6. HYPERBOLIC SURFACES

6.1 Models for the Hyperbolic Plane

Definition 6.1 *(a) The* ***hyperbolic plane*** $\mathbb{H}$ *is the geometric surface formed from the upper half-plane*

$$\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\},$$

endowed with the first fundamental form

$$\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}.$$

This is the ***Poincaré's half-plane model.***

(b) We showed earlier (Example 3.54) that the Gaussian curvature of $\mathbb{H}$ *equals* -1. *We also showed in Example 4.8 that the geodesics of* $\mathbb{H}$ *are the half-lines perpendicular to the real axis and the semicircles that meet the real axis at right angles. Note that there is a unique geodesic between any two points of* $\mathbb{H}$.

(c) As $E = G$ *in the above first fundamental form then angles are measured in* $\mathbb{H}$ *in the same way that they are in* $\mathbb{C}$.

(d) In Method 2 of Example 4.8, we showed that, given real numbers a, b, c, d *such that* $ad - bc = 1$,

$$w(z) = \frac{az + b}{cz + d}$$

is a bijective isometry of $\mathbb{H}$.

The Möbius map

$$w = \frac{z - i}{z + i}, \qquad z = \frac{i(1 + w)}{1 - w}.$$

takes $\mathbb{H}$ conformally to the disc

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}.$$

If we assign a first fundamental form to $\mathbb{D}$ in such a way that the Möbius map is an isometry then $\mathbb{D}$ is a second model for the hyperbolic plane known as **Poincaré's disc model.** Again we note the above first fundamental form can be rewritten in terms of z as

$$\frac{-4\,|\mathrm{d}z|^2}{(z - \bar{z})^2}.$$

Applying the change of variable we note that

$$|\mathrm{d}z|^2 = \left|\frac{(1 - w) + (1 + w)}{(1 - w)^2}\right|^2 |\mathrm{d}w|^2 = \frac{4\,|\mathrm{d}w|^2}{|1 - w|^4}$$

and

$$\begin{aligned}(z-\bar{z})^2 &= \left(\frac{i(1+w)}{1-w}+\frac{i(1+\bar{w})}{1-\bar{w}}\right)^2 \\ &= -\left(\frac{(1+w)(1-\bar{w})+(1-w)(1+\bar{w})}{(1-w)(1-\bar{w})}\right)^2 \\ &= .-4\left(\frac{1-w\bar{w}}{(1-w)(1-\bar{w})}\right)^2 \\ &= -4\frac{\left|1-|w|^2\right|^2}{|1-w|^4}\end{aligned}$$

Hence

$$\frac{-4\,|\mathrm{d}z|^2}{(z-\bar{z})^2}=\frac{-16\,|\mathrm{d}w|^2}{|1-w|^4}\times\frac{|1-w|^4}{-4\left|1-|w|^2\right|^2}=\frac{4\,|\mathrm{d}w|^2}{\left|1-|w|^2\right|^2}.$$

Proposition 6.2 *(a)* ***Poincaré's disc model*** *for the hyperbolic plane is the disc*

$$\mathbb{D}=\{z\in\mathbb{C}\mid |z|<1\}$$

endowed with the first fundamental form

$$\frac{4\,|\mathrm{d}z|^2}{\left|1-|z|^2\right|^2}.$$

(b) The Gaussian curvature equals -1 and the geodesics are the diameters and the circular arcs that meet the unit circles in right angles. Angles are measured in $\mathbb{D}$ the same way they are in $\mathbb{H}$.

(c) The orientation-preserving isometries of $\mathbb{D}$ take the form

$$f(z)=\frac{e^{i\theta}\,(z-a)}{1-\bar{a}z}$$

where $|a|<1$ and $\theta\in\mathbb{R}$.

Proof. (a) was proved in the discussion previous to this proposition and (b) follows as the map w is an isometry and conformal map from $\mathbb{H}$ to $\mathbb{D}$ which preserves curvature, geodesics and angles.

To prove (c) we first need to show that the circle $|z|=1$ maps to itself; if $|z|=1$ then

$$|f(z)|=\left|\frac{e^{i\theta}\,(z-a)}{1-\bar{a}z}\right|=\left|\frac{\bar{z}(z-a)}{1-\bar{a}z}\right|=\left|\frac{1-a\bar{z}}{1-\bar{a}z}\right|=1.$$

And as $f(a)=0$ then f maps $\mathbb{D}$ bijectively onto $\mathbb{D}$. Also f is orientation-preserving as it is holomorphic. Further

$$|\mathrm{d}f|=\left|\frac{(1-\bar{a}z)+\bar{a}\,(z-a)}{(1-\bar{a}z)^2}\right||\mathrm{d}z|=\left|\frac{1-|a|^2}{(1-\bar{a}z)^2}\right||\mathrm{d}z|$$

and then

$$\begin{aligned}
\frac{4\,|\mathrm{d}f|^2}{\left|1-|f(z)|^2\right|^2} &= 4\,|\mathrm{d}z|^2\left|\frac{1-|a|^2}{(1-\bar{a}z)^2}\right|^2\left|1-\left|\frac{z-a}{1-\bar{a}z}\right|^2\right|^{-2} \\
&= 4\,|\mathrm{d}z|^2\left|\frac{1-|a|^2}{(1-\bar{a}z)^2}\right|^2\left|\frac{(1-\bar{a}z)^2}{|1-\bar{a}z|^2-|z-a|^2}\right|^2 \\
&= \frac{4\,|\mathrm{d}z|^2\left(1-|a|^2\right)^2}{(1+a\bar{a}z\bar{z}-z\bar{z}-a\bar{a})^2} \\
&= \frac{4\,|\mathrm{d}z|^2\left(1-|a|^2\right)^2}{\left(1-|a|^2\right)^2\left(1-|z|^2\right)^2} \\
&= \frac{4\,|\mathrm{d}z|^2}{\left|1-|z|^2\right|^2},
\end{aligned}$$

showing f is an isometry of $\mathbb{D}$. Further these are *all* the isometries of $\mathbb{D}$. Given an isometry g of $\mathbb{D}$ then, by setting $a = g(0)$ and choosing θ appropriately, we note $f^{-1} \circ g$ is an isometry which sends 0 to 0 and the interval $(0,1)$ to itself. For orientation, distance and angles to be preserved, it follows that $f^{-1} \circ g$ is the identity and hence $g = f$. ■

Example 6.3 *(a) Let $0 < r < 1$. Find the distance in $\mathbb{D}$ between 0 and r as measured along the real axis.*
(b) Find the distance in $\mathbb{D}$ between $a, b \in \mathbb{D}$.
(c) Deduce a formula for the distance in $\mathbb{H}$ between $p, q \in \mathbb{H}$.

Solution. (a) The distance between 0 and r equals

$$\begin{aligned}
\int_0^r \frac{2\mathrm{d}x}{1-x^2} &= \int_0^r \left(\frac{1}{1-x}+\frac{1}{1+x}\right)\mathrm{d}x \\
&= \left[\log\left(\frac{1+x}{1-x}\right)\right]_0^r \\
&= \log\left(\frac{1+r}{1-r}\right) \\
&= 2\tanh^{-1} r.
\end{aligned}$$

(b) Given points $a, b \in \mathbb{D}$ then the Möbius map

$$\frac{e^{i\theta}(z-a)}{1-\bar{a}z}$$

is an isometry of $\mathbb{D}$ which takes a to 0 and for an appropriate choice of θ takes b to the positive real axis. Hence the distance between a and b equals

$$d_{\mathbb{D}}(a,b) = 2\tanh^{-1}\left|\frac{b-a}{1-\bar{a}b}\right|,$$

as measured along the geodesic between them.
(c) Recall that the map $(z-i)/(z+i)$ is an isometry between $\mathbb{H}$ and $\mathbb{D}$. So given $p,q\in\mathbb{H}$ the distance between them equals

$$\begin{aligned} d_{\mathbb{H}}(p.q) &= d_{\mathbb{D}}\left(\frac{p-i}{p+i},\frac{q-i}{q+i}\right) \\ &= 2\tanh^{-1}\left|\frac{\left(\frac{q-i}{q+i}\right)-\left(\frac{p-i}{p+i}\right)}{1-\left(\frac{\bar{p}+i}{\bar{p}-i}\right)\left(\frac{q-i}{q+i}\right)}\right| \\ &= 2\tanh^{-1}\left|\frac{(\bar{p}-i)\left((q-i)(p+i)-(p-i)(q+i)\right)}{(p+i)\left((q+i)(\bar{p}-i)-(\bar{p}+i)(q-i)\right)}\right| \\ &= 2\tanh^{-1}\left|\frac{-2ip+2iq}{2i\bar{p}-2iq}\right| \\ &= 2\tanh^{-1}\left|\frac{q-p}{q-\bar{p}}\right|. \end{aligned}$$

■

Remark 6.4 *Poincare's models for the hyperbolic plane date to 1882. There were other models for the hyperbolic plane, most notably one due to Eugenio Beltrami (1868) and Felix Klein (1871). This model again uses the open unit disc, the geodesics are the line segments in the disc, but the model is not conformal with both distance and angle being measured in a non-Euclidean fashion.*

6.2 Hyperbolic geometry and trigonometry

We have yet to show that the hyperbolic distance function $d_{\mathbb{D}}$ is a metric. Certainly $d_{\mathbb{D}}(z,w)\geqslant 0$ and $d_{\mathbb{D}}(z,w)=0$ if and only if $z=w$. Also symmetry follows as for $z,w\in\mathbb{D}$ then $|w-z|=|z-w|$ and $|1-\bar{z}w|=|1-\bar{w}z|$ as they are conjugates of one another. As there is an isometry of $\mathbb{D}$ taking any of a triangle's vertices to 0, the triangle inequality follows from:

Proposition 6.5 *For $z,w\in\mathbb{D}$,*

$$d_{\mathbb{D}}(z,w)\leqslant d_{\mathbb{D}}(0,z)+d_{\mathbb{D}}(0,w),$$

with equality if and only if z/w is real and negative.

This is turn will follow from:

Proposition 6.6 *(**Hyperbolic cosine rule**) Consider a hyperbolic triangle with vertices $0,z,w$. Write*

$$a=d_{\mathbb{D}}(0,z),\qquad b=d_{\mathbb{D}}(0,w),\qquad c=d_{\mathbb{D}}(z,w)$$

and angle C at 0. Then

$$\cosh c=\cosh a\cosh b-\sinh a\sinh b\cos C.$$

Note that if the approximations $\cosh x \approx 1 + x^2/2$ *and* $\sinh x \approx x$ *apply then the hyperbolic cosine rule approximates to*

$$c^2 = a^2 + b^2 - 2ab\cos C$$

which is the usual Euclidean cosine rule.

Proof. (Of the triangle inequality) This follows from the cosine rule as $\cos C \geqslant -1$ so that

$$\begin{aligned} \cosh c &\leqslant \cosh a \cosh b + \sinh a \sinh b \\ &= \cosh(a+b). \end{aligned}$$

As cosh is strictly increasing for non-negative arguments then

$$c \leqslant a + b.$$

Further we only have equality when $\cos C = -1$ and $C = \pi$ in which case w/z is real and negative. ■

Proof. (Of the cosine rule) Without loss of generality we may assume that z is positive. Then

$$z = \tanh \frac{d_{\mathbb{D}}(0,z)}{2}, \qquad w = e^{iC} \tanh \frac{d_{\mathbb{D}}(0,w)}{2}.$$

By the hyperbolic tangent half-angle formulae (the hyperbolic 't-formulae') then

$$\cosh d_{\mathbb{D}}(0,z) = \frac{1+z^2}{1-z^2}, \qquad \cosh d_{\mathbb{D}}(0,w) = \frac{1+|w|^2}{1-|w|^2},$$

and by definition

$$\tanh \frac{1}{2} d_{\mathbb{D}}(z,w) = \left| \frac{w-z}{1-zw} \right|.$$

So

$$\begin{aligned} \cosh d_{\mathbb{D}}(a,b) &= \frac{|1-zw|^2 + |w-z|^2}{|1-zw|^2 - |w-z|^2} \\ &= \frac{(1+z^2)\left(1+|w|^2\right) - 2\left(zw + z\bar{w}\right)}{(1-z^2)\left(1-|w|^2\right)} \\ &= \left(\frac{1+z^2}{1-z^2}\right)\left(\frac{1+|w|^2}{1-|w|^2}\right) - \left(\frac{2z}{1-z^2}\right)\left(\frac{2|w|}{1-|w|^2}\right)\left(\frac{w+\bar{w}}{2|w|}\right) \\ &= \cosh a \cosh b - \sinh a \sinh b \cos C, \end{aligned}$$

recalling

$$\sinh a = \frac{2z}{1-z^2}, \qquad \sinh b = \frac{2|w|}{1-|w|^2}, \qquad \cos \arg w = \frac{\operatorname{Re} w}{|w|}.$$

■

Remark 6.7 *(**Dual hyperbolic cosine rule**) As with spherical geometry, there is a second 'dual' cosine rule which has no equivalent in Euclidean geometry. In the hyperbolic case this reads as*

$$\cos C = -\cos A \cos B + \sin A \sin B \cosh c.$$

Proposition 6.8 *(**Hyperbolic sine rule**) For a hyperbolic triangle in $\mathbb{D}$ with angles A, B, C and sides a, b, c then*

$$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}.$$

Solution. This is Sheet 3, Part C, Exercise 1. ■

Example 6.9 *In $\mathbb{D}$ a circle of radius R has area $4\pi \sinh^2(R/2)$ and circumfernece $2\pi \sinh R$. Note that, for small values of R, these formulae approximate to πR^2 and $2\pi R$.*

Solution. The circle of radius R, centred on the origin in $\mathbb{D}$, corresponds to the circle $|z| = \tanh(R/2)$. So its interior has area equalling

$$\begin{aligned}
&\iint_{|z| \leqslant \tanh(R/2)} \sqrt{EG - F^2}\, \mathrm{d}x\, \mathrm{d}y \\
= &\int_0^{\tanh(R/2)} \int_0^{2\pi} \frac{4}{(1-r^2)^2}\, r\, \mathrm{d}r\, \mathrm{d}\theta \\
= &\ 2\pi \left[\frac{2}{1-r^2}\right]_0^{\tanh(R/2)} \\
= &\ 4\pi \left(\cosh^2\left(\frac{R}{2}\right) - 1\right) \\
= &\ 4\pi \sinh^2\left(\frac{R}{2}\right).
\end{aligned}$$

The circle can be parameterized as $x = \tanh(R/2)\cos t$ and $y = \tanh(R/2)\sin t$, so it has circumference

$$\begin{aligned}
&\int_{t=0}^{2\pi} \sqrt{E\dot{x}^2 + G\dot{y}^2}\, \mathrm{d}t \\
= &\ \tanh(R/2) \int_{t=0}^{2\pi} \frac{2}{1-\tanh^2(R/2)}\, \mathrm{d}t \\
= &\ \frac{4\pi \tanh(R/2)}{\operatorname{sech}^2(R/2)} \\
= &\ 4\pi \sinh\left(\frac{R}{2}\right) \cosh\left(\frac{R}{2}\right) \\
= &\ 2\pi \sinh R.
\end{aligned}$$

■

Remark 6.10 *It follows that a circle in $\mathbb{D}$ is a Euclidean circle – this is because isometries of $\mathbb{D}$ are Möbius maps which send circles to circles. However the centre of a hyperbolic circle will not in general coincide with the Euclidean centre. Further it is also now clear, that through three non-collinear points there need not be a circle in $\mathbb{D}$. For example the points $0, 1-\varepsilon$ and $i(1-\varepsilon)$ are not collinear – as they lie on two different diameters – but any hyperbolic (and so Euclidean) circle passing through these three points will not entirely lie in $\mathbb{D}$.*

Theorem 6.11 *(**Lambert's Formula**) Given a triangle T in $\mathbb{D}$, bounded by geodesics, its area equals*

$$\pi - \alpha - \beta - \gamma$$

where α, β, γ are the three angles.

Proof. The local Gauss-Bonnet theorem states

$$\int_{\gamma} k_g \, \mathrm{d}s + \iint_{R} K \, \mathrm{d}A + \sum_{i=1}^{n} \alpha_i = 2\pi.$$

As $k_g = 0$ on a geodesic, and recalling that $K = -1$, we find

$$0 - A + (\pi - \alpha) + (\pi - \beta) + (\pi - \gamma) = 2\pi$$

which rearranges to the required result. Note that the maximal area π can be achieved by having all three vertices on the boundary of $\mathbb{D}$. Such a triangle is called *triasymptotic*. ∎

Proposition 6.12 *(**The angle of parallelism**) Let l be a line in $\mathbb{D}$ and P be a point at distance $d > 0$ from l. This distance d is measured along a perpendicular from P to a point O on l. Then a line through P will meet l if the angle the line makes with OP is less than*

$$\Pi(d) = \sin^{-1} \operatorname{sech} d.$$

$\Pi(d)$ is known as the angle of parallelism.

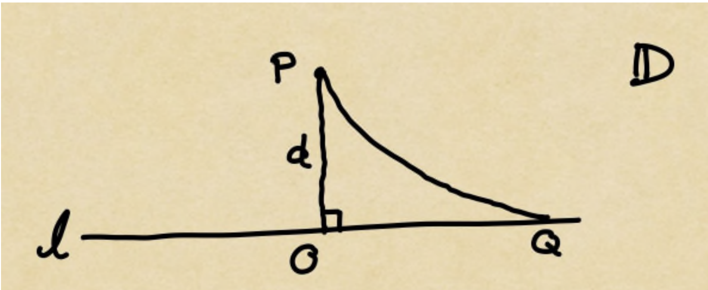

Figure 6.1 – the angle of parallelism

Proof. Without loss of generality we can take l to be the real axis and P to be on the positive imaginary axis, so represented by the complex number $\tanh(d/2)$. The point O is then the origin. Say that a second line passes through P, making an angle θ, and intersects l at the point Q. By the sine rule

$$\sin\theta = \frac{\sinh d_{\mathbb{D}}(O,Q)}{\sinh d_{\mathbb{D}}(P,Q)}.$$

And by the hyperbolic cosine rule we have

$$\cosh d_{\mathbb{D}}(P,Q) = \cosh d_{\mathbb{D}}(O,Q)\cosh d.$$

Eliminating $d_{\mathbb{D}}(O,Q)$ we find

$$\begin{aligned}\sin\theta &= \frac{1}{\sinh d_{\mathbb{D}}(P,Q)}\sqrt{\frac{\cosh^2 d_{\mathbb{D}}(P,Q)}{\cosh^2 d} - 1} \\ &= \sqrt{\coth^2 d_{\mathbb{D}}(P,Q)\,\mathrm{sech}^2 d - \mathrm{cosech}^2 d_{\mathbb{D}}(P,Q)}.\end{aligned}$$

Now as $d_{\mathbb{D}}(P,Q) \to \infty$ then $\coth^2 d_{\mathbb{D}}(P,Q) \to 1$ and $\mathrm{cosech} d_{\mathbb{D}}(P,Q) \to 0$. So the limiting case for when we can solve for θ is when

$$\sin\theta = \mathrm{sech} d.$$

This θ is the required formula for the angle of parallelism. ■

6.3 Compact Hyperbolic Surfaces

As commented earlier, a closed geometric surface with constant curvature $K = -1$ is necessarily a torus with genus $g > 1$. This is a consequence of the global Gauss-Bonnet theorem. It is not hard to appreciate how such a surface might be made from a polygon. The canonical identification space for such a torus is a $4g$-gon with edges identified as

$$a_1 a_2 a_1^{-1} a_2^{-1} \cdots a_{2g-1} a_{2g} a_{2g-1}^{-1} a_{2g}^{-1}.$$

When forming a topological surface the edges are identified by homeomorphisms and nothing further needs to be required. However, to create a hyperbolic surface we need to begin with a polygon that is already a geometric surface with boundary – so we take a regular polygon from the hyperbolic plane – and then the identifications need to be made using isometries. Further, the internal angles of the polygon, that are identified as the same vertex, need to add up to a whole angle.

Example 6.13 *Consider a regular octagon in $\mathbb{D}$, such as the one sketched as in Figure 6.2.*

Figure 6.2 – an identified octagon from $\mathbb{D}$

For any d in the range $0 < d \leqslant \infty$ such an octagon can be constructed which has the vertices at hyperbolic distance d from the origin. Recall for a regular hyperbolic octagon

$$-\textit{area of octagon} = \iint_R K \, \mathrm{d}A = 8\beta - 6\pi,$$

where β is the common internal angle. As $d \to 0$ then $\beta \to 3\pi/4$ and when $d \to \infty$ then $\beta \to 0$. For any such octagon, a topological surface can be formed by identifying the eight edges as depicted and the eight vertices are then all identified to the same vertex. However to form a geometric surface we need to identify the edges with isometries and need $\beta = \pi/4$ so that the internal angles sum to a whole angle. But this is possible for a unique choice of d as β is a decreasing function of d.

For this particular choice of d, the global Gauss-Bonnet theorem tells us that

$$-4\pi = 2\pi\chi(X) = \iint_X K \, \mathrm{d}A = -\textit{area of } X.$$

So the surface's area equals 4π. More generally, for $g > 1$, a regular $2g$-gon can be identified to form a hyperbolic surface of genus g.

Example 6.14 *Find the distance of the vertices from the origin of the octagon in Figure 6.2 and the complex numbers representing those vertices.*

Solution. We have noted that the internal angles of the octagon are $\pi/4$, so the octagon can be naturally divided into 8 isosceles triangles with angles $A = \pi/4, B = C = \pi/8$. The equal length sides are then b and c. The dual hyperbolic cosine rule states

$$\cos C = -\cos A \cos B + \sin A \sin B \cosh c$$

and so

$$\cosh c = \frac{\cos\frac{\pi}{8} + \cos\frac{\pi}{4}\cos\frac{\pi}{8}}{\sin\frac{\pi}{4}\sin\frac{\pi}{8}} = \cot\left(\frac{\pi}{8}\right)\left(\frac{1+\cos(\pi/4)}{\sin(\pi/4)}\right).$$

Noting $\cot(\pi/8) = 1 + \sqrt{2}$ and $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$ we then have

$$\cosh c = \left(1 + \sqrt{2}\right)^2.$$

The right-most vertex is then at $z = \tanh(c/2)$. By the hyperbolic tangent half-angle formulae we have

$$\frac{1+z^2}{1-z^2} = \left(1 + \sqrt{2}\right)^2.$$

Solving for z we find $z = 2^{-1/4}$. Thus the vertices of the octagon are $2^{-1/4}\omega^k$ where $\omega = e^{i\pi/4}$ and $0 \leqslant k \leqslant 7$. ■

Example 6.15 ***(A non-orientable hyperbolic surface)*** *We can create a hyperbolic surface X, that is homeomorphic to a torus of genus 3, by identifying the edges of a regular dodecagon* *in $\mathbb{D}$, centred on the origin, in the canonical way. X can then be embedded in $\mathbb{R}^3$ – as a smooth surface – and in such a way that is symmetric about the origin. The antipodal map $\sigma(x, y, z) = (-x, -y, -z)$ is a self-inverse diffeomorphism of X with $X/\langle\sigma\rangle$ being non-orientable – for example, a symmetric band within the torus would become a Möbius band. But $X/\langle\sigma\rangle$ can be endowed with the hyperbolic structure that X has. Specifically $X/\langle\sigma\rangle$ is the sphere with four cross-caps.*

Example 6.16 ***(Pseudosphere)*** *The tractoid is a hyperbolic surface, but not a complete one as its geodesics cannot be extended indefinitely. Omitting one meridian, it is isometric to the semi-infinite strip $(0, 2\pi) \times (1, \infty)$ and we see that the geodesic $x = \pi$ cannot be extended. The completion of the tractoid is the pseudosphere $\mathbb{H}/\Gamma$ where Γ is the group of isometries generated by $z \mapsto z + 2\pi$. (See Figure 6.3.)*

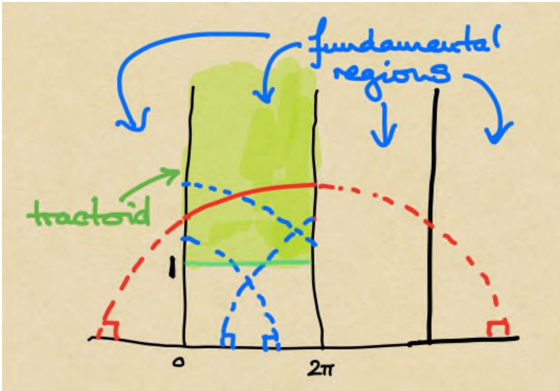

Figure 6.3 – the pseudosphere

The pseudosphere is complete but not compact.

The subject of hyperbolic surfaces is treated in detail in Stillwell. I include here just some of the key theorems.

Theorem 6.17 ***(Killing-Hopf theorem)*** *(Stillwell, p.111) Each complete, connected hyperbolic surface is of the form $\mathbb{H}/\Gamma$ where Γ is a discontinuous group of isometries of $\mathbb{H}$ which acts freely on $\mathbb{H}$.*

To say that Γ is ***discontinuous*** *means that no orbit (of Γ's action) has a limit point.*

To say that Γ acts ***freely*** *means that if $g.x = x$ for $g \in \Gamma$ and $x \in \mathbb{H}$ then g is the identity.*

Definition 6.18 *Given a free, discontinuous action of Γ on $\mathbb{H}$ then a* ***fundamental region*** *$R \subseteq \mathbb{H}$ for the action is a region of $\mathbb{H}$ which contains a representative of each orbit such that the interior of R contains at most one element of an orbit. Thus $\mathbb{H}/\Gamma$ is represented by R with some identifications on its boundary.*

Theorem 6.19 *(Stillwell, p.123) A hyperbolic surface is formed from a hyperbolic polygon provided*

(i) the edges are pairwise identified with isometries and

(ii) the sum of the internal angles, around vertices that are identified together, equals a whole angle.

Theorem 6.20 *(Stillwell, p.130) For any compact hyperbolic surface* $\mathbb{H}/\Gamma$ *there is a polygonal fundamental region for* Γ.

Theorem 6.21 *(Poincaré, 1882 – Stillwell p.180) A compact polygon* P*, satisfying the edge and angle conditions (i) and (ii) above, is a fundamental region for the group* Γ *generated by the edge-pairing transformations of* P.

Example 6.22 *Find the edge-pairing isometry which identifies the edge* a_1 *with* a_1^{-1} *as in Figure 6.2,*

Solution. Recall that the orientation-preserving isometries of $\mathbb{D}$ take the form

$$f(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}$$

where $a \in \mathbb{D}$ and $0 \leqslant \theta < 2\pi$. The vertices of the octagon are $\alpha\omega^k$ where $\alpha = 2^{-1/4}$ and $\omega = e^{i\pi/4}$. The map

$$f_1(z) = \frac{z-\alpha i}{1+\alpha i z}$$

takes the rear of edge a_1 to the origin and the front of the edge to

$$\begin{aligned}
\frac{\alpha\omega-\alpha i}{1+\alpha i\alpha\omega} &= \alpha\left(\frac{\frac{1+i}{\sqrt{2}}-i}{(1+i)/2}\right) \\
&= \frac{\alpha}{\sqrt{2}}\left(1+(1-\sqrt{2})i\right)(1-i) \\
&= \frac{\alpha}{\sqrt{2}}\left(\left(2-\sqrt{2}\right)-\sqrt{2}i\right) \\
&= \alpha\left(\left(\sqrt{2}-1\right)-i\right)
\end{aligned}$$

which has argument

$$\tan^{-1}\left(\frac{-1}{\sqrt{2}-1}\right) = -\tan^{-1}\left(\sqrt{2}+1\right) = -\frac{3\pi}{8}.$$

Thus the function $g_1(z) = e^{3\pi i/8}f_1(z)$ takes a_1 to the positive real axis from 0. We can argue the same to find a function $g_2(z)$ which takes the rear of a_1^{-1} to the origin with image along the positive real axis (details omitted). The edge-pairing isometry we are seeking is then $g_2^{-1} \circ g_1$. ∎