## Exercise sheet 2. Chapters 1-8.

### Part A

**Question 2.1.** Give an example of a noetherian topological space of infinite dimension.

**Solution.** Second part. Consider the natural numbers $\mathbb{N}$ with the topology in which the closed subsets are the subsets

$$C_n := \{1, 2, 3, \dots, n\}$$

($n \geqslant 0$) (where we set $C_0 = \emptyset$). In this topology, the sets $C_n$ are irreducible. Indeed, if $C_n = C_m \cup C_t$, then either $C_m \subseteq C_t$ or $C_t \subseteq C_m$. So $C_n$ cannot be written as the union of two proper closed subsets not contained in each other, ie $C_n$ is irreducible. Also, this topology on $\mathbb{N}$ is noetherian. Indeed, since all the closed subsets are finite, any descending sequence of closed subsets must stabilise for reasons of cardinality. Finally, we have

$$C_1 \subsetneq C_2 \subsetneq C_3 \subsetneq \dots$$

so that $\mathbb{N}$ has infinite dimension.

**Question 2.2.** (1) Let $P(x_0, \dots, x_n)$ be a homogenous polynomial. Show that all the irreducible factors of $P$ are also homogenous.

(2) Let $D \subseteq \mathbb{P}^n(k)$ be a closed subvariety. Suppose that $D$ is irreducible and that $\mathrm{cod}(D, \mathbb{P}^n(k)) = 1$. Show that there is a homogenous irreducible polynomial $P \in k[x_0, \dots, x_n]$ such that $D = \mathrm{Z}(P)$.

**Solution.** (1) Let $Q$ be an irreducible factor of $P$. Since $P$ is homogenous, the polynomial $Q(tx_0, \dots, tx_n)$ is also an irreducible factor of $P$ (up to a constant) for all $t \in k\backslash\{0\}$. Since there are only finitely many irreducible factors, we thus see that for infinitely many $t \in k$, we have $Q(tx_0, \dots, tx_n)/Q_0 \in k$ for some (irreducible) polynomial $Q_0$. In other words, for infinitely many $t \in k$, we have

$$Q(tx_0, \dots, tx_n) = \sum_{i \geqslant 0} t^i Q_{[i]}(x_0, \dots, x_n) = c(t) Q_0$$

for some $c(t) \in k$ . By the same Vandermonde argument used in question 2.4, we conclude that each $Q_{[i]}$ is a multiple of $Q_0$ by a constant. In particular, each non zero $Q_{[i]}$ has the same degree so that only one $Q_{[i]}$ can be non zero. In particular, $Q$ is homogenous.

(2) Let $D_i := D \cap U_i$, where $U_i$ a coordinate chart such that $D \cap U_i \neq \emptyset$. By question 2.7 (1), we may suppose that $\mathrm{cod}(D \cap U_i, U_i) = 1$. Then $u_i^{-1}(D_i)$ corresponds to a prime ideal $\mathfrak{p}$ of height one in $k[x_0, \dots, \breve{x}_i, \dots, x_n]$ (use Lemma 2.5 and the fact that $D_i \cap U_i$ is irreducible).

We claim that $\mathfrak{p}$ is principal. To see this, let $P \in \mathfrak{p}$. Since $k[x_0, \dots, \breve{x}_i, \dots, x_n]$ is a UFD, we can write $P = \prod_j P_j$, where $P_j$ is irreducible. Since $\mathfrak{p}$ is prime we have $\mathfrak{p} \supseteq (P_{j_0})$ for some $j_0$. However, since $P_{j_0}$ is irreducible, the ideal $(P_{j_0})$ is prime. Since $\mathfrak{p}$ has height one (and because the 0 ideal is prime), we thus have $\mathfrak{p} = (P_{j_0})$, proving the claim.

Now write $Q := P_{j_0}$. Note that since $D$ is irreducible, the closure of $D \cap U_i$ in $\mathbb{P}^n(k)$ is $D$. On the other hand, by question 2.9 (2) the closure of $D \cap U_i$ in $\mathbb{P}^n(k)$ is

$$\mathrm{Z}(x_i^{\deg(Q)} Q(x_0/x_i, \dots, x_{i-1}/x_i, \widetilde{x_i/x_i}, x_{i+1}/x_i, \dots, x_n/x_i)) =: \mathrm{Z}(\beta_i(Q))$$

(see question 2.9 for the notation). We contend that $\beta_i(Q)$ is also irreducible. To see this, suppose for contradiction that $\beta_i(Q)$ is not irreducible. Then $\beta_i(Q) = S_1 S_2$, where $S_1$ and $S_2$ are non constant. By (1) we may assume that $S_1$ and $S_2$ are homogenous. Also $\beta_i(Q)$ is by construction not divisible by $x_i$ and so neither are $S_1$ and $S_2$. We then have

$$Q = \beta_i(Q)(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) = S_1(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)S_2(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$$

so that either $S_1(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$ or $S_2(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$ is constant (since $Q$ is irreducible). Say $S_1(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$ is constant. Since we have

$$\beta_i(S_1(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)) = S_1$$

(because $S_1$ is homogenous and not divisible by $x_i$) we conclude that $S_1$ is a constant, which is a contradiction. So $\beta_i(Q)$ is irreducible.

So we have $D = \mathrm{Z}(\beta_i(Q))$, where $\beta_i(Q)$ is irreducible, which is what we wanted to prove.

## Part B

**Question 2.3.** Let $V$ (resp. $W$) be a closed subvariety of $\mathbb{P}^n(k)$ (resp. $\mathbb{P}^t(k)$). Let $V_0 \subseteq V$ (resp. $W_0 \subseteq W$) be an open subset of $V$ (resp. and open subset of $W$). View $V_0$ (resp. $W_0$) as an open subvariety of $V$ (resp. $W$). Let $Q_0, \ldots, Q_t \in k[x_0, \ldots, x_n]$ be homogenous polynomials of the same degree. Suppose that $V_0 \cap \mathrm{Z}((Q_0, \ldots, Q_t)) = \emptyset$. Let $f : V_0 \to \mathbb{P}^t(k)$ be the map given by the formula $f(\bar{v}) := [Q_0(\bar{v}), \ldots, Q_t(\bar{v})]$. Suppose finally that $f(V_0) \subseteq W_0$. Show that the induced map $V_0 \to W_0$ is a morphism of varieties.

**Solution.** By Lemma 5.3, we may and do assume that $W_0 = \mathbb{P}^t(k)$. Furthermore, we may and do assume that $V_0 = \mathbb{P}^n(k)\backslash\mathrm{Z}((Q_0, \ldots, Q_t))$ (since $f$ arises as a restriction of a map defined on $\mathbb{P}^n(k)\backslash\mathrm{Z}((Q_0, \ldots, Q_t))$). Now note that for any $j \in \{0, \ldots, t\}$, the image of $f|_{V_0\backslash\mathrm{Z}(Q_j)}$ lies in the coordinate chart $U_j^t$ of $\mathbb{P}^t(k)$. Since a map is a morphism iff it is a morphism everywhere locally, we thus only have to check that the map $f|_{(V_0\backslash\mathrm{Z}(Q_j))\cap U_i^n} : (V_0\backslash\mathrm{Z}((Q_j))) \cap U_i^n \to U_j^t$ is a morphism for any $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, t\}$. In the coordinates of $U_i^n$ and $U_j^t$, the map has the form

$$f(X_0, \ldots, \breve{X_i}, \ldots, X_n) =$$
$$\left(\frac{Q_0(X_0, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n)}{Q_j(X_0, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n)}, \ldots, \underbrace{\frac{Q_j(X_0, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n)}{Q_j(X_0, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n)}}, \ldots, \frac{Q_t(X_0, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n)}{Q_j(X_0, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n)}\right)$$

and thus by Proposition 4.5 and question 1.7, the map $f|_{(V_0\backslash\mathrm{Z}((Q_j))\cap U_i^n} : (V_0\backslash\mathrm{Z}((Q_j)) \cap U_i^n \to U_j^t$ is a morphism.

**Question 2.4.** Prove Lemma 7.1.

**Solution.** Recall the statement. Let $I \subseteq k[x_0, \ldots x_n]$ be an ideal.

(1) $I$ is homogenous iff for all $P \in I$ and all $i \geqslant 0$, we have $P_{[i]} \in I$.

(2) If $I$ is homogenous then its radical $\mathfrak{r}(I)$ is also homogenous.

We prove (1). The $\Leftarrow$ direction is clear so we only have to establish the $\Rightarrow$ direction of the equivalence. Suppose that $I = (H_1, \ldots, H_l)$ where the $H_j$ are homogenous. If $P \in I$ then $P = Q_1 H_1 + \cdots + Q_l H_l$ for some polynomials $Q_i$ and we compute

$$P_{[i]} = \sum_j Q_{j,[i-\deg(H_j)]} H_j$$

so that $P_{[i]} \in I$.

(2) Let $t \in K\backslash\{0\}$ and let $\rho_t : k[x_0, \dots, x_n] \to k[x_0, \dots, x_n]$ be the map of $k$-algebras sending $x_i$ to $tx_i$. Since $\rho_{1/t} \circ \rho_t = \mathrm{Id}$, the map $\rho_t$ is a bijection. Note that since $I$ is homogenous, we have $\rho_t(I) = I$. Now

$$\rho_t(\mathfrak{r}(I)) = \rho_t(\cap_{\mathfrak{p}\in \mathrm{Spec}(k[x_0,\dots,x_n]),\ \mathfrak{p}\supseteq I}\ \mathfrak{p}) = \cap_{\mathfrak{p}\in \mathrm{Spec}(k[x_0,\dots,x_n]),\ \mathfrak{p}\supseteq I}\ \rho_t(\mathfrak{p}) = \cap_{\mathfrak{p}\in \mathrm{Spec}(k[x_0,\dots,x_n]),\ \mathfrak{p}\supseteq I}\ \mathfrak{p} = \mathfrak{r}(I)$$

In particular, if $P \in \mathfrak{r}(I)$ then

$$\rho_t(P) = \sum_{i=0}^{\deg(P)} t^i P_{[i]} \in \mathfrak{r}(I)$$

for all $t \in K\backslash\{0\}$. If we let $t$ run through $\deg(P) + 1$ different values, we obtain a system of linear equations with maximal rank (a Vandermonde matrix), with a unique solution. Hence $P_{[i]} \in \mathfrak{r}(I)$ for all $i \geqslant 0$.

**Question 2.5.** Let $T$ be a topological space.

(1) Let $S \subseteq T$ be a subset. Suppose that $S$ is irreducible. Show that the closure of $S$ in $T$ is also irreducible.

(2) Suppose that $T$ is noetherian. Show that $T$ is Hausdorff iff $T$ is finite and discrete.

(3) Let $V$ be a variety. Show that $V$ is irreducible iff the ring $\mathcal{O}_V(U)$ is an integral domain for all open subsets $U \subseteq V$.

(4) Suppose $T$ is noetherian. Show that $T$ is quasi-compact.

**Solution.** (1) $S$ is dense in its closure $\bar{S}$ by construction. Hence any open subset $U$ of $\bar{S}$ (for the induced topology) meets $S$. By assumption $U \cap S$ is dense in $S$ and hence $U$ is also dense in $\bar{S}$.

(2) By Lemma 8.1 (= question 2.6 below), we may suppose wrog that $T$ is irreducible (recall that any subset of a noetherian space is noetherian in the induced topology). In that case, $T$ is Hausdorff iff $T$ is one point. Indeed if $U_1$ and $U_2$ are neighbourhoods of two distinct points then they must meet (by irreducibility), so that $T$ is not Hausdorff if it has more than one point.

(3) Suppose first that $V$ is reducible. Then there are two disjoint non empty open subsets $O_1, O_2 \subseteq V$. Possibly choosing smaller open sets $O_1$ and $O_2$, we may assume that $O_1$ and $O_2$ are affine. We have $\mathcal{O}_V(O_1 \cup O_2) = \mathcal{O}_V(O_1) \times \mathcal{O}_V(O_2)$ by the sheaf property and neither of the rings $\mathcal{O}_V(O_1)$ and $\mathcal{O}_V(O_2)$ is the 0-ring because $O_1$ and $O_2$ are affine and non-empty (use Theorem 3.7). Hence $\mathcal{O}_V(O_1 \cup O_2)$ is not an integral domain. Conversely, suppose that there is an open subset $O$, such that $\mathcal{O}_V(O)$ is not integral. Let $f_1, f_2 \in \mathcal{O}_V(O)$ be such that $f_1 f_2 = 0$ and $f_1, f_2 \neq 0$. Then there is by the sheaf property an open affine subset $O' \subseteq O$ such that $f|_{O'} \neq 0$. If $O'$ arises from an algebraic set $V' \subseteq k^n$ then $f$ will vanish nowhere on $V'\backslash \mathrm{Z}(f|_{O'})$, so there exists an open subset $O'' \subseteq V$ such that $f|_{O''}$ vanishes nowhere. Similarly there exists an open subset $U''$ such that $f_2|_{U''}$ vanishes nowhere. Since $f_1 f_2 = 0$, the sets $O''$ and $U''$ are disjoint, so $V$ is reducible.

(4) Let $\{U_i\}_{\in I}$ be an open covering of $T$. Let $J$ be the set of all finite subcoverings of $\{U_i\}$ (so that $J$ can be identified with the set of finite subsets of the indexing set $I$). If $A \in J$ then write $\mathrm{Un}(A)$ for the union of all the open sets which appear in $A$. Suppose for contradiction that $T$ is not quasi-compact so that $\mathrm{Un}(A) \neq T$ for all $A \in J$. Construct a sequence $A_1, A_2, \dots \in J$ in the following way. Let $A_1 \in J$ be arbitrary. Since $\mathrm{Un}(A_1) \neq T$ and $\cup_i U_i = T$, there is $A_2$ so that $\mathrm{Un}(A_2) \supsetneq \mathrm{Un}(A_1)$. Proceed in the same way for $A_2$, $A_3$ etc. The sequence will be infinite for otherwise $\{U_i\}_{\in I}$ has a finite subcovering. Hence

$$T\backslash A_1 \supsetneq T\backslash A_2 \supsetneq \dots$$

is an infinite descending sequence of closed sets, contradicting noetherianity. Hence $T$ is quasi-compact.

**Question 2.6.** Prove Lemma 8.1.

**Solution**. We recall the statement. Suppose $T$ is noetherian and non empty. Then there is a unique finite collection $\{T_i\}$ of irreducible closed subsets of $T$ such that

(a) $T = \cup_i T_i$.

(b) $T_i \not\subseteq \cup_{j \neq i} T_j$ for all $i$.

We first prove that $T$ is the union of a finite collection of irreducible closed subsets. Denote this statement by $S(T)$. To prove $S(T)$, we first make the assumption (*) that for any proper closed subset $C$ of $T$ (ie such that $C \neq T$) we have $S(C)$. Now if $T$ is irreducible then $S(T)$ holds tautologically. If $T$ is not irreducible, then $T$ has a non dense open subset and so $T$ contains two disjoint non empty open subsets $U_1$ and $U_2$, so that $T = (T \backslash U_1) \cup (T \backslash U_2)$, where $(T \backslash U_1)$ and $(T \backslash U_2)$ are not contained in each other. Since we know that $S(T \backslash U_1)$ and $S(T \backslash U_2)$ hold, we obtain a presentation of $T$ as a union of finitely many irreducible closed subsets.

So the claim is proven under assumption (*). Now suppose for contradiction that (*) does not hold. Then $T$ has a proper closed subset $C_1$ so that $S(C_1)$ does not hold. Hence $C_1$ is in particular not irreducible and thus can be written as union of two proper closed subsets $C'$ and $C''$. We know that either $S(C')$ or $S(C'')$ does not hold (otherwise $S(C_1)$ would hold), so suppose that $S(C')$ does not hold. Write $C_2 := C'$. Repeating the same process for $C_2$ in place of $C_1$ and continuing in the same way, we obtain a sequence

$$C_1 \supsetneq C_2 \supsetneq C_3 \supsetneq \dots$$

contradiction the fact that $C_1$ is noetherian. So we conclude that (*) holds and in particular we have established $S(T)$.

So $T$ is a union of finitely many irreducible closed subsets. Since this set of irreducible closed subsets is finite, it has maximal elements for the relation of inclusion and also any element of it is contained in a maximal element. Let $\{T_i\}$ be the set of maximal elements. Then we have $T = \cup_i T_i$ (since any element is contained in a maximal element) and we have $T_i \not\subseteq \cup_{j \neq i} T_j$ for all $i$, because if $T_i \subseteq \cup_{j \neq i} T_j$ for some index $i$, then $T_i$ is contained in $T_j$ for some $j \neq i$ (because $T_i$ is irreducible), which contradicts maximality. So we have established the existence of a collection $\{T_i\}$ with properties (a) and (b).

We now prove uniqueness. So suppose that we have a collection $\{T_i\}$ with properties (a) and (b). We note that the $T_i$ are precisely the closed irreducible subsets of $T$, which are maximal among all irreducible closed subsets of $T$ for the relation of inclusion. Indeed, let $C \subseteq T$ be an irreducible closed subset and suppose that $C \supseteq T_i$ for some $i$. Since $C \subseteq T_j$ for some $j$ (because $C \subseteq \cup_i T_i$ and $C$ is irreducible), we have $T_i \subseteq T_j$. This implies that $T_i = T_j$ by property (b). Hence $C = T_i$. So the $T_i$ are maximal among all irreducible closed subsets of $T$ for the relation of inclusion. On the other hand, suppose that $C$ is a closed irreducible subset of $T$, which is maximal for the relation of inclusion. Then $C \subseteq T_i$ for some $i$ (again just because $C$ is irreducible) and thus $C = T_i$ by maximality. Hence he $T_i$ are precisely the closed irreducible subsets of $T$, which are maximal among all irreducible closed subsets of $T$ for the relation of inclusion. But this determines the $T_i$ uniquely.

**Question 2.7.** Let $T$ be a topological space. Let $\{V_i\}$ be an open covering of $T$. Let $C \subseteq T$ be an irreducible closed subset (hence non empty).

(1) Show that $C \cap V_i$ is irreducible if $C \cap V_i \neq \emptyset$ and that $\sup_{i, C\cap V_i \neq \emptyset} \operatorname{cod}(C \cap V_i, V_i) = \operatorname{cod}(C, T)$ and $\sup_i \dim(V_i) = \dim(T)$.

(2) Prove Proposition 8.6.

**Solution.** (1) First part. We first show that $C \cap V_i$ is irreducible if $C \cap V_i$ is not empty. If $U \subseteq C \cap V_i$ is open (for the induced topology) and non empty it is also open in $C$, and hence it is dense in $C$, and hence dense in $C \cap V_i$. So $C \cap V_i$ is also irreducible. To show that $\sup_{i, C\cap V_i \neq \emptyset} \operatorname{cod}(C \cap V_i, V_i) = \operatorname{cod}(C, T)$, note first that from the definitions, we have $\operatorname{cod}(C \cap V_i, V_i) \leqslant \operatorname{cod}(C, T)$ for all $i$ such that $C \cap V_i \neq \emptyset$. So we only have to prove that there is an $i_0 := i_0(C)$ such that $\operatorname{cod}(C \cap V_{i_0}, V_{i_0}) \geqslant \operatorname{cod}(C, T)$. So let

$$C \subsetneq C_1 \subsetneq C_2 \subsetneq \cdots \subsetneq C_{\operatorname{cod}(C,T)}$$

be an ascending sequence of closed irreducible subsets. Let $i_0$ be an index such that $C \cap V_{i_0} \neq \emptyset$. We then have $C_j \cap V_{i_0} \neq \emptyset$ for all $j \geqslant 1$ and moreover $C_j$ is the closure of $C_j \cap V_{i_0}$ in $T$ since $C_j$ is irreducible and closed in $T$. Hence we also have

$$C \cap V_{i_0} \subsetneq C_1 \cap V_{i_0} \subsetneq C_2 \cap V_{i_0} \subsetneq \cdots \subsetneq C_{\operatorname{cod}(C,T)} \cap V_{i_0}$$

So we have $\operatorname{cod}(C \cap V_{i_0}, V_{i_0}) \geqslant \operatorname{cod}(C, T)$ as required.

Second part. Using the first part, we have

$$\begin{aligned} \dim(T) &= \sup_{C \text{ irreducible closed in } T} \operatorname{cod}(C, T) = \sup_{C \text{ irreducible closed in } T} \; \sup_{i, C\cap V_i \neq \emptyset} \operatorname{cod}(C \cap V_i, V_i) \\ = \; & \sup_i \sup_{C \text{ irreducible closed in } T,\, C\cap V_i \neq \emptyset} \operatorname{cod}(C \cap V_i, V_i) = \sup_i \sup_{\widetilde{C} \text{ irreducible closed in } V_i} \operatorname{cod}(\widetilde{C}, V_i) = \sup_i \dim(V_i) \end{aligned}$$

where we have used question 2.5 (1) in the equality before last. So the second part follows from the first part.

(2) This follows from (1) and the fact that the dimension of a polynomial ring is finite (Theorem 8.4). Indeed an affine variety is topologically a closed subset of $k^n$ for some $n \geqslant 0$ and thus all its closed subsets have finite dimension and codimension because $k^n$ has finite dimension. The general case then follows from (1) and the fact that a variety has a finite covering by affine open subvarieties.

**Question 2.8.** (1) Show that any element of $\mathrm{GL}_{n+1}(k)$ (= group of $(n+1) \times (n+1)$-matrices with entries in $k$ and with non zero determinant) defines an automorphism of $\mathbb{P}^n(k)$.

(2) Show that if $V$ is a projective variety, then for any two points $v_1, v_2 \in V$, there is an open affine subvariety $V_0 \subseteq V$ such that $v_1, v_2 \in V_0$.

**Solution.** (1) Let $M \in \mathrm{GL}_{n+1}(k)$. Define a map $\phi_M : \mathbb{P}^n(k) \to \mathbb{P}^n(k)$ by the formula

$$\phi_M([\bar{v}]) = [M \cdot \bar{v}] = [\sum_j M_{0j} v_j, \sum_j M_{1j} v_j, \ldots, \sum_j M_{nj} v_j]$$

By question 2.3, this is a morphism from $\mathbb{P}^n(k)$ to $\mathbb{P}^n(k)$. Its inverse is by construction given by $\phi_{M^{-1}}$. This construction in fact gives a homomorphism of groups $\mathrm{GL}_{n+1}(k) \to \mathrm{Aut}(\mathbb{P}^n(k))$.

(2) It is sufficient to prove this for $V = \mathbb{P}^n(k)$.

Indeed, suppose that $V$ is a closed subvariety of $\mathbb{P}^n(k)$. Suppose that $U \subseteq \mathbb{P}^n(k)$ is an open affine subvariety such that $v_1, v_2 \in U$. Note that $C \cap U$ is a closed subset of $U$ and an open subset of $C$. By construction, the sheaf of functions that $C \cap U$ inherits from $C$ as an open subvariety of $C$ is the same as the sheaf of functions it inherits from $U$ as closed subvariety of $U$ (this follows from the definitions). On the other hand, if one endows $C \cap U$ with the sheaf of functions it inherits from $U$ as a closed subvariety, then the resulting Topskf is affine by Lemma 5.4. Hence $C \cap U$ is an affine open subvariety of $C$. Furthermore, $C \cap U$ contains both $v_1$ and $v_2$.

So we prove the statement for $\mathbb{P}^n(k)$. Suppose that $v_1 = [\bar{v}_1]$ and $v_2 = [\bar{v}_2]$, where $\bar{v}_1, \bar{v}_2 \in k^{n+1} \backslash \{0\}$. We may suppose that $v_1 \neq v_2$ (otherwise take $V_0$ to be any open subvariety containing $v_1$) and so $\bar{v}_1$ and $\bar{v}_2$ are not multiples of each other. In particular, $\bar{v}_1$ and $\bar{v}_2$ are linearly independent. Now let $i \in \{0, \dots, n\}$ be such that $v_1 \in U_i$, where $U_i$ is one of the standard coordinate charts. Let $\bar{w} \in k^{n+1} \backslash \{0\}$ be such that $[\bar{w}] \in U_i \backslash \{v_1\}$. Choose $M \in \mathrm{GL}_{n+1}(k)$ such that $M(\bar{v}_1) = \bar{v}_1$ and $M(\bar{v}_2) = \bar{w}$. So in particular, $\phi_M(v_1) = v_1$ and $\phi_M(v_2) \in U_i$. Then we have $v_1, v_2 \in \phi_M^{-1}(U_i) = \phi_{M^{-1}}(U_i)$. Now $\phi_{M^{-1}}(U_i)$ is an open affine subvariety of $\mathbb{P}^n(k)$ because $U_i$ is affine and $\phi_{M^{-1}}$ is an automorphism of varieties.

## Part C

**Question 2.9.** Let $i \in \{0, \dots, n\}$ and let $u_i : k^n \to \mathbb{P}^n(k)$ be the standard map (with image the coordinate chart $U_i$). Let $C \subseteq k^n$ be a closed subvariety of $k^n$ (ie an algebraic set in $k^n$). For any $P \in k[x_0, \dots, x_{i-1}, \breve{x}_i, x_{i+1}, \dots, x_n]$ let

$$\beta_i(P) := x_i^{\deg(P)} P(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{\breve{x}_i}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}) \in k[x_0, \dots, x_n].$$

(1) Let $\bar{C}$ be the closure of $u_i(C)$ in $\mathbb{P}^n(k)$. Show that $(\beta_i(\mathcal{I}(C))) = \mathcal{I}(\bar{C})$ (where $(\beta_i(\mathcal{I}(C)))$ is the ideal of $k[x_0, \dots, x_n]$ generated by all the elements of $\beta_i(\mathcal{I}(Z))$).

(2) Suppose that $\mathcal{I}(C) = (J)$ (ie $\mathcal{I}(C)$ is a principal ideal with generator $J$). Show that $(\beta_i(J)) = \mathcal{I}(\bar{C})$.

(3) Suppose that $n = 3$ and that $C$ is the variety considered in question 1.3. Describe the closure of $u_0(C)$ in $\mathbb{P}^3(k)$. Find homogenous polynomials $(H_1, \dots, H_h)$ such that $\mathrm{Z}(H_1, \dots, H_h)$ is the closure of $u_0(C)$ in $\mathbb{P}^3(k)$.

**Solution.** (1) By construction we have $u_i^{-1}(\mathrm{Z}(\beta_i(P))) = \mathrm{Z}(P)$ for any $P$ as above so the closed set $\mathrm{Z}((\beta_i(\mathcal{I}(C))))$ contains the closure of $u_i(C)$ in $\mathbb{P}^n(k)$. In particular $(\beta_i(\mathcal{I}(C))) \subseteq \mathcal{I}(\bar{C})$. So we only have to show the opposite inclusion, ie we have to show that if $Q$ is a homogenous polynomial which vanishes on $u_i(C)$ then $Q \in (\beta_i(\mathcal{I}(C)))$. So suppose that $Q = Q(x_0, \dots, x_n)$ vanishes on $u_i(C)$. Write $Q = x_i^\delta Q_0$, where $Q_0$ is not divisible by $x_0$. By construction, $Q_0$ is also homogenous and also vanishes on $u_i(C)$. In particular, $Q_0(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$ vanishes on $C$ (by the definition of the map $u_i$). But

$$\beta_i(Q_0(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)) = Q_0(x_0, \dots, x_n)$$

because $\deg(Q_0(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) = \deg(Q_0(x_0, \dots, x_n))$. So $Q_0 \in (\beta_i(\mathcal{I}(C)))$ and hence $Q \in (\beta_i(\mathcal{I}(C)))$.

(2) Notice that $\beta_i$ is multiplicative. Hence $(\beta_i(\mathcal{I}(C))) = (\beta_i(J))$.

(3) First note the following. Let $T$ be a topological space and $\{T_i\}$ be an open covering og $T$. Let $S \subseteq T$ be a subset. Write $\mathrm{cl}(S, T)$ for the closure of $S$ in $T$. Then we have $\mathrm{cl}(S, T) = \cup_i \mathrm{cl}(S \cap T_i, T_i)$. Indeed, from the definitions, we have $\mathrm{cl}(S, T) \supseteq \mathrm{cl}(S \cap T_i, T_i)$ for all $i$ and hence $\mathrm{cl}(S, T) \supseteq \cup_i \mathrm{cl}(S \cap T_i, T_i)$. We prove the opposite inclusion. So suppose that $t \in \mathrm{cl}(S, T)$. Note that this is equivalent to saying that every

open neighbourhood of $t$ meets $S$. For all the indices $i$ such that $t \in T_i$, we then have $t \in \mathrm{cl}(S \cap T_i, T_i)$. Indeed if $V$ is an open neighbourhood of $t$ in $T_i$, then $V$ is also open in $T$ (since $T_i$ is open) and thus $V$ meets $S$. Since $V$ is contained in $T_i$, $V$ thus meets $S \cap T_i$. Since $V$ was arbitrary, we thus see that $t \in \mathrm{cl}(S \cap T_i, T_i)$. Since $t$ is contained in at least one $T_i$, we conclude that $\mathrm{cl}(S, T) \subseteq \cup_i \mathrm{cl}(S \cap T_i, T_i)$ and thus that $\mathrm{cl}(S, T) = \cup_i \mathrm{cl}(S \cap T_i, T_i)$.

Using this statement, we see that $\mathrm{cl}(u_0(C), \mathbb{P}^3(k)) = \cup_{i=0}^3 u_i(\mathrm{cl}(u_i^{-1}(u_0(C)), k^3))$.

Now the variety $u_1^{-1}(u_0(C))$ is $\{(1/t, t, t^2) \,|\, t \in k \backslash \{0\}\}$ in the coordinates $x_0, x_2, x_3$. This is precisely the zero set of $(x_2 x_0 - 1, x_3 - x_2^2)$ in $k^3$ and is thus closed in $k^3$.

The variety $u_2^{-1}(u_0(C))$ is $\{(1/t^2, 1/t, t) \,|\, t \in k \backslash \{0\}\}$ in the coordinates $x_0, x_1, x_3$. This is precisely the zero set of $(x_3 x_1 - 1, x_1^2 - x_0)$ in $k^3$ and is thus closed in $k^2$.

The variety $u_3^{-1}(u_0(C))$ is $\{(1/t^3, 1/t^2, 1/t) \,|\, t \in k \backslash \{0\}\}$ in the coordinates $x_0, x_1, x_2$. This is precisely the zero set of $(x_2^2 - x_1, x_2^3 - x_0)$ in $k^3 \backslash \{0\}$. On the other hand, the zero set of $(x_2^2 - x_1, x_2^3 - x_0)$ in $k^3$ contains 0 and it is precisely the set $\{(u^3, u^2, u) \,|\, u \in k\}$. By the reasoning of question 1.3, this set is isomorphic to $k$ via the third projection. Hence $u_3^{-1}(u_0(C))$ is an open set of a closed irreducible algebraic set (namely $\mathrm{Z}(x_2^2 - x_1, x_2^3 - x_0)$) in $k^3$ and its closure is thus $\mathrm{Z}(x_2^2 - x_1, x_2^3 - x_0)$.

As $u_3(\langle 0, 0, 0\rangle) = [0, 0, 0, 1]$, we conclude that $\mathrm{cl}(u_0(C), \mathbb{P}^3(k) = u_0(C) \cup \{[0, 0, 0, 1]\}$. Now $C$ is described by the equation $x_1^2 - x_2$ and $x_1^3 - x_3$ in $k^3$ and thus

$$\mathrm{cl}(u_0(C), \mathbb{P}^3(k)) \subseteq \mathrm{Z}(\beta_0(x_1^2 - x_2), \beta_0(x_1^3 - x_3)) = \mathrm{Z}(x_1^2 - x_2 x_0, x_1^3 - x_3 x_0^2)$$

If $x_0 = 0$ then the equations $x_1^2 - x_2 x_0$ and $x_1^3 - x_3 x_0^2$ are equivalent to the equation $x_1 = 0$. So we need a third equation which vanishes on $u_0(C)$ and which forces the equation $x_2 = 0$ if $x_0 = x_1 = 0$. Consider $x_2^3 - x_3^2 x_0 = \beta_0(x_2^3 - x_3^2) = 0$. By inspection, we see that $x_2^3 - x_3^2 x_0$ vanishes on $u_0(C)$. On the other hand, if $x_0 = 0$ and $x_2^3 - x_3^2 x_0 = 0$ then $x_2 = 0$.

We thus see that

$$\mathrm{cl}(u_0(C), \mathbb{P}^3(k)) = \mathrm{Z}(x_1^2 - x_2 x_0, x_1^3 - x_3 x_0^2, x_2^3 - x_3^2 x_0).$$