

Problem Sheet 4: Solutions

Question 1. Rotating sphere in Stokes flow.

With $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{x}_0$, $r = |\mathbf{x} - \mathbf{x}_0|$ and

$$G_{ij} = \frac{\delta_{ij}}{r} + \frac{\hat{x}_i \hat{x}_j}{r^3},$$

the rotational dipole \mathbf{G}^c is defined by

$$G_{im}^c := \frac{1}{2} \epsilon_{mlj} \frac{\partial G_{ij}}{\partial x_{0,l}},$$

where

$$\epsilon_{mlj} := \begin{cases} +1 & \text{if } (m, l, j) = (1, 2, 3) \text{ or } (3, 1, 2) \text{ or } (2, 3, 1) \\ -1 & \text{if } (m, l, j) = (1, 3, 2) \text{ or } (2, 1, 3) \text{ or } (3, 2, 1) \\ 0 & \text{if any of } i, j, k \text{ are equal} \end{cases}$$

Part a. We have

$$\begin{aligned} \frac{1}{2} \epsilon_{mlj} \frac{\partial}{\partial x_{0,l}} G_{ij} &= \frac{1}{2} \epsilon_{mlj} \frac{\partial}{\partial x_{0,l}} \left[\frac{\delta_{ij}}{r} + \frac{\hat{x}_i \hat{x}_j}{r^3} \right], \\ &= \frac{1}{2} \epsilon_{mlj} \left[\delta_{ij} \frac{\partial}{\partial x_{0,l}} \left(\frac{1}{r} \right) - \frac{1}{r^3} (\delta_{ij} \hat{x}_j + \delta_{jl} \hat{x}_i) + \hat{x}_i \hat{x}_j \frac{\partial}{\partial x_{0,l}} \left(\frac{1}{r^3} \right) \right], \\ &= \epsilon_{mli} \frac{\hat{x}_l}{r^3}. \end{aligned}$$

Part b. Thus $G_{im}^c q_m$ is a solution of Stokes equations for any constant vector \mathbf{q} and it decays at spatial infinity. With \mathbf{x}_0 the centre of the sphere, the sphere is given by $r = a$, where we have

$$a^3 G_{im}^c \Omega_m = a^3 \epsilon_{iml} \Omega_m \frac{\hat{x}_l}{r^3} = \epsilon_{iml} \Omega_m \hat{x}_l,$$

which is the velocity on a rotating sphere.

Hence

$$v_i := a^3 G_{im}^c \Omega_m$$

in a solution of the Stokes flow for a rotating sphere with radius a and angular velocity $\boldsymbol{\Omega}$ and, by uniqueness, it is the solution (with constant pressure).

Part c. One can use brute force, though that would be on the long side. The stress field of the Stokes solution

$$u_i = \frac{1}{8\pi\mu} G_{ij} g_j$$

be given by $\sigma_{ij} = T_{ijp}g_p$, with symmetry in the indices i, j . You can show

$$\frac{\partial T_{ijs}}{\partial x_j} = -\delta_{is}\delta(\hat{\mathbf{x}}),$$

from the momentum balance for u_i , consistent with the notion that u_i is the flow associated with a point force.

Then the stress field associated with $\mathbf{v} = a^3\mathbf{G}^c \cdot \boldsymbol{\Omega}$, that is

$$v_m = a^3 G_{mn}^c \Omega_n,$$

is given by

$$\Sigma_{ij} = (8\pi\mu)a^3 T_{ijp}^c \Omega_p = (8\pi\mu) \frac{a^3}{2} \epsilon_{plq} \frac{\partial}{\partial x_{0,l}} T_{ijq} \Omega_p = -(8\pi\mu) \left(\frac{a^3}{2} \Omega_p \right) \epsilon_{plq} \frac{\partial}{\partial x_l} T_{ijq}.$$

Hence the s^{th} component of the moment of the sphere due to the fluid is given by

$$M_s = \int_{\text{Sphere}} \epsilon_{sri} x_r \Sigma_{ij} n_j dS = -(8\pi\mu) \left(\frac{a^3}{2} \Omega_p \right) \int_{\text{Sphere}} \epsilon_{sri} x_r \epsilon_{plq} \left(\frac{\partial}{\partial x_l} T_{ijq} \right) n_j dS.$$

Using the divergence theorem we have

$$M_s = -(8\pi\mu) \left(\frac{a^3}{2} \Omega_p \right) \int_{\text{Sphere}} \epsilon_{sji} \epsilon_{plq} \left(\frac{\partial}{\partial x_l} T_{ijq} \right) dV = -(8\pi\mu) \left(\frac{a^3}{2} \Omega_p \right) \int_{\text{Sphere}} \epsilon_{sri} x_r \epsilon_{plq} \frac{\partial}{\partial x_j} \left(\frac{\partial}{\partial x_l} T_{ijq} \right) dV.$$

The first term is zero as there is an contraction between ϵ with antisymmetry in i, j and T_{ijq} , with symmetry in i, j . Commuting derivatives and noting the above relation for the derivative of T_{ijk} we have

$$M_s = (8\pi\mu) \left(\frac{a^3}{2} \Omega_p \right) \int_{\text{Sphere}} \epsilon_{sri} x_r \epsilon_{pli} \frac{\partial}{\partial x_l} (\delta(\hat{\mathbf{x}})) dV = -(8\pi\mu) \left(\frac{a^3}{2} \Omega_p \right) \int_{\text{Sphere}} \epsilon_{sri} \epsilon_{pri} \delta(\hat{\mathbf{x}}) dV,$$

noting the additional surface integral term to deduce the final equality must be zero as the δ -function has no support on the surface of the sphere. Finally, with $\epsilon_{sri} \epsilon_{pri} = 2\delta_{ps}$, we have

$$M_s = -8\pi\mu a^3 \Omega_s,$$

as required.

Question 2. Ciliary Pumping.

Detailed calculation is not necessary to deduce the expression for U . Fourier modes decouple at the first non-trivial order the second derivatives acting on mode number n just induce a factor of n^2 . Thus one can determine the contribution to U from the Fourier mode

$$x_e - x = \epsilon(-b_n \cos(n[x + t])), \quad y_e = \epsilon c_n \sin(n[x + t]) \quad (1)$$

by identifying

$$a = -nb_n, \quad b = nc_n \quad (2)$$

in the example sheet result

$$U_2 = \frac{1}{2} (b^2 + 2ab - a^2)$$

to obtain the contribution from this mode.

We then can consider the remaining Fourier modes without detailed calculation. The mode

$$x_e - x = \epsilon a_n \sin(n[x + t]), \quad y_e = \epsilon(-d_n \cos(n[x + t]))$$

is simply a phase shift of the mode in equation (1). By considering a shifted time coordinate

$$\bar{t} = t + \frac{1}{\omega n} \frac{\pi}{2}$$

we can determine the contribution from this mode by the substitution

$$d_n \rightarrow c_n, \quad a_n \rightarrow -b_n.$$

followed by the identification (2). Hence we use

$$a = na_n, \quad b = nd_n$$

in the example sheet result

$$U_2 = \frac{1}{2} (b^2 + 2ab - a^2)$$

to obtain the contribution from this mode.

Summing all contributions, and noting $U = \epsilon^2 U_2$ to leading order, gives

$$U = \frac{1}{2} \epsilon^2 \sum_{n=1}^{\infty} n^2 [c_n^2 + d_n^2 - a_n^2 - b_n^2 + 2(a_n d_n - c_n b_n)].$$

We now determine power optimal strokes, defined as those maximising absolute velocity, subject to the constraint of a fixed power consumption W using Lagrange multipliers with the above leading order expressions. Thus we consider

$$L[\{a_n, b_n, c_n, d_n\}] = U[\{a_n, b_n, c_n, d_n\}] - \lambda(P[\{a_n, b_n, c_n, d_n\}] - W) \quad (3)$$

and the extremal conditions

$$\frac{\partial L}{\partial a_n} = \frac{\partial L}{\partial b_n} = \frac{\partial L}{\partial c_n} = \frac{\partial L}{\partial d_n} = 0. \quad (4)$$

Thus

$$a_n^2 + 2a_n d_n - d_n^2 = 0, \quad b_n^2 - 2b_n c_n - c_n^2 = 0.$$

and hence $a_n = (-1 \pm \sqrt{2})d_n$ and $b_n = (1 \pm \sqrt{2})c_n$, which yields

$$U = \epsilon^2 \sum_{n=1}^{\infty} n^2 \left((2 \pm 2\sqrt{2})c_n^2 + (2 \mp 2\sqrt{2})d_n^2 \right) \quad (5)$$

$$P = \epsilon^2 \sum_{n=1}^{\infty} n^3 \left((4 \pm 2\sqrt{2})c_n^2 + (4 \mp 2\sqrt{2})d_n^2 \right). \quad (6)$$

Therefore the optimal stroke is achieved when $a_n = b_n = c_n = d_n = 0$ for $n \geq 2$. Without loss of generality, we can set $b_1 = 0$ as it is just a phase difference, and we finally have the optimal strokes $a_1 = (-1 \pm \sqrt{2})d_1$.

As an extra to the question, note the extremal velocity is

$$U = \mp \frac{W}{\sqrt{2}}.$$

Question 3. Ciliate Motility.

Take a reference frame comoving with the swimmer oriented such that the direction of the swimmer velocity is given by $\mathbf{U} = U\mathbf{e}_z$. The non-dimensional Stokes equations are

$$\nabla^2 \mathbf{u} = \nabla p, \quad \nabla \cdot \mathbf{u} = 0,$$

with

$$\mathbf{u} = -U\mathbf{e}_z, \quad \text{as } |\mathbf{x}| \rightarrow \infty, \quad \mathbf{u} = \epsilon \frac{d\beta_1}{dt} \sin \theta \mathbf{e}_\theta = \epsilon \dot{\beta}_1 \sin \theta \mathbf{e}_\theta, \quad \text{on } r = 1,$$

where \mathbf{e}_θ is the unit vector in the direction of increasing spherical polar θ , where $r = |\mathbf{x}|$ and $z = r \cos \theta$, $x = r \sin \theta \cos \varphi$ for instance.

Show that

$$\mathbf{u} = \left[-U(t) + \frac{Q(t)}{r^3} \right] \cos \theta \mathbf{e}_r + \left[U(t) + \frac{P(t)}{r^3} \right] \sin \theta \mathbf{e}_\theta, \quad p = \text{Const}$$

is a solution of the Stokes equation for $Q(t) = 2P(t)$. To do this, you will need to consider the vector Laplacian of \mathbf{u} . This is non-trivial in non-Cartesian coordinates and you may wish to consider using a symbolic algebra package such as Mathematica:

$$\mathbf{f1}[r, \text{theta}, \text{phi}] := (-U + Q/r^3) * \text{Cos}[\text{theta}]$$

$$\mathbf{f2}[r, \text{theta}, \text{phi}] := (U + P/r^3) * \text{Sin}[\text{theta}]$$

$$\text{FullSimplify}[\text{Laplacian}[\mathbf{f1}[r, \text{theta}, \text{phi}], \{r, \text{theta}, \text{phi}\}, \text{"Spherical"}] - 2 * \mathbf{f1}[r, \text{theta}, \text{phi}]/r^2 - 2/(r^2 * \text{Sin}[\text{theta}]) * (\mathbf{f2}[r, \text{theta}, \text{phi}] * \text{Cos}[\text{theta}] + \text{Sin}[\text{theta}] * D[\mathbf{f2}[r, \text{theta}, \text{phi}], \text{theta}])]$$

$$\frac{2(-2P + Q)\text{Cos}[\text{theta}]}{r^5}$$

$$\text{FullSimplify}[\text{Laplacian}[\mathbf{f2}[r, \text{theta}, \text{phi}], \{r, \text{theta}, \text{phi}\}, \text{"Spherical"}] + 2 * D[\mathbf{f1}[r, \text{theta}, \text{phi}], \text{theta}]/r^2 - 1/(r^2 * \text{Sin}[\text{theta}] * \text{Sin}[\text{theta}]) * \mathbf{f2}[r, \text{theta}, \text{phi}]]$$

$$\frac{2(2P - Q)\text{Sin}[\text{theta}]}{r^5}$$

Given $Q(t) = 2P(t)$, the above two expressions are zero and the governing equation is satisfied. One can then read off that $U(t) = 2\epsilon \dot{\beta}_1/3$ from the boundary condition, by finding $Q(t)$ in terms of $U(t)$ from setting the coefficient of \mathbf{e}_r to zero and then equating the coefficient of \mathbf{e}_θ to the boundary conditions at the boundary.

Question 4. Resistive force theory.

For the force balance with a spherical cell body of radius a , we have

$$\mathbf{0} = (\text{Drag force on body}) + (\text{Drag force on flagellum}). \quad (7)$$

and $\mathbf{e}_t = (-1, \epsilon h_s)$, $\mathbf{e}_n = (\epsilon h_s, 1)$ and the velocity of the flagellum element is given by $\mathbf{U} = (U, V + \epsilon h_t)$.

Hence the drag force per unit length on the element ds is given by

$$\begin{aligned} \mathbf{f} &= -[C_N \mathbf{e}_n \cdot \mathbf{U} \mathbf{e}_n + C_T \mathbf{e}_t \cdot \mathbf{U} \mathbf{e}_t] = -[(C_N - C_T) \mathbf{e}_n \cdot \mathbf{U} \mathbf{e}_n + C_T \mathbf{U}] \\ &= -[(C_N - C_T) \mathbf{e}_n \otimes \mathbf{e}_n + C_T \mathbf{I}] \mathbf{U} \\ &= -\left[(C_N - C_T) \begin{pmatrix} \epsilon^2 h_s^2 & \epsilon h_s \\ \epsilon h_s & 1 \end{pmatrix} + C_T \mathbf{I} \right] \begin{pmatrix} U \\ \epsilon h_t + V \end{pmatrix} \\ &= -(C_N - C_T) \begin{pmatrix} \epsilon^2 h_s^2 U + \epsilon^2 h_s h_t + \epsilon h_s V \\ \epsilon h_s U + \epsilon h_t + V \end{pmatrix} - C_T \begin{pmatrix} U \\ \epsilon h_t + V \end{pmatrix} \end{aligned}$$

Integrating over the flagellum length, $s \in [0, L]$, gives

$$-U \begin{pmatrix} C_T L + \epsilon^2 (C_N - C_T) \int_0^L ds h_s^2 \\ \epsilon (C_N - C_T) \int_0^L ds h_s \end{pmatrix} - V \begin{pmatrix} \epsilon (C_N - C_T) \int_0^L ds h_s \\ C_N L \end{pmatrix} - \begin{pmatrix} \epsilon^2 (C_N - C_T) \int_0^L ds h_s h_t \\ \epsilon C_N \int_0^L ds h_t \end{pmatrix}$$

Clearly the term $\epsilon^2 (C_N - C_T) \int_0^L ds h_s^2$ is a lower order than $C_T L$ and hence the former is dropped.

The cell body drag follows by setting $h = 0$ and replacing parameters dependent on geometry with those of the cell body. Hence the cell body drag is

$$-U \begin{pmatrix} C_T^b L_b \\ 0 \end{pmatrix} - V \begin{pmatrix} 0 \\ C_N^b L_b \end{pmatrix}$$

Thus, using Eqn.(7),

$$(C_T^b L_b + C_T L) U = -(C_N - C_T) \left[\epsilon^2 \int_0^L ds h_s h_t + \epsilon V \int_0^L ds h_s \right],$$

and

$$V = -\frac{1}{C_N^b L_b + C_N L} \left[\epsilon C_N \int_0^L ds h_t + \epsilon U (C_N - C_T) \int_0^L ds h_s \right].$$

Substituting the expression for V into the expression for U we have

$$(C_T^b L_b + C_T L + O(\epsilon^2)) U = -\epsilon^2 (C_N - C_T) \left[\int_0^L ds h_s h_t - \frac{C_N}{C_N^b L_b + C_N L} \int_0^L ds h_s \int_0^L ds h_t \right].$$

We can drop the $O(\epsilon^2)$ on the left as it is asymptotically small relative to $C_T L$.

Hence we have at leading order

$$U = \epsilon^2 \frac{C_T - C_N}{C_N^b L_b + C_T L} \left[\int_0^L ds h_s h_t - \frac{C_N}{C_N^b L_b + C_N L} \int_0^L ds h_s \int_0^L ds h_t \right]$$

and we recover the expression in the lecture notes provided

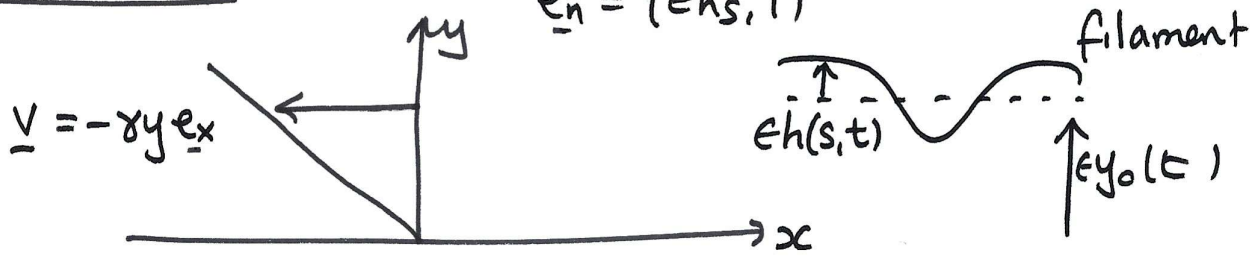
$$\frac{C_N}{C_N^b L_b + C_N L} \frac{\int_0^L ds h_s \int_0^L ds h_t}{\int_0^L ds h_s h_t} \ll 1.$$

Question 5

$$\underline{e}_t = (-1, \epsilon h_s)$$

$$\underline{e}_n = (\epsilon h_s, 1)$$

displacement y_0 is $O(\epsilon)$ on assumption there is no drift



\therefore On filament, $\underline{v} = -\gamma \epsilon (y_0 + h) \underline{e}_x$, $\underline{u} = (u, \epsilon y_{0t} + \epsilon h_t)$.

$$\underline{u} - \underline{v} = \begin{pmatrix} u - \gamma \epsilon (y_0 + h) \\ \epsilon y_{0t} + \epsilon h_t \end{pmatrix}$$

$\epsilon y_{0t} \equiv V \text{ in } \mathcal{Q}_4$

$\underline{0}$ = Drag force \underline{f} = drag per unit length

$$\underline{f} = -(C_N - C_T) \underline{e}_n \cdot (\underline{u} - \underline{v}) \underline{e}_n - C_T (\underline{u} - \underline{v})$$

$$= \underbrace{f}_{\mathcal{Q}_4} + (C_N - C_T) \underbrace{(\underline{e}_n \cdot \underline{v}) \underline{e}_n}_{-\gamma \epsilon^2 h_s (y_0 + h) \begin{pmatrix} \epsilon h_s \\ 1 \end{pmatrix}} + \underbrace{C_T \underline{v}}_{-C_T \gamma \epsilon (y_0 + h)}$$

Same expression as in \mathcal{Q}_4 with

$L_b \equiv 0$ as no cell body and $V \rightarrow \epsilon y_{0t}$

$$\int_0^L (y_0 + h) ds$$

From zero net drag

$$\underline{0} = -u \left(\begin{matrix} C_T L + O(\epsilon^2) \\ \epsilon (C_N - C_T) \int_0^L ds h_s \end{matrix} \right) - \left(\begin{matrix} \epsilon (C_N - C_T) \int_0^L ds h_s \\ C_N L \end{matrix} \right) \epsilon y_{0t}$$

$$- \left(\begin{matrix} \epsilon^2 (C_N - C_T) \int_0^L ds h_s h_t \\ \epsilon C_N \int_0^L ds h_t \end{matrix} \right) - \left(\begin{matrix} O(\epsilon^3) & + \gamma \epsilon C_T \int_0^L (y_0 + h) ds \\ \gamma \epsilon^2 (C_N - C_T) \int_0^L ds h_s (y_0 + h) \end{matrix} \right)$$

$$\begin{aligned} \therefore (C_T L + O(\epsilon^2)) U &= -\epsilon (C_N - C_T) (\epsilon y_{0t}) \int_0^L ds h_s + O(\epsilon^3) \\ &\quad - \epsilon^2 (C_N - C_T) \int_0^L ds h_s h_t + \gamma \epsilon C_T \underbrace{\int_0^L (y_0 + h) ds}_{\left(L y_0 + \int_0^L h ds \right)} \end{aligned}$$

$$\begin{aligned} (C_N L) (\epsilon y_{0t}) &= -\epsilon U (C_N - C_T) \int_0^L ds h_s - \epsilon C_N \int_0^L ds h_t \\ &\quad - \gamma \epsilon^2 (C_N - C_T) \int_0^L ds h_s (y_0 + h) \end{aligned}$$

Leading order

$$\begin{aligned} (C_T L + O(\epsilon^2)) U &= -\epsilon \left\{ \begin{array}{l} \text{from } \epsilon y_{0t} \\ \downarrow \\ O(\epsilon) \end{array} \right\} + O(\epsilon^3) + O(\epsilon^2) \\ &\quad + \gamma \epsilon C_T L y_0 + \gamma \epsilon C_T \int_0^L h ds \end{aligned}$$

$$\epsilon y_0 = \int_0^t d\bar{t} \epsilon y_{0\bar{t}} = -\frac{\epsilon}{C_N L} \left[(C_N - C_T) \int_0^t d\bar{t} \int_0^L ds h_s + C_N \int_0^t d\bar{t} \int_0^L ds h_t \right] + O(\epsilon^2)$$

assuming filament starts at $y_0 \equiv 0$

Aside. No drift entails $h(s,t)$ is such that these integrals remain $O(1)$ for large time

$$\begin{aligned} \therefore U &= -\frac{\gamma \epsilon}{C_N L} \left[(C_N - C_T) \int_0^t d\bar{t} \int_0^L ds h_s + C_N \int_0^t d\bar{t} \int_0^L ds h_t \right] \\ &\quad + \gamma \epsilon / L \int_0^L h ds + O(\epsilon^2) \end{aligned}$$