to avoid confusion about whether there is really flow (and to emphasise that hydrostatic pressure may be assumed because of the thinness of the geometry).*]

To calculate the profile of the upper surface of the lens, we note that on this surface $p = 0$. Hence, from (2) we have that it is given by

$$z = -\frac{\rho_s - \rho_0}{\rho_0} d(r),$$

which is negative since positive z correspond to distances *below* sea level.

5. (a) The scaling form of the pde is

$$\frac{H}{T} \sim \frac{H^2}{R^2}$$

while volume conservation gives us

$$T^\alpha \sim HR^2$$

so that $H \sim T^\alpha R^{-2}$. Substituting this into the first scaling relationship, we have

$$T \sim R^2/H \sim R^4/T^\alpha$$

and hence that $R \sim T^{(\alpha+1)/4}$, $H \sim T^{(\alpha-1)/2}$.

From these scalings, it is natural to consider the combinations $rt^{-(\alpha+1)/4}$ and $ht^{-(\alpha-1)/2}$ so that we make the similarity ansatz

$$h(r,t) = t^{(\alpha-1)/2} f(\eta), \quad \text{where} \quad \eta = rt^{-(\alpha+1)/4}.$$

Substitution of this ansatz into the pde gives

$$t^{(\alpha-3)/2}\left(\frac{\alpha-1}{2}f - \frac{\alpha+1}{4}\eta f'\right) = t^{-2\frac{\alpha+1}{4}}\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(t^{2\frac{\alpha-1}{2}}\eta f\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta f\frac{\mathrm{d}f}{\mathrm{d}\eta}\right) + \frac{\alpha+1}{4}\eta^2 f' + \frac{1-\alpha}{2}\eta f = 0, \tag{3}$$

as desired.

We also have the volume constraint, which becomes

$$\frac{1}{2\pi} = \int_0^{\eta_N} \eta f(\eta)\ \mathrm{d}\eta,$$

and the front condition that $h[a(t),t)] = 0$, which becomes $f(\eta_N) = 0$ with $\eta_N = a(t)/t^{(\alpha+1)/4}$.

(b) We now let $\eta = \eta_N(1-x)$ with $x \ll 1$ to examine the region within the current but close to the edge, or nose. Substituting this ansatz into (3) and linearising gives

$$\eta_N^{-2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\eta_N f\frac{\mathrm{d}f}{\mathrm{d}x}\right) + \frac{\alpha+1}{4}\eta_N^2\eta_N^{-1}f' + \frac{1-\alpha}{2}\eta_N f = 0.$$

We spot that by letting $f = \eta_N^2 F(x)$ we may write

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F\frac{\mathrm{d}F}{\mathrm{d}x}\right) + \frac{\alpha+1}{4}F' + \frac{1-\alpha}{2}F = 0 \tag{4}$$

and, since $F(x=0)=0$, we expect that we may neglect the F term here, i.e. $F \ll 1$, and check this *a posteriori*. We find that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F\frac{\mathrm{d}F}{\mathrm{d}x}\right)+\frac{\alpha+1}{4}F' \approx 0$$

so that

$$F\frac{\mathrm{d}F}{\mathrm{d}x}+\frac{\alpha+1}{4}F \approx A = 0 \qquad \text{[Since } F=0 \text{ at the nose]}$$

We therefore find that

$$F=-\frac{\alpha+1}{4}x,$$

in which case it is self-consistent to neglect the linear term (4).

Finally, we have

$$f(\eta)=\eta_N^2\frac{\alpha+1}{4}(1-\eta/\eta_N),$$

as required.