B1.1 Logic
Lecture 4

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4. Logical Equivalence

4.1 Definition

Two formulas ϕ, ψ are **logically equivalent** if $\phi \models \psi$ and $\psi \models \phi$, i.e. if $\tilde{v}(\phi) = \tilde{v}(\psi)$ for *every* valuation v.

Notation: $\phi \models\!\!\!=\!\!\!| \; \psi$

Exercise: $\phi \models\!\!\!=\!\!\!| \; \psi$ if and only if $\models (\phi \leftrightarrow \psi)$.

4.2 Lemma

(i) For any formulas ϕ, ψ

$$(\phi \vee \psi) \models\!\!\!=\!\!\!| \; \neg(\neg\phi \wedge \neg\psi).$$

(ii) Hence every formula is logically equivalent to one without '$\vee$'.

Proof. (i) Either use truth tables,
or observe that for any valuation v:

$$\begin{array}{lll} & \tilde{v}(\neg(\neg\phi \wedge \neg\psi)) = F & \\ \text{iff} & \tilde{v}((\neg\phi \wedge \neg\psi)) = T & \text{by tt } \neg \\ \text{iff} & \tilde{v}(\neg\phi) = \tilde{v}(\neg\psi) = T & \text{by tt } \wedge \\ \text{iff} & \tilde{v}(\phi) = \tilde{v}(\psi) = F & \text{by tt } \neg \\ \text{iff} & \tilde{v}(\phi \vee \psi) = F & \text{by tt } \vee \end{array}$$

(ii) Induction on the length of the formula ϕ.
Clear for length 1.
For the induction step observe that

$$\text{if } \psi \models\!\!\!=\!\!\!\!\!\mid\ \psi' \text{ then } \neg\psi \models\!\!\!=\!\!\!\!\!\mid\ \neg\psi',$$

and $(\phi \vee \psi) \models\!\!\!=\!\!\!\!\!\mid\ \neg(\neg\phi \wedge \neg\psi)$ by (i),
and for $(\phi \star \psi)$ where $\star$ is not $\vee$ observe:

$$\text{if } \phi \models\!\!\!=\!\!\!\!\!\mid\ \phi' \text{ and } \psi \models\!\!\!=\!\!\!\!\!\mid\ \psi' \text{ then}$$
$$(\phi \star \psi) \models\!\!\!=\!\!\!\!\!\mid\ (\phi' \star \psi').$$

□

4.3 Some convenient notation

If $\phi_1, \ldots, \phi_n$ are formulas, we can write their disjunction as

$$(\ldots((\phi_1 \vee \phi_2) \vee \phi_3) \ldots \vee \phi_n).$$

This is rather cumbersome notation, so we abbreviate it to

$$\bigvee_{i=1}^{n} \phi_i.$$

Formally, we make the following recursive definitions:

$$\bigvee_{i=1}^{1} \phi_i = \phi_1 \text{ and } \bigwedge_{i=1}^{1} \phi_i = \phi_1,$$

and for $n > 1$,

$$\bigvee_{i=1}^{n} \phi_i = (\bigvee_{i=1}^{n-1} \vee \phi_n) \text{ and } \bigwedge_{i=1}^{n} \phi_i = (\bigwedge_{i=1}^{n-1} \wedge \phi_n).$$

So $\tilde{v}(\bigvee_{i=1}^{n} \phi_i) = T$ iff for some i, $\tilde{v}(\phi_i) = T$ and $\tilde{v}(\bigwedge_{i=1}^{n} \phi_i) = T$ iff for all i, $\tilde{v}(\phi_i) = T$.

4.4 Some logical equivalences

Let A, B, A_i be formulas. Then

1. $\neg(A \vee B) \models\!\!\!\!\!=\!\!\!\mid (\neg A \wedge \neg B)$
   More generally,
   $$\neg \bigvee_{i=1}^{n} A_i \models\!\!\!\!\!=\!\!\!\mid \bigwedge_{i=1}^{n} \neg A_i,$$
   hence also
   $$\neg \bigwedge_{i=1}^{n} A_i \models\!\!\!\!\!=\!\!\!\mid \bigvee_{i=1}^{n} \neg A_i.$$
   These are called *De Morgan's Laws*.

2. $(A \to B) \models\!\!\!\!\!=\!\!\!\mid (\neg A \vee B)$

3. $(A \leftrightarrow B) \models\!\!\!\!\!=\!\!\!\mid ((A \to B) \wedge (B \to A))$

4. $(A \vee B) \models\!\!\!\!\!=\!\!\!\mid ((A \to B) \to B)$

5. $(\phi \wedge \bigvee_{i=1}^{n} \psi_i) \models\!\!\!\!\!=\!\!\!\mid \bigvee_{i=1}^{n}(\phi \wedge \psi_i)$
   ("$\wedge$ distributes over $\vee$";
   similarly, $\vee$ distributes over $\wedge$.)

5. Adequacy of the Connectives

The connectives $\neg$ (unary) and $\rightarrow, \wedge, \vee, \leftrightarrow$ (binary) are the *logical part* of our language for propositional calculus.

Question:

- Do we have "enough connectives"?
- That is, can we express everything which is logically conceivable using only these connectives?
- More precisely, is every possible truth table implemented by some formula of $\mathcal{L}_{\text{prop}}$?

Answer: yes.

5.1 Definition

(i) We denote by V_n the set of all functions

$$v : \{p_0, \dots, p_{n-1}\} \to \{T, F\},$$

i.e. "partial" valuations assigning values only to the first n propositional variables. Note $\#V_n = 2^n$.

(ii) An n-**ary truth function** is a function

$$J : V_n \to \{T, F\}.$$

There are precisely 2^{2^n} such functions.

(iii) Let $\text{Form}_n(\mathcal{L}_{\text{prop}})$ be the set of formulas which contain only propositional variables from the set $\{p_0, \dots, p_{n-1}\}$.

Then any $\phi \in \text{Form}_n(\mathcal{L}_{\text{prop}})$ determines the truth function

$$\begin{array}{rccc} J_\phi : & V_n & \to & \{T, F\} \\ & v & \mapsto & \widetilde{v}(\phi). \end{array}$$

(So J_ϕ corresponds to the truth table for ϕ.)

5.2 Theorem
Our language $\mathcal{L}_{\text{prop}}$ *is* **adequate**,
i.e. for every $n > 0$ *and* every *truth function* $J : V_n \to \{T, F\}$ *there is some* $\phi \in \text{Form}_n(\mathcal{L}_{\text{prop}})$ *with* $J_\phi = J$.

Proof: Let $J : V_n \to \{T, F\}$ be any n-ary truth function.

If $J(v) = F$ for all $v \in V_n$ take $\phi := (p_0 \wedge \neg p_0)$.
Then, for all $v \in V_n$: $J_\phi(v) = \tilde{v}(\phi) = F = J(v)$.

Otherwise let $U := \{v \in V_n \mid J(v) = T\} \neq \emptyset$.
For each $v \in U$ and each $i < n$ define the formula

$$\psi_i^v := \begin{cases} p_i & \text{if } v(p_i) = T \\ \neg p_i & \text{if } v(p_i) = F \end{cases}$$

and let $\psi^v := \bigwedge_{i=0}^{n-1} \psi_i^v$.

Then for any valuation $w \in V_n$ one has the following equivalence ($\star$):

$$\widetilde{w}(\psi^v) = T \quad \text{iff} \quad \begin{array}{l}\text{for all } i < n: \\ \widetilde{w}(\psi_i^v) = T\end{array} \qquad \text{(by tt } \wedge\text{)}$$
$$\text{iff} \quad w = v \qquad \text{(by def. of } \psi_i^v\text{)}$$

Now define $\phi := \bigvee_{v \in U} \psi^v$.
Then for any valuation $w \in V_n$:

$$\begin{aligned}
\widetilde{w}(\phi) = T \quad &\text{iff} \quad \text{for some } v \in U: \widetilde{w}(\psi^v) = T \quad &&\text{(by tt } \vee\text{)} \\
&\text{iff} \quad \text{for some } v \in U: w = v &&\text{(by } (\star)\text{)} \\
&\text{iff} \quad w \in U \\
&\text{iff} \quad J(w) = T
\end{aligned}$$

Hence $J_\phi(w) = J(w)$ for all $w \in V_n$;
i.e. $J_\phi = J$.

$\square$

5.3 Definition

(i) A formula which is a conjunction of p_i's and $\neg p_i$'s is called a **conjunctive clause** - e.g. ψ^v in the proof of 5.2.

(ii) A formula which is a disjunction of conjunctive clauses is said to be in **disjunctive normal form** ('**dnf**') - e.g. ϕ in the proof of 5.2.

So in fact the proof of 5.2 yields the following stronger statement:

5.4 Theorem - 'The dnf-Theorem'
For any truth function

$$J : V_n \to \{T, F\}$$

there is a formula $\phi \in \text{Form}_n(\mathcal{L}_{\text{prop}})$ **in dnf** *with* $J_\phi = J$.

In particular, every formula is logically equivalent to one in dnf.

5.5 Definition

Suppose S is a set of (truth-functional) connectives – so each $s \in S$ is given by some truth table.

(i) Write $\mathcal{L}_{\text{prop}}[S]$ for the language with connectives S instead of $\{\neg, \rightarrow, \wedge, \vee, \leftrightarrow\}$ and define $\text{Form}(\mathcal{L}_{\text{prop}}[S])$ and $\text{Form}_n(\mathcal{L}_{\text{prop}}[S])$ accordingly.

(ii) We say that S is **adequate** (or **truth-functionally complete**) if for all $n \geq 1$ and for all n-ary truth functions J there is some $\phi \in \text{Form}_n(\mathcal{L}_{\text{prop}}[S])$ with $J_\phi = J$.

5.6 Examples

1. $S = \{\neg, \wedge, \vee\}$ is adequate, by the dnf-Theorem.

2. Hence, by Lemma 4.2(i), $S = \{\neg, \wedge\}$ is adequate:

   $$(\phi \vee \psi) \models\!\!\!\dashv \neg(\neg\phi \wedge \neg\psi)$$

   Similarly, $S = \{\neg, \vee\}$ is adequate:

   $$(\phi \wedge \psi) \models\!\!\!\dashv \neg(\neg\phi \vee \neg\psi)$$

3. We can express $\vee$ in terms of $\to$ (4.4.4), so $\{\neg, \to\}$ is adequate.

4. $S = \{\vee, \wedge, \to\}$ is **not** adequate: any $\phi \in \text{Form}(\mathcal{L}_{\text{prop}}[S])$ has T in the top row of tt ϕ, so no such ϕ gives $J_\phi = J_{\neg p_0}$.

5. There are precisely two binary connectives, say $\uparrow$ and $\downarrow$, such that $S = \{\uparrow\}$ and $S = \{\downarrow\}$ are adequate.