B1.1 Logic

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Introduction

1. What is mathematical logic for?

- Provides a uniform, unambiguous **language** for mathematics;

- gives a precise formal definition of a **proof**;

- explains and guarantees **exactness, rigour and certainty** in mathematics;

- establishes the **foundations** of mathematics.

B1 (Foundations)
= B1.1 (Logic) + B1.2 (Set theory)

N.B.: Course does not teach you to think logically, but it explores what it *means* to think logically.

2. Historical motivation

- *19th cent.*:
  Search for conceptual foundations in analysis: attempts to formalise the notions of **infinity, infinitesimal, limit, ...**

  > "*The definitive clarification of the nature of the infinite has become necessary, not merely for the special interests of the individual sciences but for the honour of human understanding itself.*" – Hilbert 1926

- Hilbert's 2nd Problem, 1900 ICM address: prove consistency of an axiom system for arithmetic.

  > *"I am convinced that it must be possible to find a direct proof for the compatibility of the arithmetical axioms." – Hilbert 1900*

2. Historical motivation (cont)

- Early attempts to formalise mathematics:
  - *Cantor*'s **naive** set theory;
  - *Frege*'s Begriffsschrift and Grundgesetze.

  For any expressible property $P(x)$, Frege's system posited the existence of the set

  $$\{x : P(x)\}.$$

- **Russell's paradox**:
  consider the set $R := \{s : s \notin s\}$

  $$\begin{array}{lcll} R \in R & \Rightarrow & R \notin R & \text{contradiction} \\ R \notin R & \Rightarrow & R \in R & \text{contradiction} \end{array}$$

  $\rightsquigarrow$ *fundamental crisis in the foundations of mathematics*

3. Hilbert's Program

1. find a uniform formal **language** for all mathematics
2. find a complete system of **inference rules/ deduction rules**
3. find a complete system of mathematical **axioms**
4. prove that the resulting system is **consistent**, i.e. does not lead to contradictions

⋆ **complete:** every mathematical sentence can be proved or disproved using 2. and 3.

⋆ 1., 2. and 3. should be **finitary/effective/computable/algorithmic** so, e.g., in 3. you can't take as axioms *the system of all true sentences in mathematics*

4. Solutions to Hilbert's program

Step 1. (formal language for mathematics)
possible in the framework of
ZF = *Zermelo-Fraenkel set theory* or
ZFC = **ZF** + *Axiom of* **C***hoice*

(this is an empirical fact)
$\rightsquigarrow$ B1.2 Set Theory

Step 2. (complete proof system)
possible in **1st-order logic**:
Gödel's Completeness Theorem
$\rightsquigarrow$ B1.1 Logic - this course

Step 3. (complete axiom system)
not possible ($\rightsquigarrow$ C1.2):
Gödel's 1st Incompleteness Theorem:
there is no effective axiomatization
of arithmetic

Step 4. (proving consistency)
not possible ($\rightsquigarrow$ C1.2):
Gödel's 2nd Incompleteness Theorem

5. Decidability

Step 3. of Hilbert's program fails:
there is no effective axiomatization
for the entire body of mathematics

But: many important parts of mathematics **are** completely and effectively axiomatizable; they are **decidable**, i.e. there is an *algorithm* = *program* = *effective procedure* to decide whether a sentence is true or false
$\rightsquigarrow$ allows proofs by computer

Example: $Th(\mathbb{C};+,\cdot)$, the **1st-order theory** of the field $\mathbb{C}$.

Axioms = *field axioms*
+ *all non-constant polynomials have a zero*
+ *the characteristic is* 0

Every **algebraic** property of $\mathbb{C}$ follows from these axioms.
Similarly for $Th(\mathbb{R})$.
$\rightsquigarrow$ C1.1 Model Theory

6. Why *mathematical* logic?

1. Language and deduction rules are tailored for *mathematical objects* and mathematical ways of reasoning
   N.B.: Logic tells you what a proof *is*, not how to *find* one

2. The *method* is mathematical:
   we will develop logic as a *calculus* with sentences and formulas
   $\Rightarrow$ Logic is itself a mathematical discipline,
   not meta-mathematics or philosophy,
   no ontological questions like
   what is a number?

3. Logic has *applications* in other areas of mathematics, and also in theoretical computer science