USEFUL FACTS FROM INTEGRATION THEORY

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Approximation of measurable sets

Let E be a measurable subset of $\mathbb{R}^n$.

(i) If E is open, then E can be written as a countable union of non-overlapping dyadic cubes.

(ii) (Outer regularity) For every $\varepsilon > 0$, there exists an open set $G \supset E$ such that $|G \setminus E| \leq \varepsilon$.

(iii) (Inner regularity) Suppuse $|E| < \infty$. For every $\varepsilon > 0$, there exists a compact set $K \subset E$ such that $|E \setminus K| \leq \varepsilon$.

Approximation of measurable functions

Let E be a measurable subset of $\mathbb{R}^n$ and $f \geq 0$ be a non-negative measurable function on E. Then there exists non-negative simple functions f_j such that $f_j \nearrow f$ a.e.

Interchanging limit and integral

(i) Fatou's lemma.

(ii) Monotone convergence theorem.

(iii) Dominated convergence theorem.

Modes of convergence

Let E be a measurable subset of $\mathbb{R}^n$, $f, f_1, f_2, \ldots$ be measurable functions on E.

(i) If $f_j \to f$ a.e., then $f_j \to f$ in measure.

(ii) If $f_j \to f$ in measure, then $f_{j_k} \to f$ a.e. for some subsequence (f_{j_k}).

(iii) Let $1 \leq p < \infty$ and assume in addition that $(f_j) \subset L^p(E)$ and $f \in L^p(E)$. If $f_j \to f$ in $L^p(E)$, then

- $f_j \rightharpoonup f$ in $L^p(E)$,
- $f_j \to f$ in measure, and
- $f_{j_k} \to f$ a.e. for some subsequence (f_{j_k}).

(Weak) compactness

Let E be a measurable subset of $\mathbb{R}^n$, $1 \leq p < \infty$, $f, f_1, f_2, \ldots \in L^p(E)$.

(i) If $f_j \rightharpoonup f$ in $L^p(E)$, then (f_j) is bounded in $L^p(E)$.

(ii) (Weak sequential compactness) If $1 < p < \infty$ and (f_j) is bounded in $L^p(E)$, then a subsequence (f_{j_k}) converges weakly in $L^p(E)$.