Exercise sheet 3. Chapters 1-12.

Part A

Question 3.1. Show that k^2 is not homeomorphic to $\mathbb{P}^2(k)$.

Solution. Suppose for contradiction that there is a homeomorphism $h : k^2 \to \mathbb{P}^2(k)$. Let $V_0 := Z(x_1)$ and $V_1 = Z(x_1 - 1)$. We have $\dim(V_0) = \dim(V_1) = 1$ and V_0 and V_1 are irreducible. Noting that both dimension and irreducibility only depend on the underlying topology, we see that $h(V_0)$ and $h(V_1)$ are irreducible closed subsets of dimension 1 of $\mathbb{P}^2(k)$. Thus we have $h(V_0) \cap h(V_1) \neq \emptyset$ by Proposition 11.2. However, by construction we have $V_0 \cap V_1 = \emptyset$, so this is a contradiction.

Part B

Question 3.2. Let $V_0 = Z(x_0x_3 - x_1^2) \subseteq \mathbb{P}^3(k)$ and $V_1 = Z(x_1x_3 - x_2^2) \subseteq \mathbb{P}^3(k)$. Let $C := V_0 \cap V_1 \subseteq \mathbb{P}^3(k)$. Let $U := \mathbb{P}^3 \backslash Z(x_0, x_1, x_2)$ and endow U with its structure of open subvariety of $\mathbb{P}^3(k)$. Let $g : U \to \mathbb{P}^2(k)$ be the morphism such that $g([X_0, X_1, X_2, X_3]) = [X_0, X_1, X_2]$ for all $[X_0, X_1, X_2, X_3] \in U$ (see question 2.3).

(1) Show that the morphism $g|_{C\cap U} : C \cap U \to \mathbb{P}^2(k)$ extends to a morphism $f : C \to \mathbb{P}^2(k)$.

(2) Show that $f(C)$ is closed and that $f(C) = Z(z_0z_2^2 - z_1^3)$.

(3) Show that the induced map $f : C \to f(C)$ is an isomorphism.

Solution. (1) We have $Z(x_0, x_1, x_2) = \{[0, 0, 0, 1]\} \subseteq C$. So $g|_{C\cap U}$ is only undefined at one point. Now assuming that $X_1, X_2, X_3 \neq 0$ and $[X_0, X_1, X_2, X_3] \in C$, we have

$$\begin{aligned} & g([X_0, X_1, X_2, X_3]) = [X_0, X_1, X_2] = [X_0X_3, X_1X_3, X_2X_3] = [X_1^2, X_2^2, X_2X_3] \\ = \; & [X_1^2X_2, X_2^3, X_2^2X_3] = [X_1^2X_2, X_1X_2X_3, X_1X_3^2] = [X_1X_2, X_2X_3, X_3^2]. \end{aligned}$$

So the map $h : C \backslash (Z(x_1x_2, x_2x_3, x_3^2) \cap C) \to \mathbb{P}^2(k)$ given by the formula

$$h([X_0, X_1, X_2, X_3]) = [X_1X_2, X_2X_3, X_3^2]$$

coincides with g on $C \backslash ((Z(x_1x_2x_3) \cup Z(x_0, x_1, x_2)) \cap C)$. Now $[0, 0, 0, 1] \notin Z(x_1x_2, x_2x_3, x_3^2)$ so h extends $g|_{U\cap C}$ in a neighborhood of $[0, 0, 0, 1]$.

(2) The fact that $f(C) \subseteq Z(z_0z_2^2 - z_1^3)$ follows from the fact that if $[X_0, X_1, X_2, X_3] \in C$ then

$$X_1^3 = X_1X_0X_3 = X_0(X_1X_3) = X_0X_2^2.$$

The fact that $f(C)$ is closed follows from Corollary 12.10 (and the fact that projective varieties are complete). It also follows from (3) of this question.

(3) We shall construct an inverse map. Suppose that $[Z_0, Z_1, Z_2] \in Z(z_0z_2^2 - z_1^3)$. Suppose first that $Z_0, Z_1 \neq 0$. Then we have

$$[Z_0, Z_1, Z_2, Z_2^2/Z_1] = [Z_0Z_1, Z_1^2, Z_2Z_1, Z_2^2] = [Z_0, Z_1, Z_2, Z_1^2/Z_0] = [Z_0^2, Z_1Z_0, Z_2Z_0, Z_1^2]$$

(because $Z_2^2/Z_1 = Z_1^2/Z_0$ since $[Z_0, Z_1, Z_2] \in Z(z_0z_2^2 - z_1^3)$). Next, plugging either member of this equality into the equations for C we see that

$$[Z_0Z_1, Z_1^2, Z_2Z_1, Z_2^2] \in C$$

if either $Z_0Z_1 \neq 0$ or $Z_1^2 \neq 0$ or $Z_2Z_1 \neq 0$ or $Z_2^2 \neq 0$ and

$$[Z_0^2, Z_1Z_0, Z_2Z_0, Z_1^2] \in C$$

if either $Z_0^2 \neq 0$ or $Z_1Z_0 \neq 0$ or $Z_2Z_0 \neq 0$ or $Z_1^2 \neq 0$.

Let $\iota_2 : \mathbb{P}^2(k)\backslash \mathrm{Z}(z_0z_1, z_1^2, z_2z_1, z_2^2) \to \mathbb{P}^3(k)$ be the map such that

$$\iota_2([Z_0, Z_1, Z_2]) = [Z_0Z_1, Z_1^2, Z_2Z_1, Z_2^2]$$

and $\iota_1 : \mathbb{P}^2(k)\backslash \mathrm{Z}(z_0^2, z_1z_0, z_2z_0, z_1^2) \to \mathbb{P}^3(k)$ be the map such that

$$\iota_1([Z_0, Z_1, Z_2]) = [Z_0^2, Z_1Z_0, Z_2Z_0, Z_1^2].$$

We have shown that these two maps coincide on $\mathrm{Z}(z_0z_2^2 - z_1^3)$ whenever $Z_0, Z_1 \neq 0$. If $[Z_0, Z_1, Z_2] \in \mathrm{Z}(z_0z_2^2 - z_1^3)$ and $Z_0 = 0$ then $Z_1 = 0$ so $[Z_0, Z_1, Z_2] = [0, 0, 1]$. By the above ι_2 is defined at $[0, 0, 1]$. On the other hand, if $Z_1 = 0$ then either $Z_0 = 0$ or $Z_2 = 0$ so that either $[Z_0, Z_1, Z_2] = [0, 0, 1]$ or $[Z_0, Z_1, Z_2] = [1, 0, 0]$. Again, ι_1 is defined at $[1, 0, 0]$. Hence ι_1 and ι_2 together define a map $\phi : \mathrm{Z}(z_0z_2^2 - z_1^3) \to C$. Also, by construction, we have

$$\phi|_{\mathrm{Z}(z_0z_2^2 - z_1^3)\backslash \mathrm{Z}(z_0z_2^2 - z_1^3, z_0z_1)} \circ f|_{C\backslash \mathrm{Z}(x_0x_1)} = \mathrm{Id}_{C\backslash \mathrm{Z}(x_0x_1)}$$

and

$$f|_{C\backslash \mathrm{Z}(x_0x_1)} \circ \phi|_{\mathrm{Z}(z_0z_2^2 - z_1^3)\backslash \mathrm{Z}(z_0z_2^2 - z_1^3, z_0z_1)} = \mathrm{Id}_{\mathrm{Z}(z_0z_2^2 - z_1^3)\backslash \mathrm{Z}(z_0z_2^2 - z_1^3, z_0z_1)}$$

(use the equalities at the beginning of the solution to (3)). Finally, we check by hand

$$f(\phi([1, 0, 0])) = f(\iota_1([1, 0, 0])) = f([1, 0, 0, 0]) = [1, 0, 0]$$

$$f(\phi([0, 0, 1])) = f(\iota_2([0, 0, 1])) = f([0, 0, 0, 1]) = h([0, 0, 0, 1]) = [0, 0, 1].$$

Suppose $[X_0, X_1, X_2, X_3] \in C$. If $X_0 = 0$ then $X_1 = X_2 = 0$. Also, if $X_1 = 0$ then $X_2 = 0$ and either $X_0 = 0$ or $X_3 = 0$. So we have either $[X_0, X_1, X_2, X_3] = [0, 0, 0, 1]$ or $[X_0, X_1, X_2, X_3] = [1, 0, 0, 0]$. Again we check

$$\phi(f([0, 0, 0, 1])) = \phi([0, 0, 1]) = \iota_2([0, 0, 1]) = [0, 0, 0, 1]$$

$$\phi(f([1, 0, 0, 0])) = \phi([1, 0, 0]) = \iota_1([1, 0, 0]) = [1, 0, 0, 0]$$

So we have shown that $f : C \to f(C)$ and ϕ are inverse to each other.

Question 3.3. (1) Let $f : X \to Y$ be a surjective morphism of quasi-projective varieties. Suppose that X is complete. Show that Y is also complete.

(2) Show that a noetherian topological space only has finitely many connected components.

(3) Let $(V, \mathcal{O}_V)$ be a projective variety. Show that the k-vector space $\mathcal{O}_V(V)$ is finite-dimensional.

Solution. (1) We have to show that for any quasi-projective variety B and any closed subset $C \subseteq Y \times B$ the projection $\pi_B(C)$ of C on the second factor is closed. Now the natural map $f \times \mathrm{Id}_B : X \times B \to Y \times B$ is surjective, so we have $\pi_B(C) = \pi_B((f \times \mathrm{Id}_B)^{-1}(C))$ and $\pi_B((f \times \mathrm{Id}_B)^{-1}(C))$ is closed since X is complete.

(2) Recall that the connected components of a topological space T are the connected subsets of T, which are maximal (with respect to inclusion) among all such subsets. One can show that the connected components of T cover T (if this is not known to the students, this can be part of the exercise. See any standard textbook in topology for the solution, which is not difficult).

Note that if $C \subseteq T$ is connected then so is its closure $\bar{C}$. Indeed, if $\bar{C} = C_1 \cup C_2$ where C_1 and C_2 are disjoint, non empty and open in $\bar{C}$, then $C = (C_1 \cap C) \cup (C_2 \cap C)$ and again $C_1 \cap C$ and $C_2 \cap C$ are disjoint, non empty and open in C. Hence the connected components of T are closed. This fact is not needed for (2) but will be used in (3).

Now suppose for contradiction that T has infinitely many connected components. In particular, T is not connected. So $T = T_1 \cup T_2$, where T_1 and T_2 are open, non empty and disjoint. In particular T_1 and T_2 are closed. Now either T_1 or T_2 is not connected. Indeed, from the definitions any connected components of T is contained in either T_1 or T_2. If T_1 and T_2 are connected, then each component is equal to either T_1 or T_2 (by maximality) and so there would be only finitely many. So suppose that T_1 is not connected. Repeating the same reasoning, we obtain a closed and open subset T_{11} in T_1, which is not equal to T_1 and which is not connected (otherwise all the connected components would be one of T_2, T_{11} or T_{12}). Continuing in this way, we obtain a decreasing sequence

$$T \supsetneq T_1 \supsetneq T_{11} \supsetneq T_{111} \supsetneq \dots$$

of closed subsets, contradicting the noetherian condition.

(3) Recall that a regular function on a variety V defines a morphism $f : V \to k$. The image of f is a closed subset of k because V is complete. See Corollary 12.10 for this. Hence $f(V)$ is either finite or it is k (see question 1.1). The second case cannot occur because by (1) the variety k would then be complete, which is not true. To see that k is not complete, let $\pi_2 : k^2 \to k$ be the second projection. Note that $\pi_2(\mathrm{Z}(x_1x_2 - 1)) = k\backslash\{0\}$ and that $k\backslash\{0\}$ is not closed in k. Hence k is not complete (see Definition 12.7).[1] Now if C is a connected component of V, then $f(C)$ is connected and thus $f(C)$ is a point. So f is constant on each connected component of V (use Lemma 12.8). On the other hand, we know by (2) that V only has finitely many connected components so the connected components of V are also open.

Hence we have an isomorphism of k-algebras

$$\mathcal{O}_V(V) \simeq \bigoplus_{C \text{ connected comp. of } V} \mathcal{O}_V(C) \simeq \bigoplus_{C \text{ connected comp. of } V} k$$

Question 3.4. Let V and W be quasi-projective varieties. Suppose that V is irreducible. Let $\mathrm{Mor}(V, W)$ be the set of morphisms from V to W and let $\rho : \mathrm{Mor}(V, W) \to \mathrm{Rat}(V, W)$ be the natural map (ie ρ sends a morphism to the rational map it represents). Show that ρ is injective.

Solution. We have to show that if $U \subseteq V$ is an open subvariety and $f, g : V \to W$ are two morphisms such that $f|_U = g|_U$, then $f = g$. Now suppose that there is $v_0 \in V\backslash U$ such that $f(v_0) \neq g(v_0)$. Let $f \times g : V \to W \times W$ be the morphism of varieties such that $(f \times g)(v) = (f(v), g(v))$ for all $v \in V$. Let $\Delta_W \subseteq W \times W$ be the diagonal, which we know to be closed because W is separated (by Proposition 12.5). In particular, the set $(f \times g)^{-1}(W \times W\backslash\Delta_W)$ is open and contains v_0. In particular, there is an open set $O \subseteq V$ such that $f(v) \neq g(v)$ for all $v \in O$. But O must meet U, since V is irreducible. This is a contradiction so $f(v) = g(v)$ for all $v \in V$.

Question 3.5. (1) Show that for any $m, n \geqslant 0$, $k^m \prod k^n \simeq k^{n+m}$.

(2) Let $V \subseteq k^m$ and $W \subseteq k^n$ be algebraic sets. Show that $V \times W \subseteq k^{n+m}$ is an algebraic set and describe $\mathcal{I}(V \times W)$. Show that the affine variety associated with the algebraic set $V \times W \subseteq k^{n+m}$ is a product of the affines varieties associated with V and W.

[1] I am grateful to Dragos Ghioca for providing this argument.

Solution. (1) We proceed as in the proof of Theorem 10.2. The projections from $\pi_1 : k^{n+m} \to k^n$ and $\pi_2 : k^{n+m} \to k^m$ are clearly morphisms since they are polynomial maps. Now let V be a variety and let $a : V \to k^n$ and $b : V \to k^m$ be morphisms. If there is a morphism of varieties $a \prod b : V \to k^{n+m}$ such that $\pi_1 \circ a \prod b = a$ and $\pi_2 \circ a \prod b = b$ then $a \prod b = a \times b$, since as a set k^{n+m} is the Cartesian product of k^n and k^m. Hence we only have to show that $a \times b$ is a morphism a varieties. Since a map between varieties is a morphism iff it is a morphism in a neighborhood of all points of the source, we may assume without restriction of generality that V is affine. So suppose that V is an algebraic set in k^t (say). By definition a (resp. b) is then the restriction to V of a polynomial map $A : k^t \to k^n$ (resp. $B : k^t \to k^m$). The map $A \times B : k^t \to k^{n+m}$ is polynomial by definition and $a \times b$ is the restriction to V of $A \times B$. Hence $a \times b$ is a morphism.

(2) The ideal $\mathcal{I}(V \times W)$ is described at the beginning of the proof of Proposition 10.8 and it is also shown there that $V \times W$ is closed in k^{n+m} (because it is a vanishing set of an explicit ideal). It then follows from Corollary 10.6 that $V \times W \subseteq k^{n+m}$ is a product of V and W.

Question 3.6. Let $a : X \to Y$ be a rational map between two quasi-projective varieties. Suppose that Y is quasi-projective. Show that there is a unique representative $f : O \subseteq X$ of a (where $O \subseteq X$ is an open subvariety of X) such that if $f : U \to Y$ is a representative of a then $U \subseteq O$. The open set O is called the *open set of definition* of a.

Solution. Let $\{f_i : O_i \to Y\}$ be the set of all representatives of a. Let $O := \cup_i O_i$. Define the morphism $f : O \to Y$ in the following way. Let $o \in O$ and let $f_i : O_i \to Y$ be a representative of a such that $o \in O_i$. Define $f(o) := f_i(o)$. To show that this definition makes sense, we have to show that $f_i(o) = f_j(o)$ if $f_j : O_j \to Y$ is any other representative of a such that $o \in O_j$. To see this, note that by definition we have $f_i|_{O_i \cap O_j} = f_j|_{O_i \cap O_j}$ by Question 3.5 so that $f_i(o) = f_j(o)$. To see that f is a morphism note that by construction $f|_{O_i} = f_i$ is a morphism for all i. Since the O_i cover O, f is a morphism because a (ordinary) map between varieties is a morphism iff it is everywhere locally a morphism.

Question 3.7. Let $n \geqslant 0$ and let $q : k^{n+1}\backslash\{0\} \to \mathbb{P}^n(k)$ be the map such that $q(\bar{v}) = [\bar{v}]$ for all $\bar{v} \in k^{n+1}\backslash\{0\}$. Let $V \subseteq \mathbb{P}^n(k)$ be a closed subset. Endow $k^{n+1}\backslash\{0\}$ with the structure of variety it inherits from k^{n+1} as an open subset.

(1) Show that q is a morphism of varieties.

(2) Show that $\mathcal{I}(V)$ is prime iff V is irreducible.

(3) Show that $q^{-1}(V)$ is irreducible iff V is irreducible.

Solution. (1) Let $i \in \{0, \ldots, n\}$. We then have

$$q^{-1}(U_i) = \{\langle X_0, \ldots, X_n \rangle \in k^{n+1}\backslash\{0\} \,|\, X_i \neq 0\}$$

and so $q^{-1}(U_i)$ is an open subset of k^{n+1}. The map $q|_{q^{-1}(U_i)} : q^{-1}(U_i) \to U_i$ is given by the formula $\langle X_0, \ldots, X_n \rangle \mapsto \langle X_0/X_i, \ldots, \widetilde{X_i/X_i}, \ldots, X_n/X_i \rangle$ and so by Proposition 4.5, $q|_{q^{-1}(U_i)} : q^{-1}(U_i) \to U_i$ is a morphism. Since the $q^{-1}(U_i)$ cover $k^{n+1}\backslash\{0\}$, we conclude that q is a morphism.

(2) We first show that the minimal prime ideals containing $\mathcal{I}(V)$ are homogenous. So let $\{\mathfrak{p}_i\}_{i \in I}$ be the minimal prime ideals containing $\mathcal{I}(V)$. Write $c := \#I$. We proceed as in question 2.4. So let $t \in K\backslash\{0\}$ and let $\rho_t : k[x_0, \ldots, x_n] \to k[x_0, \ldots, x_n]$ be the map of k-algebras sending x_i to tx_i. Since $\rho_{1/t} \circ \rho_t = \mathrm{Id}$, the

map ρ_t is a bijection. Note that since $\mathcal{I}(V)$ is homogenous, we have $\rho_t(\mathcal{I}(V)) = \mathcal{I}(V)$. Now

$$\rho_t(\mathcal{I}(V) = \rho_t(\cap_i \mathfrak{p}_i) = \cap_i \rho_t(\mathfrak{p}_i)$$

and thus by unicity ρ_t permutes the ideals $\mathfrak{p}_i$ (use Theorem 2.4). We conclude that $\rho_{t^{c!}}(\mathfrak{p}_i) = \mathfrak{p}_i$ for all $i \in I$ (since the permutation group on c elements has $c!$ elements). We now reason as in question 2.4. Let $P \in \mathfrak{p}_i$. Let $\delta := \deg(P)$. Since k is infinite, we can find $t_0, \dots, t_\delta \in k$ such that the elements $t_0^{c!}, \dots, t_\delta^{c!} \in k$ are distinct. Then we have

$$\rho_{t_l^{c!}}(P) = \sum_{j \geqslant 0} t_l^{j \cdot c!} P_{[j]} \in \mathfrak{p}_i$$

for all $l = 0, \dots, \delta$. This gives a linear system (a Vandermonde matrix) with a unique solution in the $P_{[j]}$ and so we conclude that $P_{[j]} \in \mathfrak{p}_i$. Hence $\mathfrak{p}_i$ is a homogenous ideal.

So now suppose that $\mathcal{I}(V)$ is a prime ideal. Suppose for contradiction that V is not irreducible. By the discussion after Lemma 7.4, we see that there is a $r > 1$ and radical ideals $I_1, \dots, I_r$ such that $\mathcal{I}(V) = \cap_i I_i$ and $I_i \supsetneq \cap_{j \neq i} I_j$ and $I_i \subsetneq \cap_{j \neq i} I_j$ for all i. In particular, there are elements $r_1 \in I_1$ and $r_2 \in \cap_{j \neq 1} I_j$ such that $r_1 \notin \cap_{j \neq 1} I_j$ and $r_2 \notin I_1$. In particular, $r_1, r_2 \notin \mathcal{I}(V) = \cap_i I_i$. However we have $r_1 r_2 \in \mathcal{I}(V)$ so $\mathcal{I}(V)$ is not prime. This is a contradiction.

Conversely, suppose that $\mathcal{I}(V)$ is not a prime ideal. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal prime ideals of $\mathcal{I}(V)$. By assumption, we have $r > 1$ and by the above claim, the $\mathfrak{p}_i$ are also homogenous. By Theorem 2.4, we know that $\mathcal{I}(V) = \cap_i \mathfrak{p}_i$ and that for all i we have $\mathfrak{p}_i \supsetneq \cap_{j \neq i} \mathfrak{p}_j$ and $\mathfrak{p}_i \subsetneq \cap_{j \neq i} \mathfrak{p}_j$. Now note that neither $\mathfrak{p}_1$ nor $\cap_{j \neq i} \mathfrak{p}_j$ is the irrelevant ideal. Indeed, the irrelevant ideal contains all the non trivial homogenous ideals and if either $\mathfrak{p}_1$ or $\cap_{j \neq i} \mathfrak{p}_j$ were the irrelevant ideal then the previous two equalities would not be satisfied. Thus, applying $\mathrm{Z}(\cdot)$ to the same last two last equalities, we conclude that V is the union of two proper closed subsets which are not contained in each other. So V is reducible.

(3) Suppose that V is not empty (otherwise, there is nothing to prove). Then $V = \mathrm{Z}(\mathcal{I}(V))$ by Proposition 7.3. In particular, $\mathcal{I}(V)$ is a homogenous ideal which contains no non zero constants. Also, by construction $q^{-1}(V)$ is precisely the zero set of $\mathcal{I}(V)$ in $k^{n+1} \backslash \{0\}$. On the other hand, the zero set of $\mathcal{I}(V)$ in k^{n+1} is $q^{-1}(V) \cup \{0\}$ since any non constant homogenous polynomial vanishes at 0. Since $q^{-1}(V)$ is not empty, it contains the intersection of a line with $k^{n+1} \backslash \{0\}$. By the reasoning in the second part of the proof of Proposition 11.2, the Zariski closure of this line contains 0. Hence the Zariski closure of $q^{-1}(V)$ is $q^{-1}(V) \cup \{0\}$.

Now suppose that $q^{-1}(V)$ is irreducible (in $k^{n+1} \backslash \{0\}$). We conclude from question 2.5 and the last paragraph that $q^{-1}(V) \cup \{0\}$ is closed and irreducible. The last paragraph also implies that the ideal of $q^{-1}(V) \cup \{0\}$ in k^{n+1} is $\mathcal{I}(V)$. Hence $\mathcal{I}(V)$ is prime (by Lemma 2.5). We conclude from (2) that V is irreducible.

Conversely, suppose that V is irreducible. Then $\mathcal{I}(V)$ is prime by (2). By the first paragraph, $q^{-1}(V) \cup \{0\}$ is then closed and irreducible. Hence $q^{-1}(V)$ is irreducible (in $k^{n+1} \backslash \{0\}$), since it is the intersection of an irreducible set and an open set.

Part C

Question 3.8. (1) Let $U \subseteq \mathbb{P}^1(k)$ be an open subset (for the Zariski topology). Let $f : U \to \mathbb{P}^1(k)$ be a morphism of varieties. Show that there exists a morphism of varieties $g : \mathbb{P}^1(k) \to \mathbb{P}^1(k)$ such $g|_U = f$.

(2) Show that every automorphism of $\mathbb{P}^1(k)$ is of the form described in question 2.8.

(3) Show that k is not isomorphic to any of its proper open subvarieties (an open subvariety is proper if it is not equal to k).

Solution. (1) First note the following. Let $O \subseteq U$ be an open subset. Let $a : O \to \mathbb{P}^1(k)$ be a morphism. Then: if a extends to a morphism $U \to \mathbb{P}^1(k)$, then this extension is unique. This follows from question 3.4. We may thus without restriction of generality replace U by one of its open subsets.

Now since the coordinate charts U_0 and U_1 cover $\mathbb{P}^1(k)$ we know that $f^{-1}(U_i) \neq \emptyset$ for either $i = 0$ or $i = 1$. Supposing that $f^{-1}(U_0) \neq \emptyset$, we may thus replace U by $f^{-1}(U_0)$ and thus suppose that $f(U) \subseteq U_0$. Further, replacing U by $U \cap U_0$ we may also suppose that $U \subseteq U_0$. Finally, by Lemma 4.1, we may without restriction of generality suppose that $u_0^{-1}(U)$ is an open affine subvariety with coordinate ring isomorphic to $k[x_1][h^{-1}]$, where $h \in k[x_1]$. Let now $j : u_0^{-1}(U) \to k$ be the map such that $u_0 \circ j = f \circ u_0$.

By Theorem 3.7, the map j is induced by a map of k-algebras $j^* : k[x_1] \to k[x_1][h^{-1}]$. Let $P(x_1)/h^l(x_1) =: j^*(x_1)$, where $l \geqslant 0$ and where we suppose without restriction of generality that P and h are coprime. If $P = 0$ then j and therefore f is a constant map and then g can be defined on all of $\mathbb{P}^1(k)$ to be the constant map with the same value. So we may suppose that $P \neq 0$.

Now by construction, we have

$$j(X_1) = P(X_1)/h^l(X_1)$$

for all $X_1 \in u_0^{-1}(U) \subseteq k$ (see Corollary 4.4 for more details about this). Let $\delta := \max(\deg(P), l \cdot \deg(h))$. Let $A(x_0, x_1) := x_0^\delta P(x_1/x_0)$ and $B(x_0, x_1) := x_0^\delta h^l(x_1/x_0)$. Note that A and B are homogenous. Note also that either $A(0,1) \neq 0$ or $B(0,1) \neq 0$ because we have either $\delta = \deg(P)$ or $\delta = \deg(h^l)$.

Now define a map $\mathbb{P}^1(k) \to \mathbb{P}^1(k)$ by the formula

$$[X_0, X_1] \mapsto [B(X_0, X_1), A(X_0, X_1)]$$

for all $X_0, X_1 \in k$, not both zero. Note that if $X_0 = 0$ then $\langle A(X_0, X_1), B(X_0, X_1)\rangle \neq 0$ since either $A(0,1) \neq 0$ or $B(0,1) \neq 0$. If $X_0 \neq 0$ then $\langle A(X_0, X_1), B(X_0, X_1)\rangle \neq 0$ for all X_1 because P and h are coprime. So this map is well-defined. If $[X_0, X_1] \in U$ then by assumption we have $X_0 \neq 0$ and $h(X_1/X_0) \neq 0$ so that

$$\begin{aligned} & [B(X_0, X_1), A(X_0, X_1)] = [X_0^\delta h^l(X_1/X_0), X_0^\delta P(X_1/X_0)] = [h^l(X_1/X_0), P(X_1/X_0)] \\ = \; & [1, P(X_1/X_0)/h^l(X_1/X_0)] = [1, j(X_1/X_0)] = u_0(j(X_1/X_0)) = f(u_0(X_1/X_0)) = f([1, X_1/X_0]) = f([X_0, X_1]) \end{aligned}$$

so the map $[X_0, X_1] \mapsto [B(X_0, X_1), A(X_0, X_1)]$ is a morphism $\mathbb{P}^1(k) \to \mathbb{P}^1(k)$ extending f.

(2) Let $A : \mathbb{P}^1(k) \to \mathbb{P}^1(k)$ be an automorphism. Write $\infty := [0, 1] \in \mathbb{P}^1(k)$. We saw in the solution of question 2.8 that any point of $\mathbb{P}^1(k)$ can be moved to ∞ (or any other point) by an automorphism of the required type (ie given by an invertible 2×2-matrix). Composing A with a suitable automorphism of the required type, we may thus suppose that $A(\infty) = \infty$. In that case, the restriction of A to U_0 gives an automorphism $U_0 \simeq U_0$ (since $\mathbb{P}^1(k) = U_0 \cup \{\infty\}$). Now note that by Theorem 3.7, an automorphism of $U_0 \simeq k$ corresponds to a k-algebra automorphism ϕ of $\mathcal{C}(U_0) \simeq k[x_1]$. Note that for any polynomial $P(x_1) \in k[x_1]$, $\deg(P) = \dim_k k[x_1]/(P(x_1))$. Since ϕ induces an isomorphism of k-algebras (and hence k-vector spaces) $k[x_1]/(P(x_1)) \simeq k[x_1]/(\phi(P(x_1)))$ we thus have $\deg(\phi(P(x_1))) = \deg(P(x_1))$. So any automorphism of $k[x_1]$ sends x_1 to $ax_1 + b$ for some $a, b \in k$ with $a \neq 0$. We conclude that there are elements $a, b \in k$ such that $a \neq 0$ and such that

$$A([1, X_1]) = [1, aX_1 + b]$$

for all $X_1 \in k$. Thus, if $X_0 \neq 0$ we have

$$A([X_0, X_1]) = A([1, X_1/X_0]) = [1, a(X_1/X_0) + b] = [X_0, aX_1 + bX_0].$$

Now consider the matrix

$$M := \begin{pmatrix} 1 & 0 \\ b & a \end{pmatrix}$$

This matrix has determinant a and thus lies in $\mathrm{GL}_2(k)$. By construction, the automorphism a_M defined by M restricts to A on U_0 and hence by question 3.4 we have $A = a_M$.

(3) Suppose for contradiction that $U \subsetneq k$ is a proper open subvariety and that $f : k \to U$ is an isomorphism. Identify k with U_0, where $U_0 \subseteq \mathbb{P}^1(k)$ is the well-known coordinate chart. By composition, f induces a morphism $\phi : U_0 \to \mathbb{P}^1(k)$. By (1), ϕ extends to a morphism $g : \mathbb{P}^1(k) \to \mathbb{P}^1(k)$, such that $g(U_0) = U$. We know that $g(\mathbb{P}^1(k))$ is closed by Corollary 12.10. Since $g(\mathbb{P}^1(k))$ also contains a non-empty subset, we see that $g(\mathbb{P}^1(k)) = \mathbb{P}^1(k)$. In particular, we must have $g([0, 1]) = [0, 1]$, otherwise g is not surjective. But then $g(\mathbb{P}^1(k))$ does not contain $U_0 \backslash U$ and so g is not surjective, which is a contradiction.

Alternatively, note that if $U \neq k$, then there are non constant invertible regular functions on U. Indeed let $a \in k \backslash U$. Then the function $x - a$ never vanishes on U. On the other hand there are no non constant invertible functions on k. So k cannot be isomorphic to U.