B1.1 Logic
Lecture 3

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2. Valuations

In natural language, the **truth** or **falsity** of a sentence using logical connectives is determined by the truth or falsity of its subclauses:
"Socrates is dead or Socrates is a vampire" is true because "Socrates is dead" is true.

The propositional calculus abstracts this to a recursive definition of the **truth value** T ('true') or F ('false') of a formula ϕ in terms of the truth values of the propositional variables occuring in ϕ.

2.1 Definition

1. A **valuation** v is a function

$$v : \{p_0, p_1, p_2, \ldots\} \to \{T, F\}.$$

2. Given a valuation v we extend v uniquely to a function

$$\tilde{v} : \text{Form}(\mathcal{L}_{\text{prop}}) \to \{T, F\}.$$

(Form($\mathcal{L}_{\text{prop}}$) denotes the set of all formulas of $\mathcal{L}_{\text{prop}}$)

defined recursively as follows:

(i) If ϕ is a formula of length 1, i.e. a propositional variable, then $\tilde{v}(\phi) := v(\phi)$.

(ii) If ϕ is a formula of length $n > 1$, and $\tilde{v}$ has been defined on formulas of length $< n$:
by the Unique Readability Theorem,
either $\phi = \neg\psi_1$ for a unique ψ_1,
or $\phi = (\psi_1 \star \psi_2)$ for a unique pair ψ_1, ψ_2 and a unique $\star \in \{\to, \wedge, \vee, \leftrightarrow\}$.
Then the ψ_i are formulas of length $< n$, and we define $\tilde{v}(\phi)$ in terms of the $\tilde{v}(\psi_i)$ by the **truth tables** on the following slide.

Truth Tables

Define $\widetilde{v}(\phi)$ by the following truth tables:

Negation

ψ	$\neg\psi$
T	F
F	T

i.e. if $\widetilde{v}(\psi) = T$ then $\widetilde{v}(\neg\psi) = F$
and if $\widetilde{v}(\psi) = F$ then $\widetilde{v}(\neg\psi) = T$

Binary Connectives

ψ	χ	$\psi \to \chi$	$\psi \wedge \chi$	$\psi \vee \chi$	$\psi \leftrightarrow \chi$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	T	F	T	F
F	F	T	F	F	T

so, e.g., if $\widetilde{v}(\psi) = F$ and $\widetilde{v}(\chi) = T$
then $\widetilde{v}(\psi \vee \chi) = T$ etc.

Remark: These truth tables correspond roughly to our ordinary use of the words *'not'*, *'if - then'*, *'and'*, *'or'* and *'if and only if'*, except, perhaps, the truth table for implication ($\to$).

2.2 Example

Construct the full truth table for the formula

$$\phi := ((p_0 \vee p_1) \to \neg(p_1 \wedge p_2))$$

$\widetilde{v}(\phi)$ only depends on $v(p_0), v(p_1)$ and $v(p_2)$.

p_o	p_1	p_2	$(p_0 \vee p_1)$	$(p_1 \wedge p_2)$	$\neg(p_1 \wedge p_2)$	ϕ
T	T	T	T	T	F	F
T	T	F	T	F	T	T
T	F	T	T	F	T	T
T	F	F	T	F	T	T
F	T	T	T	T	F	F
F	T	F	T	F	T	T
F	F	T	F	F	T	T
F	F	F	F	F	T	T

2.3 Example Truth table for

$$\phi := ((p_0 \to p_1) \to (\neg p_1 \to \neg p_0))$$

p_0	p_1	$(p_0 \to p_1)$	$\neg p_1$	$\neg p_0$	$(\neg p_1 \to \neg p_0)$	ϕ
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

3. Logical Validity

3.1 Definition

- A valuation v **satisfies** a formula ϕ if $\widetilde{v}(\phi) = T$.

- A formula ϕ is **logically valid** if ϕ is satisfied by *every* valuation (e.g. Example 2.3, not Example 2.2). Such a ϕ is also called a **tautology**. *Notation:* $\models \phi$

- A formula ϕ is **satisfiable** if ϕ is satisfied by *some* valuation. So:

  ϕ is satisfiable iff $\neg\phi$ is *not* a tautology.

- A formula ϕ is a **logical consequence** of a formula ψ if, for *every* valuation v:

  $$\text{if } \widetilde{v}(\psi) = T \text{ then } \widetilde{v}(\phi) = T.$$

  Notation: $\psi \models \phi$

3.2 Lemma $\psi \models \phi$ *if and only if* $\models (\psi \to \phi)$.

Proof. '$\Rightarrow$': Assume $\psi \models \phi$.
Let v be any valuation.

- If $\tilde{v}(\psi) = T$ then (by def.) $\tilde{v}(\phi) = T$, so then $\tilde{v}((\psi \to \phi)) = T$ by tt $\to$.

  ('tt $\star$' refers to the truth table of the connective $\star$)

- If $\tilde{v}(\psi) = F$ then $\tilde{v}((\psi \to \phi)) = T$ by tt $\to$.

Thus, for every valuation v, $\tilde{v}((\psi \to \phi)) = T$, so $\models (\psi \to \phi)$.

'$\Leftarrow$': Conversely, suppose $\models (\psi \to \phi)$.
Let v be any valuation s.t. $\tilde{v}(\psi) = T$.
Since $\tilde{v}((\psi \to \phi)) = T$, also $\tilde{v}(\phi) = T$ by tt $\to$.
Hence $\psi \models \phi$. $\square$

3.3 Definition Let Γ be any (possibly infinite) set of formulas and let ϕ be any formula.
Then ϕ is a **logical consequence** of Γ if, for every valuation v:

$$\text{If } \tilde{v}(\psi) = T \text{ for all } \psi \in \Gamma \text{ then } \tilde{v}(\phi) = T.$$

Notation: $\Gamma \models \phi$

Note:

$$\models \phi \;\Leftrightarrow\; \emptyset \models \phi,$$
$$\psi \models \phi \;\Leftrightarrow\; \{\psi\} \models \phi.$$

Lemma 3.2 generalises to:

3.4 Lemma
$\Gamma \cup \{\psi\} \models \phi$ *if and only if* $\Gamma \models (\psi \to \phi)$.

Proof. Similar to the proof of Lemma 3.2. Exercise. □

3.5 Example

	$\models ((p_0 \to p_1) \to (\neg p_1 \to \neg p_0))$	(Ex. 2.3)
Hence	$(p_0 \to p_1) \models (\neg p_1 \to \neg p_0)$	by 3.2
Hence	$\{(p_0 \to p_1), \neg p_1\} \models \neg p_0$	by 3.4

3.6 Example

$$\phi \models (\psi \to \phi)$$

Proof. For any v:
if $\widetilde{v}(\phi) = T$ then, by tt $\to$, $\widetilde{v}((\psi \to \phi)) = T$
(no matter what $\widetilde{v}(\psi)$ is). $\square$