More on Interpolation: Splines

M.Sc. in Mathematical Modelling & Scientific Computing,
Practical Numerical Analysis

Michaelmas Term 2023, Lecture 2

1D Interpolation

Recall the canonical 1D interpolation problem: given a set of nodes x_i, $0 \leq i \leq n$ and data at those nodes $f(x_i)$, construct a function $p(x)$ such that

$$p(x_i) \;=\; f(x_i)$$

for $0 \leq i \leq n$.

Last time we looked at the Lagrange interpolant which was global, in the sense that it was defined by the same function on the whole interval.

This time we look for piecewise polynomial interpolants — functions which are polynomials on each subinterval $[x_{i-1}, x_i]$, $1 \leq i \leq n$, and satisfy certain continuity conditions. These are known as splines.

Splines

Again we are given a set of nodes x_i, $0 \leq i \leq n$ and data at those nodes $f(x_i)$. When talking about splines, the x_i are often known as knots. The interpolating spline:

- is a polynomial of degree k in each subinterval $[x_{i-1}, x_i]$, $1 \leq i \leq n$;
- is continuous and has continuous derivatives up to order $k-1$;
- satisfies the interpolation conditions.

Linear Splines

The simplest splines are linear splines (i.e. $k = 1$). The continuity and interpolation conditions are enough to determine them uniquely since

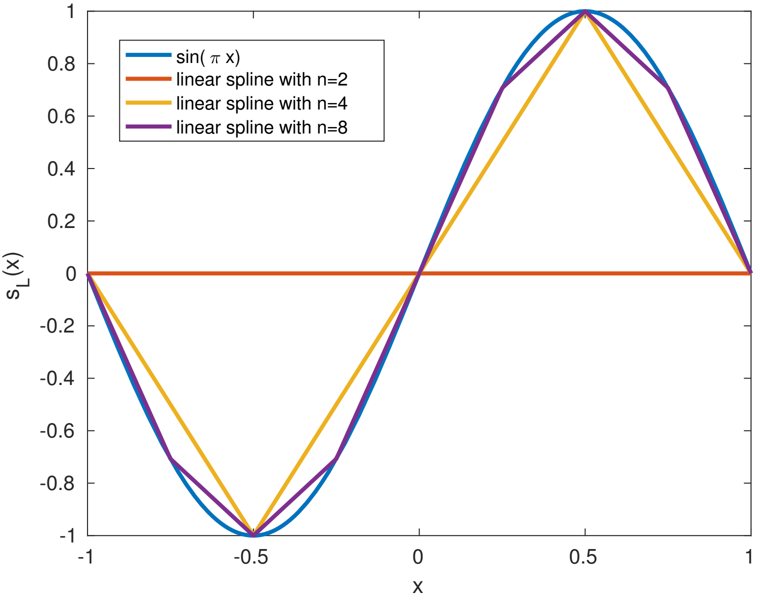
- the number of unknowns is $2n$ (there are n intervals and 2 unknowns required to determine a linear function in each subinterval)
- the number of constraints is $2n$ made up of
  - $n + 1$ interpolation conditions (at x_i, $0 \leq i \leq n$)
  - $n - 1$ continuity conditions (at x_i, $1 \leq i \leq n - 1$)

We can write the linear spline, $s_L(x)$, as

$$s_L(x) = \frac{x_i - x}{x_i - x_{i-1}} f(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} f(x_i)$$

for $x \in [x_{i-1}, x_i]$, $1 \leq i \leq n$.

Example



sin(π x)
linear spline with n=2
linear spline with n=4
linear spline with n=8
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
s_L(x)
-1
-0.5
0
0.5
1
x

Convergence

Theorem 1

Suppose $f \in C^2[a, b]$ and let $s_L(x)$ be the linear spline that interpolates f at the knots $a = x_0 < x_1 < \ldots < x_n = b$, then

$$\|f - s_L\|_\infty \quad \leq \quad \frac{1}{8}h^2\|f''\|_\infty \, .$$

where $h = \max_i h_i$ and $h_i = x_i - x_{i-1}$.

This tells us that if we use a uniform grid, then every time we double n (and thus we halve h) we expect the error to decrease by a factor of 4.

Minimisation Property

The linear spline also has a nice minimisation property as follows:

Theorem 2

Let s_L be the linear spline that interpolates a function $f \in C[a, b]$ at the knots $a = x_0 < x_1 < \ldots < x_n = b$. Then for any function v in $H^1(a, b)$ that also interpolates f at the knots,

$$\|s_L'\|_2 \quad \leq \quad \|v'\|_2 \; .$$

In other words, this theorem tells us that, among all functions in $H^1(a, b)$ that interpolate f at the knots, the linear spline $s_L(x)$ is the flattest, in the sense that its average slope is the smallest.

Global Form of Linear Splines

We may also write a global expression for the linear interpolating spline as a sum of basis functions:

$$s_L(x) \;=\; \sum_{i=0}^{n} \phi_i(x) f(x_i) \, ,$$

where the basis functions $\phi_i(x)$ are defined as

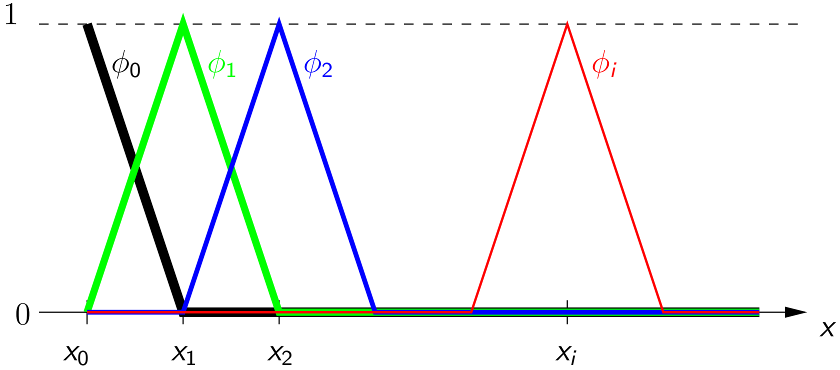
$$\phi_i(x) \;=\; \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & \text{if } x_{i-1} \leq x \leq x_i \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & \text{if } x_i \leq x \leq x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i \leq n-1$ and

$$\phi_0(x) \;=\; \begin{cases} \frac{x_1 - x}{x_1 - x_0} & \text{if } x_0 \leq x \leq x_1 \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_n(x) \;=\; \begin{cases} \frac{x - x_{n-1}}{x_n - x_{n-1}} & \text{if } x_{n-1} \leq x \leq x_n \\ 0 & \text{otherwise} \end{cases}$$

Basis Functions



Note that the basis functions are nodal, i.e. $\phi_i(x_j) = \delta_{i,j}$. Thus

$$s_L(x_j) \;=\; \sum_{i=0}^{n} \phi_i(x_j) f(x_i) \;=\; \sum_{i=0}^{n} \delta_{i,j} f(x_i) \;=\; f(x_j)\,.$$

Cubic Splines

Cubic splines are also popular due to their increased regularity over linear splines. This time

- the number of unknowns is $4n$ (there are n intervals and 4 unknowns required to determine a cubic function in each subinterval)
- the number of constraints is $4n - 2$ made up of
  - $n + 1$ interpolation conditions (at x_i, $0 \leq i \leq n$)
  - $3(n - 1)$ continuity conditions ($s(x)$, $s'(x)$ and $s''(x)$ must be continuous at x_i, $1 \leq i \leq n - 1$)

Thus we need two more conditions to determine the cubic spline uniquely.

Natural Cubic Splines

For natural cubic splines the two final conditions are

$$s''(x_0) \;=\; s''(x_n) \;=\; 0\;.$$

Such splines have a minimisation property analagous to linear splines and are characterised as follows:

Theorem 3

Let s be the natural cubic spline that interpolates a function $f \in C[a,b]$ at the knots $a = x_0 < x_1 < \ldots < x_n = b$. Then for any function v in $H^2(a,b)$ that also interpolates f at the knots,

$$\|s''\|_2 \;\leq\; \|v''\|_2\;.$$

This theorem essentially means that the natural cubic spline minimises the 'average curvature' over functions in $H^2(a,b)$ that interpolate f at the knots.

Construction of Natural Cubic Splines

Let $\sigma_i = s''(x_i)$ for $0 \leq i \leq n$ (note these are unknown). Then we can write

$$s''(x) \;=\; \frac{x_i - x}{h_i}\sigma_{i-1} + \frac{x - x_{i-1}}{h_i}\sigma_i \;, \quad \text{for } x \in [x_{i-1}, x_i] \;.$$

Integrate twice to get

$$s(x) \;=\; \frac{(x_i - x)^3}{6h_i}\sigma_{i-1} + \frac{(x - x_{i-1})^3}{6h_i}\sigma_i + \alpha_i(x - x_{i-1}) + \beta_i(x_i - x) \;,$$

for $x \in [x_{i-1}, x_i]$. Here the α_i and β_i are constants of integration to be determined. The interpolation conditions become

$$s(x_{i-1}) \;=\; \frac{1}{6}\sigma_{i-1}h_i^2 + h_i\beta_i \;=\; f(x_{i-1}) \;, \qquad (1)$$

$$s(x_i) \;=\; \frac{1}{6}\sigma_i h_i^2 + h_i\alpha_i \;=\; f(x_i) \;. \qquad (2)$$

Construction of Natural Cubic Splines

Using the interpolation conditions, the definition of $s(x)$ and the continuity of s' at the knots gives, after some algebra,

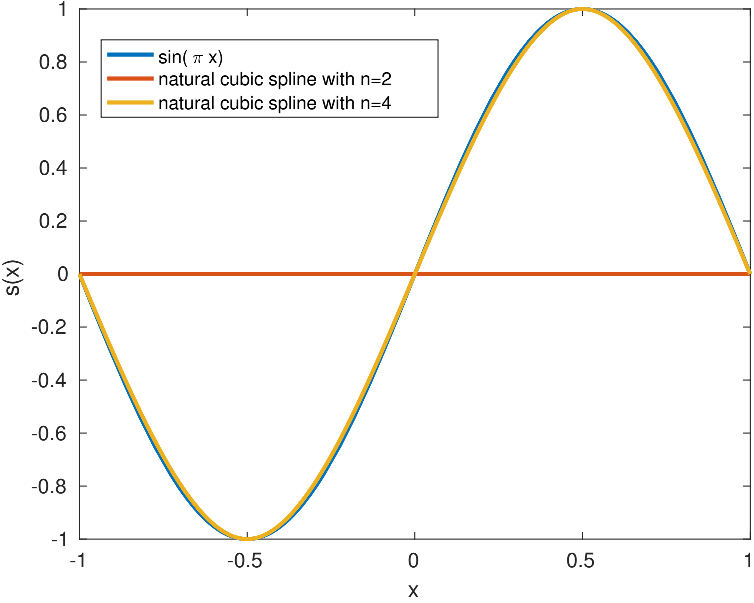
$$h_i\sigma_{i-1} + 2(h_{i+1} + h_i)\sigma_i + h_{i+1}\sigma_{i+1} = 6\left(\frac{f(x_{i+1}) - f(x_i)}{h_{i+1}} - \frac{f(x_i) - f(x_{i-1})}{h_i}\right)$$

along with

$$\sigma_0 = \sigma_n = 0$$

which is a nonsingular tridiagonal system for the σ_i. Once we know the σ_i we can also compute the α_i and the β_i coefficients, using (1) and (2), and hence the natural cubic spline in each subinterval.

Example



sin(π x)
natural cubic spline with n=2
natural cubic spline with n=4
s(x)
x
1
0.8
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0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1