Supplementary Applied Mathematics MT2023

Problem Sheet 1

Eigenfunction expansions

1. Adjoint problems. Use the adjoint relation, $\langle w, Ly\rangle = \langle L^* w, y\rangle$, to determine the differential operator and boundary conditions for the adjoint problem. In each case, state clearly if the operator and/or the full system is self-adjoint.

   (a) $Ly \equiv \dfrac{d^2y}{dx^2}, \qquad 2y(0) + y'(0) = 0, \qquad y(1) + y'(1) = 0.$

   (b) $Ly \equiv \dfrac{d^4y}{dx^4} - \dfrac{dy}{dx}, \qquad y'(0) - y''(0) = 0, \qquad y'''(0) = 0, \qquad y(1) = 0, \qquad y'(1) - y'''(1) = 0.$

2. Adjoint condition. Let

$$Ly \equiv a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x), \quad a < x < b.$$

   Determine a condition on the functions $a_i(x)$, $i = 0, 1, 2$, for which $L^* = L$. Show further that the system is fully self adjoint for any general class of boundary conditions of the form

$$\begin{aligned} \alpha_1 y(a) + \alpha_2 y'(a) = 0, \\ \beta_1 y(b) + \beta_2 y'(b) = 0, \end{aligned} \tag{1}$$

   where the constants α_j, β_j $(j = 1, 2)$ are distinct.

3. An inhomogeneous problem.

   (a) Find the general homogeneous solution of the Cauchy-Euler equation

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + (1+\alpha)y = 0, \tag{2}$$

   where α is a given positive constant.

   (b) Use (a) to determine the eigenvalues and eigenfunctions of the self-adjoint problem

$$\frac{d}{dx}\left(x^3\frac{dy}{dx}\right) + \lambda xy = 0 \qquad y(1) = 0 \qquad y(\mathrm{e}) = 0. \tag{3}$$

   [*Hint: you may find it helpful to consider* $\lambda = 1 + \alpha$.]

   (c) Use (b) to obtain the eigenfunction expansion for the solution of the inhomogeneous problem

$$\frac{d}{dx}\left(x^3\frac{dy}{dx}\right) = x \qquad y(1) = 0 \qquad y(\mathrm{e}) = 0. \tag{4}$$

   State the coefficients explicitly (i.e. compute the integrals).

4. Eigenfunction expansion gone wrong. What is wrong with the below argument?

   Let L be a second-order differential operator as usual, with boundary conditions BC. Its eigenfunctions satisfy $Ly_k = \lambda_k y_k$ (or $L^* w_k = \lambda_k w_k$) with homogeneous boundary conditions $BC = 0$ (or $BC^* = 0$) and are complete. The solution of any inhomogeneous problem $Ly = f$, $BC \neq 0$, can then be written $y(x) = \sum_k c_k y_k(x)$, and in lectures we gave a method for finding c_k in terms of f and the boundary conditions.

Now suppose I say

$$
\begin{aligned}
Ly &= f \\
\Rightarrow L\sum_k c_k y_k &= f \\
\Rightarrow \sum_k c_k L y_k &= f \\
\Rightarrow \sum_k c_k \lambda_k y_k &= f \\
\Rightarrow w_j \sum_k c_k \lambda_k y_k &= w_j f \\
\Rightarrow \sum_k c_k \lambda_k \langle w_j, y_k \rangle &= \langle w_j, f \rangle \\
\Rightarrow c_j &= \frac{\langle w_j, f \rangle}{\lambda_j \langle w_j, y_j \rangle}
\end{aligned}
$$

where I have (legitimately) used orthogonality to get the last line. The answer must be wrong because it contains no reference to the boundary conditions, but what is wrong? (Carefully consider the justification for going from one line to the next).

[You might try going through the steps with the system $Ly \equiv y'' = 0$, $y(0) = 0$, $y(1) = 1$. Here the problem is self-adjoint with eigenfunctions $\sin n\pi x$, and the recipe above gives $c_k = 0$, which are obviously not the Fourier sine coefficients of the solution $y = x$. The recipe in lectures *does* give the right answer, of course.]

5. Optional question (extra practice).

$$Ly = x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} = f(x), \quad \text{with } y(1) = 0 = y(2). \tag{5}$$

What are the eigenfunctions and eigenvalues of the above problem? What are the eigenfunctions and eigenvalues of the adjoint operator? Explain briefly how you would use the eigenvalues and eigenfunctions of the adjoint operator to solve equation (5) for an arbitrary function $f(x)$.