Exercise sheet 3. Chapters 1-12.

Part A

Question 3.1. Show that k^2 is not homeomorphic to $\mathbb{P}^2(k)$.

Part B

Question 3.2. Let $V_0 = \mathrm{Z}(x_0x_3 - x_1^2) \subseteq \mathbb{P}^3(k)$ and $V_1 = \mathrm{Z}(x_1x_3 - x_2^2) \subseteq \mathbb{P}^3(k)$. Let $C := V_0 \cap V_1 \subseteq \mathbb{P}^3(k)$. Let $U := \mathbb{P}^3 \backslash \mathrm{Z}(x_0, x_1, x_2)$ and endow U with its structure of open subvariety of $\mathbb{P}^3(k)$. Let $g : U \to \mathbb{P}^2(k)$ be the morphism such that $g([X_0, X_1, X_2, X_3]) = [X_0, X_1, X_2]$ for all $[X_0, X_1, X_2, X_3] \in U$ (see question 2.3).

(1) Show that the morphism $g|_{C\cap U} : C \cap U \to \mathbb{P}^2(k)$ extends to a morphism $f : C \to \mathbb{P}^2(k)$.

(2) Show that $f(C)$ is closed and that $f(C) = \mathrm{Z}(z_0z_2^2 - z_1^3)$.

(3) Show that the induced map $f : C \to f(C)$ is an isomorphism.

Question 3.3. (1) Let $f : X \to Y$ be a surjective morphism of quasi-projective varieties. Suppose that X is complete. Show that Y is also complete.

(2) Show that a noetherian topological space only has finitely many connected components.

(3) Let $(V, \mathcal{O}_V)$ be a projective variety. Show that the k-vector space $\mathcal{O}_V(V)$ is finite-dimensional.

Question 3.4. Let V and W be quasi-projective varieties. Suppose that V is irreducible. Let $\mathrm{Mor}(V, W)$ be the set of morphisms from V to W and let $\rho : \mathrm{Mor}(V, W) \to \mathrm{Rat}(V, W)$ be the natural map (ie ρ sends a morphism to the rational map it represents). Show that ρ is injective.

Question 3.5. (1) Show that for any $m, n \geqslant 0$, $k^m \prod k^n \simeq k^{n+m}$.

(2) Let $V \subseteq k^m$ and $W \subseteq k^n$ be algebraic sets. Show that $V \times W \subseteq k^{n+m}$ is an algebraic set and describe $\mathcal{I}(V \times W)$. Show that the affine variety associated with the algebraic set $V \times W \subseteq k^{n+m}$ is a product of the affines varieties associated with V and W.

Question 3.6. Let $a : X \to Y$ be a rational map between two quasi-projective varieties. Suppose that Y is quasi-projective. Show that there is a unique representative $f : O \subseteq X$ of a (where $O \subseteq X$ is an open subvariety of X) such that if $f : U \to Y$ is a representative of a then $U \subseteq O$. The open set O is called the *open set of definition* of a.

Question 3.7. Let $n \geqslant 0$ and let $q : k^{n+1}\backslash\{0\} \to \mathbb{P}^n(k)$ be the map such that $q(\bar{v}) = [\bar{v}]$ for all $\bar{v} \in k^{n+1}\backslash\{0\}$. Let $V \subseteq \mathbb{P}^n(k)$ be a closed subset. Endow $k^{n+1}\backslash\{0\}$ with the structure of variety it inherits from k^{n+1} as an open subset.

(1) Show that q is a morphism of varieties.

(2) Show that $\mathcal{I}(V)$ is prime iff V is irreducible.

(3) Show that $q^{-1}(V)$ is irreducible iff V is irreducible.

Part C

Question 3.8. (1) Let $U \subseteq \mathbb{P}^1(k)$ be an open subset (for the Zariski topology). Let $f : U \to \mathbb{P}^1(k)$ be a morphism of varieties. Show that there exists a morphism of varieties $g : \mathbb{P}^1(k) \to \mathbb{P}^1(k)$ such $g|_U = f$.

(2) Show that every automorphism of $\mathbb{P}^1(k)$ is of the form described in question 2.8.

(3) Show that k is not isomorphic to any of its proper open subvarieties (an open subvariety is proper if it is not equal to k).