C3.4 Algebraic Geometry (chap. 1-12 of the notes)

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Introduction

Classical algebraic geometry is the study of the sets of of simultaneous solutions of collections of polynomial equations in several variables with coefficients in an algebraically closed field. Such sets are called algebraic varieties. So eg the set of simultaneous solutions of the equations

$$x^2 + y^2 - 1 = 0, xy = 0$$

in $\mathbb{C}^2$ is an algebraic variety.

Because they are so easy to define, algebraic varieties appear in almost every area of mathematics. They play a crucial role in number theory, in topology, in differential geometry and complex geometry (ie the theory of complex manifolds). When the base field is $\mathbb{C}$, an algebraic variety defines a complex manifold provided it has "no kinks" (we shall give a precise definition later).

A basic reference for classical algebraic geometry is chap. I of D. Mumford's book

The Red Book of Varieties and Schemes (Springer Lecture Notes in Mathematics 1358).

Another reference is chap. I of R. Hartshorne's book

Algebraic Geometry (Springer).

One might also consult the book by M. Reid

Undergraduate algebraic geometry (London Mathematical Society Student Texts **12**, Cambridge University Press 1988).

An updated free version of M. Reid's lectures can be found online.

The natural generalisation of classical algebraic geometry is the *theory of schemes*, which will be taught in Hilary Term.

In Grothendieck's theory of schemes, the base field can be replaced by any commutative ring but the absence of Hilbert's Nullstellensatz, which is at the root of the material presented here, means that different techniques have to be used.

There are three important tools, which will not be presented in this course:

- The theory of sheaves
- Cohomological techniques
- The technique of base change

These tools are very powerful but there will not be enough time to present them in these lectures. Also, the best framework for them is the theory of schemes (although they could also be used in the restricted setting of this text).

There is also a tool from Commutative Algebra, which will not be used here but which is very useful in Algebraic Geometry: the tensor product of modules over a ring. Tensor products are ubiquitous in the theory of schemes.

The prerequisites for this course are the part A course *Rings and Modules* and the part B course *Commutative Algebra*.

It is assumed that the reader is familiar with the terminology used in the notes of the commutative algebra course. We shall often quote results proven in that course, referring to it as "CA". I have put the CA notes on the web page of the present course for easy reference.

Throughout the course, we shall work over a fixed algebraically closed field k.

As in the CA course, a ring will be a commutative ring with unit, unless stated otherwise.

The reader may assume that for any $n \geqslant 1$, the ring of polynomials $k[x_1, \ldots, x_n]$ is a UFD (Unique Factorisation Domain).

It can also be assumed that the localisation $k[x_1, \ldots, x_n]_S$ is a UFD for any multiplicative set $S \subseteq k[x_1, \ldots, x_n]$.

Hilbert's Nullstellensatz and algebraic sets

Let $n \geqslant 0$ and let $R_n := k[x_1, \ldots, x_n]$.

Let $\Sigma \subseteq R_n$.

The *algebraic set* associated with Σ is

$$\mathrm{Z}(\Sigma) = \text{zero set of } \Sigma := \{(t_1, \ldots, t_n) \in k^n \,|\, \forall P \in \Sigma : P(t_1, \ldots, t_n) = 0\}$$

If we let ΣR_n be the ideal generated by Σ in R_n then we clearly have

$$\mathrm{Z}(\Sigma) = \mathrm{Z}(\Sigma R_n).$$

We now recall two basic results in commutative algebra.

Theorem 1.1 (Hilbert's basis theorem; see Th. 7.6 in CA)

The ring $k[x_1, \ldots, x_n]$ is noetherian.

Recall that a noetherian ring is a ring all of whose ideals are finitely generated. In particular any algebraic set in k^n is the zero set of a finite number of polynomials.

Theorem 1.2 (Hilbert's strong Nullstellensatz; see Cor. 9.5 in CA)

For any ideal $I \subseteq R_n$ we have

$$\mathfrak{r}(I) = \{P \in R_n \,|\, \forall (t_1, \ldots, t_n) \in \mathrm{Z}(I) : P(t_1, \ldots, t_n) = 0\}$$

Here $\mathfrak{r}(I)$ is the radical (or nilradical) of I.

If $A \subseteq k^n$ is subset, we shall write

$$\mathcal{I}(A) := \{P \in R_n \,|\, \forall (t_1, \ldots, t_n) \in A : P(t_1, \ldots, t_n) = 0\}.$$

The set $\mathcal{I}(A)$ is clearly and ideal in R_n.

Note that the strong HNS implies that $\mathcal{I}(\mathrm{Z}(I)) = \mathfrak{r}(I)$ for any ideal of R_n.

We may now prove the basic

Proposition 1.3

Let $V \subseteq k^n$ be an algebraic set and let $I \subseteq R_n$ be an ideal. Then the identities

$$\mathrm{Z}(I) = \mathrm{Z}(\mathfrak{r}(I)), \mathcal{I}(\mathrm{Z}(I)) = \mathfrak{r}(I) \text{ and } \mathrm{Z}(\mathcal{I}(V)) = V$$

hold.

In particular, the two maps

$$\{\text{algebraic sets in } k^n\} \overset{\mathcal{I}}{\underset{Z}{\rightleftarrows}} \{\text{radical ideals in } R_n\}$$

are inverse to each other.

Note that in this correspondence, we have

$$V_1 \subseteq V_2 \iff \mathrm{Z}(V_1) \supseteq \mathrm{Z}(V_2)$$

for any two algebraic sets V_1 and V_2.

Proof. (of Proposition 1.3) The identity $\mathrm{Z}(I) = \mathrm{Z}(\mathfrak{r}(I))$ follows from the definitions.

The identity $\mathcal{I}(\mathrm{Z}(I)) = \mathfrak{r}(I)$ was already proven.

We thus only have to prove that $\mathrm{Z}(\mathcal{I}(V)) = V$.

To see this, note that by definition we have $V \subseteq \mathrm{Z}(\mathcal{I}(V))$.

On the other hand, by definition $V = \mathrm{Z}(J)$ for some ideal J in $k[x_1, \ldots, x_n]$.

By construction, we have $J \subseteq \mathcal{I}(V)$, so $\mathrm{Z}(J) = V \supseteq \mathrm{Z}(\mathcal{I}(V))$.

Hence $V = \mathrm{Z}(\mathcal{I}(V))$. $\square$

We also note the following identities, whose proof is straighforward:

(1) $\mathcal{I}(V_1 \cup V_2) = \mathcal{I}(V_1) \cap \mathcal{I}(V_2)$

(2) $\mathcal{I}(\cap_i V_i) = \mathfrak{r}(\sum_i \mathcal{I}(V_i))$

(3) $\mathrm{Z}(I_1 \cap I_2) = \mathrm{Z}(I_1) \cup \mathrm{Z}(I_2)$

(4) $\mathrm{Z}(\sum_i I_i) = \cap_i \mathrm{Z}(I_i)$

In view of the properties (4) and (3) above, the algebraic sets in k^n can be viewed as the closed sets of a topology on k^n, called the *Zariski topology*.

If $V \subseteq k^n$ is an algebraic set, we endow V with the topology induced by the Zariski topology of k^n.

This topology is called the *Zariski topology* of V.

We can refine the correspondence above as follows.

Say that an algebraic set $V \subseteq k^n$ is *reducible* if

$$V = V_1 \cup V_2,$$

where $V_1, V_2 \subseteq k^n$ are non empty algebraic sets, $V_1 \not\subseteq V_2$ and $V_2 \not\subseteq V_1$.

An algebraic set $V \subseteq k^n$ is said to be *irreducible* if it is not reducible.

One verifies from the definition that an algebraic set is irreducible iff all its non empty open subsets are dense.

For the following two lemmata, we shall need the following result from CA:

Theorem 1.4

Let R be a noetherian commutative ring and let $I \subseteq R$ be a radical ideal.

Then there is unique finite set of prime ideals $\{\mathfrak{p}_l\}$ such that

$$I = \bigcap_l \mathfrak{p}_l$$

and such that for all indices l we have $\mathfrak{p}_l \not\supseteq \cap_{j \neq l} \mathfrak{p}_j$.

Furthermore, the $\mathfrak{p}_l$ are the prime ideals of R, which are minimal for the inclusion relation among the prime ideals containing I.

Proof. This follows from the Lasker-Noether theorem (see Prop. 7.8 in CA) and the remark after Th. 6.7 in CA. □

Lemma 1.5

Let $V \subseteq k^n$ be an algebraic subset. Then V is irreducible iff $\mathcal{I}(V)$ is a prime ideal.

Proof. "$\Leftarrow$": Suppose that V is reducible. Then $V = V_1 \cup V_2$, where V_1 and V_2 are two algebraic subsets not contained in each other (and in particular not empty).

By property (1) above, we have $\mathcal{I}(V) = \mathcal{I}(V_1) \cap \mathcal{I}(V_2)$, where $\mathcal{I}(V_1)$ and $\mathcal{I}(V_2)$ are two ideals not contained in each other.

In particular, there is $a_1 \in \mathcal{I}(V_1)$ such that $a_1 \notin \mathcal{I}(V_2)$ and $a_2 \in \mathcal{I}(V_2)$ such that $a_2 \notin \mathcal{I}(V_1)$. In particular $a_1, a_2 \notin \mathcal{I}(V)$.

On the other hand $a_1 a_2 \in \mathcal{I}(V)$ so that $\mathcal{I}(V)$ is not prime.

"$\Rightarrow$": Suppose that $\mathcal{I}(V)$ is not prime.

Let $\{\mathfrak{p}_l\}_{l\in\Lambda}$ be the set of prime ideals in R, which are minimal among the prime ideals containing $\mathcal{I}(V)$.

By Theorem 1.4 we know that Λ is finite and that $\mathcal{I}(V) = \cap_l \mathfrak{p}_l$.

Hence $\#\Lambda > 1$ since $\mathcal{I}(V)$ is not prime. Let l_1 be any element of Λ.

By Theorem 1.4 again (or Prop. 6.1 (ii) in CA and the minimality of the $\mathfrak{p}_l$) we have $\mathfrak{p}_{l_1} \not\supseteq \cap_{l\neq l_1}\mathfrak{p}_l$.

On the other hand, we also have $\mathfrak{p}_{l_1} \not\subseteq \cap_{l\neq l_1}\mathfrak{p}_l$ by minimality.

Hence $\mathrm{Z}(\mathfrak{p}_{l_1}) \not\subseteq \mathrm{Z}(\cap_{l\neq l_1}\mathfrak{p}_l)$ and $\mathrm{Z}(\mathfrak{p}_{l_1}) \not\supseteq \mathrm{Z}(\cap_{l\neq l_1}\mathfrak{p}_l)$.

Finally, we have $\mathrm{Z}(\mathcal{I}(V)) = V = \mathrm{Z}(\mathfrak{p}_{l_1}) \cup \mathrm{Z}(\cap_{l\neq l_1}\mathfrak{p}_l)$ by (3) above and Proposition 1.3 so that V is reducible. □

Lemma 1.6

Let $V \subseteq k^n$ be an algebraic set.

Then there is a unique finite collection $\{V_l\}_{l \in \Lambda}$ of irreducible algebraic subsets of k^n such that

(1) $V = \cup_l V_l$;

(2) $\forall l : V_l \not\subseteq \cup_{j \neq l} V_j$.

Furthermore, the V_l are the irreducible algebraic sets in k^n, which are maximal among the irreducible algebraic sets contained in V.

Proof. In view of the remark after Prop. 1.3, the properties (1)...(4) above and Lemma 1.5, this is equivalent to Theorem 1.4 for $R = R_n$. ☐

Proposition 1.7

Let $V \subseteq k^n$ *be an algebraic set defined by a radical ideal* I*.*

Let $\bar{v} = \langle v_1, \ldots, v_n \rangle \in V$ *and let* $\mathfrak{m}$ *be a maximal ideal of* R_n*.*

Suppose that $\mathfrak{m} \supseteq I$*. Then*

(1) $\mathcal{I}(\{\bar{v}\}) \supseteq I$ *and* $\mathcal{I}(\{\bar{v}\})$ *is a maximal ideal of* R_n*;*

(2) $\mathbb{Z}(\mathfrak{m})$ *consists of one point* $\bar{u} = \langle u_1, \ldots, u_n \rangle \in V$ *and* $\bar{u} \in V$*;*

(3) $\mathfrak{m} = (x_1 - u_1, \ldots, x_n - u_n)$ *where* $\bar{u}$ *is as in (2).*

Proof. Unravel the definitions and use the correspondence between radical ideals and algebraic sets. ☐

The last proposition in particular provides a correspondence between the points of V and the maximal ideals of R_n containing $\mathcal{I}(V)$, or equivalently with the maximal ideals of $R_n/\mathcal{I}(V)$.

In other words, if we write for any ring R

$$\mathrm{Spm}(R) := \{\text{maximal ideals of } R\}$$

then there is a natural bijection $\mathrm{Spm}(R_n/\mathcal{I}(V)) \simeq V$.

Lemma 1.8

Let $V \subseteq k^n$ be an algebraic set.

Under the bijection

$$\mathrm{Spm}(R_n/\mathcal{I}(V)) \simeq V,$$

the closed subsets of V correspond the subsets of $\mathrm{Spm}(R_n/\mathcal{I}(V))$ of the form

$$\mathrm{Z}(S) := \{\mathfrak{m} \in \mathrm{Spm}(R_n/\mathcal{I}(V)) \,|\, \mathfrak{m} \supseteq S\}$$

where $S \subseteq R_n/\mathcal{I}(V)$.

The closed subsets of V are in one to one correspondence with the radical ideals of $R_n/\mathcal{I}(V)$ via $\mathrm{Z}(\cdot)$.

Proof. Left to the reader. Unroll the definitions. ☐

Note that the set

$$\{\mathfrak{m} \in \mathrm{Spm}(R_n/\mathcal{I}(V)) \,|\, \mathfrak{m} \supseteq S\}$$

corresponds in V to the set $\mathrm{Z}(S') \cap V$ for any lifting of S to R_n.

So the notation $\mathrm{Z}(S)$ will not lead to any confusion.

Also, if $C \subseteq V$ is a closed subset, then we have

$$C = \mathrm{Z}(\mathcal{I}(C) \,(\mathrm{mod}\, \mathcal{I}(V))) = \mathrm{Z}(\mathcal{I}(C)) \cap V.$$

So we will sometimes use the shorthand $\mathcal{I}(C)$ for

$$\mathcal{I}(C) \,(\mathrm{mod}\, \mathcal{I}(V)) \subseteq R_n/\mathcal{I}(V)$$

if C is a closed subset of V.

With this notation, the properties (1),. . . , (4) listed above are also valid for the correspondence described in Lemma 1.8.

Regular maps between algebraic sets

Let $n, t \geqslant 0$.

A map $\phi : k^n \to k^t$ is said to be *polynomial* if there are elements

$$P_1(x_1, \ldots, x_n), \ldots, P_t(x_1, \ldots, x_n) \in R_n = k[x_1, \ldots, x_n],$$

such that

$$\phi(a_1, \ldots, a_n) = \langle P_1(a_1, \ldots, a_n), \ldots, P_t(a_1, \ldots, a_n) \rangle$$

for all $\langle a_1, \ldots, a_n \rangle \in k^n$.

Note that the polynomials P_i define a map of k-algebras $\phi^* : R_t \to R_n$ by the formula

$$\phi^*(Q(y_1, \ldots, y_t)) := Q(P_1(x_1, \ldots, x_n), \ldots, P_t(x_1, \ldots, x_n))$$

On the other hand, if we are given a map of k-algebras

$$\Phi : k[y_1, \ldots, y_t] = R_t \to R_n = k[x_1, \ldots, x_n],$$

then we can define polynomials $T_1(x_1, \ldots, x_n), \ldots, T_t(x_1, \ldots, x_n) \in R_n$ by the formula

$$T_i(x_1, \ldots, x_n) := \Phi(y_i)$$

and these two processes are obviously inverse to each other.

So to give polynomials P_i as above is equivalent to giving a map of k-algebras $R_t \to R_n$.

If $\Phi : R_t \to R_n$ is a map of k-algebras, we shall write

$$\mathrm{Spm}(\Phi) : k^n \to k^t$$

for the corresponding polynomial map.

Note that from definitions we see that the composition of two polynomials maps is a polynomial map.

Lemma 1.9

The map

$$\mathrm{Spm} : \{\textit{maps of } k\textit{-algebras } R_t \to R_n\} \to \{\textit{polynomial maps } k^n \to k^t\}$$

is bijective.

Proof. The surjectivity of Spm is a tautology so we only have to prove injectivity.

Let $\Phi_1, \Phi_2 : R_t \to R_n$ be two maps of k-algebras.

Suppose that $\mathrm{Spm}(\Phi_1) = \mathrm{Spm}(\Phi_2)$. We have to prove that $\Phi_1 = \Phi_2$.

Suppose that Φ_1 (resp. Φ_2) is defined by polynomials $P_{11}(x_1, \ldots, x_n), \ldots, P_{1t}(x_1, \ldots, x_n)$ (resp. $P_{21}(x_1, \ldots, x_n), \ldots, P_{2t}(x_1, \ldots, x_n)$).

Let $i \in \{1, \ldots, t\}$. If $\mathrm{Spm}(\Phi_1) = \mathrm{Spm}(\Phi_2)$ then the polynomial $P_{1i} - P_{2i}$ vanishes for all the values of its variables.

This implies that $P_{1i} = P_{2i}$. Since i was arbitrary, we conclude that $\Phi_1 = \Phi_2$. $\square$

In view of the lemma, for any polynomial map

$$\phi : k^n \to k^t,$$

there is a unique map of k-algebras

$$\phi^* : R_t \to R_n$$

such that $\mathrm{Spm}(\phi^*) = \phi$.

Note that the operation $(\cdot)^*$ (resp. $\mathrm{Spm}(\cdot)$) is compatible with composition of polynomial maps (resp. composition of maps of k-algebras). This follows from the definitions.

Let now $V \subseteq k^n$ and $W \subseteq k^t$ be algebraic sets in k^n and k^t, respectively.

A map

$$\psi : V \to W$$

is said to be *regular* if there is a polynomial map

$$\phi : k^n \to k^t$$

such that $\phi(V) \subseteq W$ and such that $\psi(v) = \phi(v)$ for all $v \in V$.

Note that if ψ is given, there might be several different ϕ inducing ψ.

Note also that a regular map is continuous for the Zariski topology.

Finally, note that a composition of regular maps is regular.

In the next slides we shall generalise Lemma 1.9 to algebraic sets.

Definition 1.10

Let $V \subseteq k^n$ be an algebraic set. The coordinate ring $\mathcal{C}(V)$ of V is the ring

$$\mathcal{C}(V) := R_n/\mathcal{I}(V).$$

Note that since $\mathcal{I}(V)$ is a radical ideal, the ring $\mathcal{C}(V)$ is a reduced ring, ie the only nilpotent element of $\mathcal{C}(V)$ is the zero element.

We also recall that any finitely generated algebra over a field is a Jacobson ring (see Cor. 9.4 in CA).

In particular, $\mathcal{C}(V)$ is a Jacobson ring. Recall that a Jacobson ring R is a ring such that for any ideal $I \subseteq R$, we have

$$\cap_{\mathfrak{m}\in\mathrm{Spm}(R),\mathfrak{m}\supseteq I} = \cap_{\mathfrak{p}\in\mathrm{Spec}(R),\mathfrak{p}\supseteq I} =: \mathfrak{r}(I)$$

where $\mathrm{Spec}(R)$ is the set of prime ideals of R (see section 4 in CA).

Let now $V \subseteq k^n$ and $W \subseteq k^t$ be algebraic sets in k^n and k^t, respectively.

Let

$$\psi : V \to W$$

be a regular map and let

$$\phi : k^n \to k^t$$

be a polynomial map inducing ψ, as above.

Suppose that

$$\phi = \mathrm{Spm}(\Phi)$$

for the map of k-algebras $\Phi : R_t \to R_n$.

Lemma 1.11

We have $\Phi(\mathcal{I}(W)) \subseteq \mathcal{I}(V)$.

Proof. Suppose Φ is given by elements

$$P_1(x_1, \ldots, x_n), \ldots, P_t(x_1, \ldots, x_n) \in R_n = k[x_1, \ldots, x_n],$$

as above. By assumption, for all $\bar{v} \in V$, we have

$$\langle P_1(\bar{v}), \ldots, P_t(\bar{v}) \rangle \in W$$

and so for any $Q(y_1, \ldots, t_t) \in \mathcal{I}(W)$ and any $\bar{v} \in V$, we have $Q(P_1(\bar{v}), \ldots, P_t(\bar{v})) = 0$. In other words,

$$\Phi(Q) = Q(P_1(x_1, \ldots, x_n), \ldots, P_t(x_1, \ldots, x_n)) \in \mathcal{I}(V)$$

as required. $\square$

From the lemma, we see that Φ induces a map of k-algebras $\Phi_{V,W} : \mathcal{C}(W) \to \mathcal{C}(V)$.

The next lemma is needed in the next proposition.

Lemma 1.12

If $\bar{v} := \langle v_1, \ldots, v_n \rangle \in V$ then the maximal ideal of $\mathcal{C}(W)$ corresponding to $\psi(\bar{v})$ is the ideal

$$\begin{aligned} & \Phi_{V,W}^{-1}((x_1 - v_1, \ldots, x_n - v_n) \pmod{\mathcal{I}(V)}) \\ = \quad & \Phi^{-1}((x_1 - v_1, \ldots, x_n - v_n)) \pmod{\mathcal{I}(W)}. \end{aligned}$$

In particular, $\Phi_{V,W}^{-1}$ sends maximal ideals to maximal ideals and $\Phi_{V,W}$ entirely determines $\psi : V \to W$.

Proof. Note first that $\Phi^{-1}((x_1 - v_1, \ldots, x_n - v_n))$ is maximal in R_t because there is by construction an injection of k-algebras

$$R_t/\Phi^{-1}((x_1 - v_1, \ldots, x_n - v_n)) \hookrightarrow R_n/(x_1 - v_1, \ldots, x_n - v_n) \simeq k$$

so that $R_t/\Phi^{-1}((x_1 - v_1, \ldots, x_n - v_n)) \simeq k$ (isomorphism of k-algebras).

On the other hand, any maximal ideal in $R_t = k[y_1, \ldots, y_t]$ is likewise of the form $(y_1 - u_1, \ldots, y_t - u_t)$ by Proposition 1.7.

So in order to determine the ideal $\Phi^{-1}((x_1 - v_1, \ldots, x_n - v_n))$ we only need to find $u_1, \ldots, u_t \in k$ such that

$$\Phi(y_i - u_i) \in (x_1 - v_1, \ldots, x_n - v_n). \tag{1}$$

By the correspondence between algebraic sets and radical ideals, condition (1) is equivalent to the condition that the polynomial $\Phi(y_i - u_i)$ vanishes on $\langle v_1, \ldots, v_n \rangle$.

We compute

$$\Phi(y_i - u_i)(\langle v_1, \ldots, v_n\rangle) = \Phi(y_i)(\langle v_1, \ldots, v_n\rangle) - u_i = \phi_i(\langle v_1, \ldots, v_n\rangle) - u_i$$

where ϕ_i is the projection of the map $\phi : k^n \to k^t$ to the i-th coordinate.

We thus see that $\Phi(y_i - u_i)$ vanishes on $\langle v_1, \ldots, v_n\rangle$ for all $i \in \{1, \ldots, t\}$ iff

$$\phi(\langle v_1, \ldots, v_n\rangle) = \langle u_1, \ldots, u_t\rangle.$$

Hence

$$\Phi^{-1}((x_1 - v_1, \ldots, x_n - v_n)) = (y_1 - \phi_1(\bar{v}), \ldots, y_t - \phi_t(\bar{v})).$$

In particular, the maximal ideal of $\mathcal{C}(W)$ corresponding to $\psi(\bar{v})$ is the ideal

$$\Phi_{V,W}^{-1}((x_1 - v_1, \ldots, x_n - v_n) \pmod{\mathcal{I}(V)}).$$

□

We now have the

Proposition 1.13

The map $\Phi_{V,W} : \mathcal{C}(W) \to \mathcal{C}(V)$ depends only on ψ.

Proof. Suppose that ψ is also induced by another polynomial map

$$\phi' : k^n \to k^t,$$

associated with a map of k-algebras $\Phi' : R_t \to R_n$.

Let

$$\Phi'_{V,W} : \mathcal{C}(W) \to \mathcal{C}(V)$$

be the map of k-algebras induced by ϕ' via Lemma 1.11.

Let $\mathfrak{m} \in \mathrm{Spm}(V)$.

By the above lemma and the assumptions, we have

$$(\Phi')^{-1}_{V,W}(\mathfrak{m}) = \Phi^{-1}_{V,W}(\mathfrak{m}) \in \mathrm{Spm}(\mathcal{C}(W)).$$

Let

$$\mathfrak{n} := (\Phi')^{-1}_{V,W}(\mathfrak{m}) = \Phi^{-1}_{V,W}(\mathfrak{m}).$$

Let $r \in \mathcal{C}(W)$. We have commutative diagrams

$$\begin{tikzcd}
\mathcal{C}(W) \arrow[r, "\Phi_{V,W}"] \arrow[d] & \mathcal{C}(V) \arrow[d] \\
\mathcal{C}(W)/\mathfrak{n} \arrow[r] & \mathcal{C}(V)/\mathfrak{m} \\
k \arrow[r, "="] \arrow[u, "\simeq"] & k \arrow[u, "\simeq"]
\end{tikzcd}$$

and also

$$\begin{tikzcd}
\mathcal{C}(W) \arrow[r, "\Phi'_{V,W}"] \arrow[d] & \mathcal{C}(V) \arrow[d] \\
\mathcal{C}(W)/\mathfrak{n} \arrow[r] & \mathcal{C}(V)/\mathfrak{m} \\
k \arrow[r, "="] \arrow[u, "\simeq"] & k \arrow[u, "\simeq"]
\end{tikzcd}$$

In particular, we see that $\Phi_{V,W}(r) \pmod{\mathfrak{m}} = \Phi'_{V,W}(r) \pmod{\mathfrak{m}}$.

Since $\mathfrak{m}$ was an arbitrary maximal ideal of $\mathcal{C}(V)$, we conclude that $\Phi_{V,W}(r) - \Phi'_{V,W}(r)$ lies in the Jacobson radical of $\mathcal{C}(V)$.

Since $\mathcal{C}(V)$ is a Jacobson ring and is reduced, we thus see that

$$\Phi_{V,W}(r) = \Phi'_{V,W}(r).$$

Since $r \in \mathcal{C}(W)$ was arbitrary, we conclude that $\Phi_{V,W} = \Phi'_{V,W}$. ☐

From the last lemma, we see that we may write

$$\Phi_{V,W} =: \psi^*.$$

Lemma 1.14

Let

$$\Lambda : \mathcal{C}(W) \to \mathcal{C}(V)$$

be a map of k-algebras. Then there is a regular map

$$\lambda : V \to W$$

such that $\lambda^* = \Lambda$.

We skip the proof, which is straightforward (see the notes for details).

From the last lemma, Lemma 1.12 and Proposition 1.13, we see that given a map of k-algebras

$$\Lambda : \mathcal{C}(W) \to \mathcal{C}(V),$$

there is a unique regular map

$$\mathrm{Spm}(\Lambda) : V \to W$$

such that $\mathrm{Spm}(\Lambda)^* = \Lambda$.

On the other hand, by Proposition 1.13, Lemma 1.12 and the previous lemma, given a regular map $\lambda : V \to W$, the map of k-algebras

$$\lambda^* : \mathcal{C}(W) \to \mathcal{C}(V)$$

is the unique one such that

$$\mathrm{Spm}(\lambda^*) = \lambda.$$

We conclude that there is a bijection from the set of regular maps

$$V \to W$$

to the set of maps of k-algebras

$$\mathcal{C}(W) \to \mathcal{C}(V),$$

which sends $\lambda : V \to W$ to λ^* and who inverse is given by $\mathrm{Spm}(\cdot)$.

Finally note that any finitely generated reduced k-algebra is isomorphic as a k-algebra to the coordinate ring of some algebraic set.

All this leads to an intrinsic characterisation of algebraic sets and regular maps between them.

We may view algebraic sets as a category whose objects are pairs (V, n) $(n \geqslant 0)$, where V is the zero set in k^n of a set of k-polynomials in n variables.

The categorical arrows from (V, n) to (W, t) are the maps from V to W, which are restrictions of polynomial maps from k^n to k^t.

The following theorem is a categorical summary of the previous discussion.

Theorem 1.15

The category of algebraic sets is antiequivalent to the category of finitely generated reduced k-algebras.

Varieties

Let $V \subseteq k^n$ be an algebraic set.

Note that from Theorem 1.15, there is a natural identification between the regular maps from V to k (where k is viewed as an algebraic set) and the elements of $\mathcal{C}(V)$.

Indeed the elements of $\mathcal{C}(V)$ are in one-to-one correspondence with the morphisms of k-algebras $k[x] \to \mathcal{C}(V)$ and in turn these morphisms correspond to regular maps $V \to k$.

More concretely, let $f \in \mathcal{C}(V) = R_n/\mathcal{I}(V)$ and let $\widetilde{f}$ be an arbitrary lifting of f to $R_n = k[x_1, \ldots, x_n]$.

The regular function $V \to k$ corresponding to f is then the restriction of the map $k^n \to k$ given by the polynomial $\widetilde{f}$.

We would also like to make sense of regular maps from open subsets of V to k.

Definition 1.16

Let $U \subseteq V$ be an open subset. A function

$$u : U \to k$$

is said to be regular if for any regular map of algebraic sets

$$\tau : T \to V$$

such that $\tau(T) \subseteq U$, the function $\tau \circ u$ is regular on T (ie corresponds to an element of $\mathcal{C}(T)$).

To show that this definition is useable, we shall need the following

Lemma 1.17

Any open set in V is a union of open subsets of the form $V \backslash \mathrm{Z}(f)$, for $f \in \mathcal{C}(V)$.

Proof. Left to the reader. Unroll the definitions. ☐

Lemma 1.18

Suppose that the regular map $h : V' \to V$ makes $\mathcal{C}(V')$ isomorphic to $\mathcal{C}(V)[f^{-1}]$ as a $\mathcal{C}(V)$-algebra for some $f \in \mathcal{C}(V)$. Then

(1) *h is injective and h is a homeomorphism onto $V \backslash \mathrm{Z}(f)$;*

(2) *if $g : V'' \to V$ is a regular map such that $g(V'') \subseteq V \backslash \mathrm{Z}(f)$, then there is a unique regular map $g' : V'' \to V'$ such that $g = h \cdot g'$.*

We skip the proof. Note that this can be translated into a problem of commutative algebra. See the notes for details.

Corollary 1.19

Let $f \in \mathcal{C}(V)$*.*

The regular functions on

$$V \backslash \mathrm{Z}(f)$$

are the restrictions of the functions $k^n \to k$ *which are of the form*

$$\frac{P(x_1, \ldots, x_n)}{(F(x_1, \ldots, x_n))^l}$$

($l \geqslant 0$ *), where* $P(x_1, \ldots, x_n) \in R_n$ *and* $F(x_1, \ldots, x_n) \in R_n$ *is any lifting of* f *to* R_n*.*

Proof. Note first that $\mathcal{C}(V)[f^{-1}] \simeq \mathcal{C}(V)[t]/(tf-1)$ as a $\mathcal{C}(V)$-algebra (see Lemma 5.3 in CA).

Hence $\mathcal{C}(V)[f^{-1}]$ corresponds to the algebraic set Z in k^{n+1} given by the ideal generated by the sets $\mathcal{I}(V)$ and $tF(x_1,\dots,x_n)-1$ in $k[x_1,\dots,x_n,t]$.

The polynomial map $\phi : k^{n+1} \to k^n$ inducing the map of k-algebras $\mathcal{C}(V) \to \mathcal{C}(V)[t]/(tf-1)$ is simply given by the formula $\phi(\langle v_1,\dots,v_n,z\rangle) = \langle v_1,\dots,v_n\rangle$.

The inverse of the map $Z \overset{\phi|_Z}{\to} V\backslash \mathrm{Z}(f)$ is given by the formula $\langle v_1,\dots,v_n\rangle \mapsto \langle v_1,\dots,v_n,F(v_1,\dots,v_n)^{-1}\rangle$.

Hence a regular map on $V\backslash \mathrm{Z}(f)$ is given by the evaluation of a polynomial in the variables $x_1,\dots,x_n,t$ on the vector $\langle v_1,\dots,v_n,F(v_1,\dots,v_n)^{-1}\rangle$ (for $\langle v_1,\dots,v_n\rangle \in V\backslash \mathrm{Z}(f)$). $\square$

Proposition 1.20

Let U be an open subset of the algebraic set $V \subseteq k^n$.

A function $a : U \to k$ is regular iff for any point $\bar{u} \in U$, there is

- *a polynomial $F \in R_n$, such that $F(\bar{u}) \neq 0$*
- *a polynomial $P \in R_n$ such that a coincides with P/F in a neighbourhood of $\bar{u}$.*

This implies in particular that if a function $a : U \to k$ is regular and nowhere vanishing, then $1/a : U \to k$ is also a regular function.

In other words, the units in the ring of regular functions $U \to k$ are the nowhere vanishing regular functions.

Proof. (of Proposition 1.20). We first show the following.

Let $W \subseteq k^t$ be an algebraic set.

Let $f_1, \ldots, f_l \in \mathcal{C}(W)$ and suppose that $(f_1, \ldots, f_l) = \mathcal{C}(W)$.

Let $h : W \to k$ be a function (not assumed regular) and suppose that for each $i \in \{1, \ldots, l\}$ there is an integer $n_i \geqslant 0$ and an element $c_i \in \mathcal{C}(W)$ such that $h|_{W \setminus \mathrm{Z}(f_i)} = c_i / f_i^{n_i}$.

We claim that the function h is then regular on W (ie arises from an element of $\mathcal{C}(W)$.

To prove this, note first that we may assume that all the n_i are equal to some $m \geqslant 1$.

Indeed, if we let $m := 1 + \sup_i n_i$ then we may write

$$h|_{W \setminus \mathrm{Z}(f_i)} = c_i f_i^{m - n_i} / f_i^m$$

for all i.

Now notice that for all $i, j \in \{1, \ldots, l\}$ we have

$$h|_{W \setminus \mathrm{Z}(f_i f_j)} = c_i / f_i^m = c_j / f_j^m$$

so that

$$(f_i f_j)^m (c_i / f_i^m - c_j / f_j^m) = f_j^m c_i - c_j f_i^m = 0$$

on $W \setminus \mathrm{Z}(f_i f_j)$.

We deduce that

$$(f_i f_j) f_j^m c_i = (f_i f_j) c_j f_i^m$$

on V.

Now let $b_i \in \mathcal{C}(W)$ be functions such that

$$\sum_i b_i f_i^{2m} = 1$$

(note that we also have $(f_1^{2m}, \dots, f_l^{2m}) = \mathcal{C}(W)$ - prove this or see Lemma 12.2 in CA).

Let

$$\widetilde{h} := \sum_i b_i f_i^m c_i.$$

We compute

$$\begin{aligned} \widetilde{h} f_j^{2m} &= \sum_i b_i f_i^m f_j^{2m} c_i = \sum_i b_i (f_i f_j)^m f_j^m c_i \\ &= \sum_i b_i (f_i f_j)^m f_i^m c_j = (\sum_i b_i f_i^{2m}) f_j^m c_j = f_j^m c_j \end{aligned}$$

so that $\widetilde{h}|_{W \setminus \mathrm{Z}(f_j)} = c_j / f_j^m$. Hence $\widetilde{h} = h$. This completes the proof of the claim.

Coming back to the proposition, note that the "$\Rightarrow$" direction of the equivalence stated in the proposition is clear from Lemma 1.17 and Corollary 1.19.

Thus we only have to prove the "$\Leftarrow$" direction of the equivalence.

Since the topology of U is quasi-compact (this will be proven in exercise sheet 2, Q4 (4) - you can also prove this directly), we may reword this implication as follows.

Let $g_1, \dots, g_l \in \mathcal{C}(V)$ and suppose that $U = \cup_i (V \backslash \mathrm{Z}(g_i))$.

Let $V' \subseteq k^{n'}$ be an algebraic set and let $H : V' \to V$ be a regular map such that $H(V') \subseteq U$.

Suppose that for all $i \in \{1, \dots, l\}$ we have

$$a|_{V \backslash \mathrm{Z}(g_i)} = d_i / g_i$$

for some $n_i \geqslant 0$ and some $d_i \in \mathcal{C}(V)$.

The "$\Leftarrow$" direction of the equivalence of the proposition is then the statement that $a \circ H = H^*(a)$ is a regular function on V'.

So we only have to prove this last statement.

Note first that by construction, for all $i \in \{1, \ldots, l\}$ we have

$$H^*(a)|_{V' \setminus \mathrm{Z}(H^*(g_i))} = H^*(d_i)/H^*(g_i).$$

Also, since $H(V') \subseteq U$, we have

$$(H^*(g_1), \ldots, H^*(g_l)) = \mathcal{C}(V').$$

Hence we may apply the preceding claim to

$$W = V',\ f_i = H^*(g_i) \text{ and } h = H^*(a)$$

to conclude that $H^*(a)$ is regular on V'. $\square$

Note that in view of the previous proposition, the following property holds trivially: if $U' \subseteq U$ is an inclusion of open subsets of V, then the restriction to U' of a regular function on U is also regular.

We encapsulate this property in the following

Definition 1.21

Let T be a topological space.

A sheaf of functions $\mathcal{O}_T$ on T with values in k is an assignement, which associates with each open subset O of T a sub k-algebra $\mathcal{O}_T(O)$ of $\mathrm{Maps}(O, k)$, with the following property:

- *for any open covering $\{O_i\}$ of an open subset O, a function $f : O \to k$ lies in $\mathcal{O}_T(O)$ iff $f|_{O_i} \in \mathcal{O}_T(O_i)$ for all i.*

Here $\mathrm{Maps}(O, k)$ is the set of functions from O to k, with its natural k-algebra structure (given by pointwise multiplication and addition).

Note that if O is an open subset of topological space endowed with a sheaf of k-valued functions, O inherits a sheaf of k-valued functions from T.

Proposition 1.20 implies that for any algebraic set $V \subseteq k^n$, the regular functions on Zariski open subsets of V define a sheaf of functions $\mathcal{O}_V$ with values in k on V.

There is a natural notion of mapping between topological spaces endowed with sheaves of k-valued functions:

Definition 1.22

Let $(T, \mathcal{O}_T)$ and $(T', \mathcal{O}_{T'})$ be two topological spaces endowed with sheaves of functions with values in k.

A morphism (sometimes loosely called a map) from $(T, \mathcal{O}_T)$ to $(T', \mathcal{O}_{T'})$ is a continuous map $a : T \to T'$ such that for any open subset

$$U' \subseteq T'$$

and any element

$$f \in \mathcal{O}_{T'}(U'),$$

the function

$$f \circ a|_{a^{-1}(U')}$$

on $a^{-1}(U')$ lies in $\mathcal{O}_T(a^{-1}(U'))$.

Let T be a topological space endowed with a sheaf of functions $\mathcal{O}_T$ with values in k.

Let $t \in T$. Let

$$\widehat{\mathcal{O}}_{T,t} := \cup_{O \text{ open}, \, t \in O} \mathcal{O}_T(O)$$

(where all the $\mathcal{O}_T(O)$ are considered to be disjoint from each other).

Define an equivalence relation on $\widehat{\mathcal{O}}_{T,t}$ by declaring two functions in $\widehat{\mathcal{O}}_{T,t}$ equivalent if they coincide in some open neighbourhood of t.

The set of equivalence classes in $\widehat{\mathcal{O}}_{T,t}$ has a natural k-algebra structure and we denote it by $\mathcal{O}_{T,t}$.

The k-algebra $\mathcal{O}_{T,t}$ is called *the local ring at t*.

Note that by definition, for any open neighbourhood O of t, there is a natural map of k-algebras $\mathcal{O}_T(O) \to \mathcal{O}_{T,t}$.

Also, there is a natural map of k-algebras $\mathcal{O}_{T,t} \to k$, which is given by evaluation at t.

If we are given a morphism from $(T, \mathcal{O}_T)$ to $(T', \mathcal{O}_{T'})$ as in the last definition, the pull-back of functions gives a map of k-algebras $\mathcal{O}_{T,a(t)} \to \mathcal{O}_{T,t}$ for any $t \in T$.

From the very definition of regularity, we see that any regular map from an algebraic set to another induces a morphism between the associated topological spaces with sheaves of k-valued functions.

We are now ready for the definition of a general variety.

Definition 1.23

Let T be a topological space endowed with a sheaf of functions with values in k.

We say that T is a variety if there is a finite open covering $\{U_i\}$ of T, such that U_i with its induced sheaf of k-valued functions is isomorphic to an algebraic set endowed with its sheaf of regular functions.

A morphism of varieties is a morphism of the corresponding topological spaces with sheaves of k-valued functions.

Lemma 1.24

Let $V \subseteq k^n$ be an algebraic set and let $(V, \mathcal{O}_V)$ be the associated topological space with sheaf of k-valued functions. Let $\bar{v} \in V$.

Then the natural map of k-algebras

$$\mathcal{C}(V) = \mathcal{O}_V(V) \to \mathcal{O}_{V,\bar{v}}$$

extends (necessarily uniquely) to an isomorphism of k-algebras

$$\mathcal{C}(V)_{\bar{v}} \simeq \mathcal{O}_{V,\bar{v}}.$$

Here we identified $\bar{v}$ with the corresponding maximal ideal $\mathcal{I}(\{\bar{v}\})$ when writing $\mathcal{C}(V)_{\bar{v}}$ (so that $\mathcal{C}(V)_{\bar{v}}$ is the localisation of $\mathcal{C}(V)$ at the multiplicative set $\mathcal{C}(V)\backslash\mathcal{I}(\{\bar{v}\})$).

Proof. We first show that the map $\mathcal{C}(V) \to \mathcal{O}_{V,\bar{v}}$ extends to a map of k-algebras $\mathcal{C}(V)_{\bar{v}} \to \mathcal{O}_{V,\bar{v}}$.

To show this, we have to show that a regular function $f \in \mathcal{C}(V)$, which does not vanish at $\bar{v}$, maps to a unit in $\mathcal{O}_{V,\bar{v}}$.

By definition, a unit in $\mathcal{O}_{V,\bar{v}}$ is represented by a regular function in a neighbourhood of $\bar{v}$, which vanishes nowhere in that neighbourhood.

Now since f does not vanish at $\bar{v}$, it is nowhere vanishing in the set $V \backslash \mathrm{Z}(f)$, which is a neighbourhood of $\bar{v}$. So the image of f in $\mathcal{O}_{V,\bar{v}}$ is a unit.

So we have a unique extension of the map $\mathcal{C}(V) \to \mathcal{O}_{V,\bar{v}}$ to a map of k-algebras $\mathcal{C}(V)_{\bar{v}} \to \mathcal{O}_{V,\bar{v}}$.

We still have to show that this last map is injective and surjective.

We first show injectivity. Let $f/s \in \mathcal{C}(V)_{\bar{v}}$ (where $s \in \mathcal{C}(V)\backslash\mathcal{I}(\{\bar{v}\})$).

Suppose that the image of f/s in $\mathcal{O}_{V,\bar{v}}$ vanishes.

By definition, this means that the function f vanishes in a neighbourhood of $\bar{v}$.

In particular, there exists an $h \in \mathcal{C}(V)$ such that f vanishes in $V\backslash\mathrm{Z}(h)$, where h does not vanish at $\bar{v}$ (use Lemma 1.17).

In other words, the image of f in $\mathcal{C}(V)[h^{-1}]$ vanishes.

Since $h \notin \mathcal{I}(\{\bar{v}\})$, the natural map $\mathcal{C}(V) \to \mathcal{C}(V)_{\bar{v}}$ factors through $\mathcal{C}(V)[h^{-1}]$ and hence the image of f in $\mathcal{C}(V)_{\bar{v}}$ also vanishes.

This settles injectivity.

Now for surjectivity.

By Lemma 1.17, an element $\widetilde{e} \in \mathcal{O}_{V,\bar{v}}$ is represented by a regular function on $V \backslash \mathrm{Z}(h)$, for some h which does not vanish at $\bar{v}$.

Such a function corresponds to an element of $\mathcal{C}(V)[h^{-1}]$ and again since the natural map $\mathcal{C}(V) \to \mathcal{C}(V)_{\bar{v}}$ factors through $\mathcal{C}(V)[h^{-1}]$, we see that $\widetilde{e}$ lies in the image of $\mathcal{C}(V)_{\bar{v}}$.

Since $\widetilde{e} \in \mathcal{O}_{V,\bar{v}}$ was arbitrary, the natural map $\mathcal{C}(V)_{\bar{v}} \to \mathcal{O}_{V,\bar{v}}$ is surjective. □

In particular, the ring $\mathcal{O}_{V,\bar{v}}$ is local.

Also, note that the natural evaluation map $\mathcal{O}_{V,\bar{v}} \to k$ is surjective, because all constant functions are regular on V.

Hence the kernel of the map $\mathcal{O}_{V,\bar{v}} \to k$ is maximal.

Hence this kernel coincides with the unique maximal ideal of $\mathcal{O}_{V,\bar{v}}$.

For Definition 1.23 to be coherent, we need to check that we can recover an algebraic set from its associated topological space with sheaf of k-valued functions:

Lemma 1.25

Let $V \subseteq k^n$ and $W \subseteq k^t$ be two algebraic sets.

Let $(V, \mathcal{O}_V)$ and $(W, \mathcal{O}_W)$ be the associated topological spaces with sheaves of k-valued functions.

Let g be a morphism from $(V, \mathcal{O}_V)$ to $(W, \mathcal{O}_W)$.

Then g is induced by a regular map $\psi : V \to W$.

Proof. By definition, the morphism g provides a map of k-algebras $\mathcal{C}(W) \to \mathcal{C}(V)$.

Furthermore, for any $\bar{v} \in V$, we have a commutative diagram of k-algebras

$$\begin{tikzcd}
\mathcal{C}(W) \arrow[r, "g^*"] \arrow[d] & \mathcal{C}(V) \arrow[d] \\
\mathcal{O}_{W,g(\bar{v})} \arrow[r, "g^*"] & \mathcal{O}_{V,\bar{v}}
\end{tikzcd}$$

From the remark after Lemma 1.24, the ring $\mathcal{O}_{V,\bar{v}}$ is a local ring and its maximal ideal consists of the elements represented by the regular functions h defined in a neighbourhood of $\bar{v}$ such that $h(\bar{v}) = 0$.

The same is true for $\mathcal{O}_{W,g(\bar{v})}$ and $g(\bar{v})$ in place of $\bar{v}$. In particular, the map $g^* : \mathcal{O}_{W,g(\bar{v})} \to \mathcal{O}_{V,\bar{v}}$ sends the maximal ideal of $\mathcal{O}_{W,g(\bar{v})}$ into the maximal ideal of $\mathcal{O}_{V,\bar{v}}$.

Since the involved rings are local, this implies that the inverse image by g^* of the maximal ideal of $\mathcal{O}_{V,\bar{v}}$ is the maximal ideal of $\mathcal{O}_{W,g(\bar{v})}$.

We conclude that the inverse image of

$$\mathcal{I}(\{\bar{v}\}) \subseteq \mathcal{C}(V)$$

by

$$g^* : \mathcal{C}(V) \to \mathcal{C}(W)$$

is $\mathcal{I}(\{\bar{g}(\bar{v})\})$.

In particular, $g(\bar{v}) = \mathrm{Spm}(g^*)(\bar{v})$ (use Lemma 1.12).

Hence g is induced by the map of k-algebras $g^* : \mathcal{C}(W) \to \mathcal{C}(V)$ and hence by a regular map $V \to W$ (by Theorem 1.15). ☐

Open and closed subvarieties

Proposition 1.26

Let $(V, \mathcal{O}_V)$ *be a variety.*

Let $U \subseteq V$ *be an open subset and let* $\mathcal{O}_U$ *be the sheaf of* k*-valued functions induced by* $\mathcal{O}_V$*.*

Then $(U, \mathcal{O}_U)$ *is a variety and the inclusion map is a morphism of varieties.*

Proof. Let $\{V_i\}$ be an open covering of V such that each V_i is isomorphic as a Topskf to an affine variety.

Then $\{V_i \cap U\}$ is an open covering of U.

Since $V_i \cap U$ is open in V_i, there is for each i a subset $E_i \subseteq \mathcal{C}(V_i)$ such that

$$\cup_{e \in E_i}(V_i \backslash \mathrm{Z}(e)) = V_i \cap U$$

(use Lemma 1.17). Hence we only have to show that the open subset $V_i \backslash \mathrm{Z}(e)$ of V_i is isomorphic as a Topskf to an affine variety.

But this follows from Lemma 1.18. □

An open subset of a variety is called an *open subvariety* if it is endowed with the structure of Topskf described in the last Proposition.

Let $(V, \mathcal{O}_V)$ be a variety. Let $Z \subseteq V$ be a closed subset.

Endow Z with the topology induced by V.

For any open subset O of Z, define a function $f : O \to k$ to be regular if there is collection of open subsets $\{U_i\}$ of V and regular functions $g_i : U_i \to k$ such that

- $(\cup_i U_i) \cap Z = O$;

- $g_i|_{O \cap U_i} = f|_{O \cap U_i}$.

This endows Z with a structure of topological space with k-valued functions.

We shall write $\mathcal{O}_Z$ for the corresponding sheaf of k-valued functions.

The sheaf of k-valued functions $\mathcal{O}_Z$ on Z is said to be *induced* by $\mathcal{O}_V$.

Proposition 1.27

The topological space Z with sheaf of k-valued functions $\mathcal{O}_Z$ is a variety. The inclusion map $Z \to V$ is a morphism of varieties.

Proof. The inclusion map $Z \to V$ provides us with a morphism

$$(Z, \mathcal{O}_Z) \to (V, \mathcal{O}_V)$$

of Topskf by construction.

Hence we only have to show that $(Z, \mathcal{O}_Z)$ is a variety (see Definition 1.23).

Let $\{V_i\}$ be a covering of V by open subsets such that $(V_i, \mathcal{O}_{V_i})$ is isomorphic as a Topskf to an affine variety.

By definition, it is sufficient to show that for each i, the Topskf $Z \cap V_i$ is isomorphic to an affine variety.

Hence we may assume that V is affine to begin with.

Hence we are reduced to the situation where $V \subseteq k^n$ is an algebraic set and $Z \subseteq k^n$ is another algebraic set such that $Z \subseteq V$.

Endow Z with the sheaf of functions $\mathcal{O}_Z$ induced by $\mathcal{O}_V$.

We would like to show that $(Z, \mathcal{O}_Z)$ is isomorphic to an affine variety as a Topskf.

Now note that by Proposition 1.20 the sheaf $\mathcal{O}_Z$ is precisely the sheaf of regular functions on Z viewed as an algebraic subset of k^n.

So $(Z, \mathcal{O}_Z)$ is isomorphic to an affine variety as a Topskf. ☐

An closed subset of a variety V is called a *closed subvariety* if it is endowed with the structure of Topskf induced by V.

Lemma 1.28

Let $(W, \mathcal{O}_W)$ *and* $(V, \mathcal{O}_V)$ *be two varieties.*

Let Z *(resp.* O*) be a closed subset (resp. open subset) of* V*.*

Endow Z *(resp.* O*) with its structure of closed (resp. open) subvariety.*

Let $\lambda : W \to V$ *be a morphism of Topskf such that*

$$\lambda(W) \subseteq Z$$

(resp. $\lambda(W) \subseteq O$*).*

Then the induced map $W \to Z$ *(resp.* $W \to O$*) is a morphism of Topskf.*

Proof. Left to the reader. Unroll the definitions. ☐

We also record a consequence of the proof of Proposition 1.27:

Lemma 1.29

Let $V \subseteq W \subseteq k^n$, where V and W are algebraic sets in k^n.

Let $(V, \mathcal{O}_V) \to (W, \mathcal{O}_W)$ be the corresponding morphism of topological spaces with sheaves of k-valued functions.

Then $\mathcal{O}_V$ is induced by $\mathcal{O}_W$.

Projective space

Projective varieties arise when one tries to find an algebraic counterpart of the topological notion of compactness.

We will revisit this later when we consider complete varieties.

Let $n \geqslant 0$. A *line through the origin* of k^{n+1} is by definition the vector subspace $[\bar{v}]$ of k^{n+1} generated by a vector $\bar{v} \in k^{n+1} \backslash \{0\}$.

We define *projective space of dimension* n to be the set $\mathbb{P}^n(k)$ of lines through the origin of k^{n+1}.

If $\bar{v} = \langle v_0, \ldots, v_n \rangle \in k^{n+1} \backslash \{0\}$, we shall write $[v_0, \ldots, v_n]$ for $[\langle v_0, \ldots, v_n \rangle]$.

We shall endow $\mathbb{P}^n(k)$ with a variety structure.

For $i \in \{0, \ldots n\}$, define

$$U_i = \{[v_0, \ldots, v_n] \in \mathbb{P}^n(k) \mid v_i \neq 0\}.$$

In the following, we shall write the symbol $\check{}$ over a term that is to be omitted.

The map $u_i : k^n \to U_i$ such that

$$u_i(\langle v_0, \ldots, \check{v}_i, \ldots, v_n \rangle) := [v_0, \ldots, v_{i-1}, 1, v_{i+1}, \ldots v_n]$$

is clearly a bijection and we have

$$u_i^{-1}([v_0, \ldots, v_n]) = \langle \frac{v_0}{v_i}, \ldots, \check{\frac{v_i}{v_i}}, \ldots, \frac{v_n}{v_i} \rangle.$$

if $[v_0, \ldots, v_n] \in U_i$.

If $j < i$ and $v_j \neq 0$, we compute

$$(u_j^{-1} \circ u_i)(\langle v_0, \ldots, \check{v}_i, \ldots, v_n \rangle) = u_j^{-1}([v_0, \ldots, v_{i-1}, 1, v_{i+1}, \ldots v_n])$$

$$= \langle \frac{v_0}{v_j}, \ldots, \check{\frac{v_j}{v_j}}, \ldots, \frac{1}{v_j}, \frac{v_{i+1}}{v_j}, \ldots, \frac{v_n}{v_j} \rangle$$

and if $j > i$ and $v_j \neq 0$, we have similarly

$$(u_j^{-1} \circ u_i)(\langle v_0, \ldots, \check{v}_i, \ldots, v_n \rangle) = \langle \frac{v_0}{v_j}, \ldots, \frac{v_{i-1}}{v_j}, \frac{1}{v_j}, \ldots, \check{\frac{v_j}{v_j}}, \ldots, \frac{v_n}{v_j} \rangle$$

Hence, if $i \neq j$, the map $u_j^{-1} \circ u_i$ gives a map from the open subset of k^n

$$\mathcal{U}_{ij} := \{\langle v_0, \ldots, \check{v}_i, \ldots, v_n \rangle \in k^n \mid v_j \neq 0\}$$

into the open subset of k^n

$$\mathcal{U}_{ji} := \{\langle v_0, \ldots, \check{v}_j, \ldots, v_n \rangle \in k^n \mid v_i \neq 0\}$$

and $u_i(\mathcal{U}_{ij}) = U_i \cap U_j = u_j(\mathcal{U}_{ji})$.

Let $u_{ij} := u_j^{-1} \circ u_i : \mathcal{U}_{ij} \to \mathcal{U}_{ji}$.

Note that if one sees $\mathcal{U}_{ij}$ as an open subvariety of k^n, then $\mathcal{U}_{ij}$ is an affine variety associated with the coordinate ring

$$k[x_0, \dots, \check{x}_i, \dots, x_n][x_j^{-1}] \simeq k[x_0, \dots, \check{x}_i, \dots, x_n][t]/(tx_j - 1)$$

and similarly, $\mathcal{U}_{ji}$ is an affine variety associated with the coordinate ring

$$k[y_0, \dots, \check{y}_j, \dots, y_n][y_i^{-1}] \simeq k[y_0, \dots, \check{y}_j, \dots, y_n][t]/(zy_i - 1)$$

One checks from the definitions that u_{ij} arises from the polynomial map which sends z to x_j and y_l to $x_l \cdot t$ if $l \neq i$ and to t if $l = i$.

Hence u_{ij} defines a morphism of varieties from $\mathcal{U}_{ij}$ to $\mathcal{U}_{ji}$.

One checks from the just given formula that u_{ij} and u_{ji} are inverse to each other, so u_{ij} is an isomorphism of varieties.

Now we define a topology on $\mathbb{P}^n(k)$ by declaring a subset $O \subseteq \mathbb{P}^n(k)$ to be open iff $u_i^{-1}(O)$ is open in k^n for all $i \in \{0, \dots, n\}$.

Furthermore, if $O \subseteq \mathbb{P}^n(k)$ is open, we define a k-valued function

$$f : O \to k$$

to be *regular* iff

$$f \circ u_i|_{u_i^{-1}(O)}$$

is a regular function on $u_i^{-1}(O)$ for all i.

Since $(k^n, \mathcal{O}_{k^n})$ is a Topskf, we see that with this definition, $\mathbb{P}^n(k)$ becomes a Topskf.

We shall write $\mathcal{O}_{\mathbb{P}^n(k)}$ for the just defined sheaf of k-valued functions on $\mathbb{P}^n(k)$.

Proposition 1.30

The sets U_i are open in $\mathbb{P}^n(k)$ for all $i \in \{0, \ldots, n\}$.

The maps $u_i : k^n \to \mathbb{P}^n(k)$ restrict to isomorphisms of Topskf between k^n and $(U_i, \mathcal{O}_{U_i})$, where $\mathcal{O}_{U_i}$ is the sheaf of k-valued functions induced on U_i by $\mathcal{O}_{\mathbb{P}^n(k)}$.

In particular, the Topskf $(\mathbb{P}^n(k), \mathcal{O}_{\mathbb{P}^n(k)})$ is a variety.

The U_i are called the *standard coordinate charts* of $\mathbb{P}^n(k)$.

We shall sometimes write U_i^n for U_i to emphasise the dependence on n.

Proof. To show that U_i is open, we have to show that $u_j^{-1}(U_i)$ is open in k^n for all j.

We have shown above that $u_j^{-1}(U_i) = \mathcal{U}_{ji}$ is open, so U_i is open.

Next, we have to show that the map u_i is a homeomorphism onto its image.

The map u_i is continuous and injective by definition so we only have to show that u_i is an open map.

So let $O \subseteq k^n$ be an open set. We have to show that $u_i(O)$ is open, or in other words that $u_j^{-1}(u_i(O))$ is open for all j.

Now we have

$$u_j^{-1}(u_i(O)) = u_j^{-1}(u_i(O) \cap (U_i \cap U_j)) = u_j^{-1}(u_i(O \cap \mathcal{U}_{ij})) = u_{ij}(O \cap \mathcal{U}_{ij})$$

and $u_{ij}(O \cap \mathcal{U}_{ij})$ is open in $\mathcal{U}_{ji}$ since $u_{ij} : \mathcal{U}_{ij} \to \mathcal{U}_{ji}$ is a homeomorphism by the above.

On the other hand $\mathcal{U}_{ji}$ is open in U_j, so $u_{ij}(O \cap \mathcal{U}_{ij})$ is also open in U_j.

So u_i is a homeomorphism onto its image.

For the rest of the proof, see the notes. $\square$

Example. The space $\mathbb{P}^1(k)$ only has two coordinate charts, the charts U_0 and U_1.

By inspection, we see that $\mathbb{P}^1(k)\backslash U_i$ consists of only one point.

So one can see $\mathbb{P}^1(k)$ as the "compactification" of k obtained by adding a "point at ∞" to k.

If $k = \mathbb{C}$, the space $\mathbb{P}^1(k)$ can be naturally identified (as a set) with the Riemann sphere of complex analysis.

Projective varieties

What are the closed subsets of projective space? To answer this question, we shall need the following definitions.

A polynomial $P(x_0, \ldots, x_n) \in k[x_0, \ldots, x_n]$ is said to be *homogenous* if it is a sum of monomials of the same degree.

Any polynomial $P(x_0, \ldots, x_n)$ has a canonical decomposition

$$P = \sum_{i=0}^{\deg(P)} P_{[i]}$$

where $P_{[i]}$ is the sum of the monomials of degree i appearing in P (so that in particular $P_{[i]}$ is homogenous).

Example. The polynomials x_0, $x_0^2 + x_0 x_1$ are homogenous but $x_0^2 + x_1$ is not.

We have a decomposition of $k[x_0, \ldots, x_n]$ as an internal direct sum

$$k[x_0, \ldots, x_n] = \bigoplus_{l \geqslant 0} k[x_0, \ldots, x_n]_{[l]}$$

where $k[x_0, \ldots, x_n]_{[l]}$ is the k-vector space of homogenous polynomials of degree l.

In particular, we have $k[x_0, \ldots, x_n]_{[0]} = k$.

This decomposition into a direct sum makes $k[x_0, \ldots, x_n]$ into a graded ring in the sense of section 11.2 of CA.

Example. We have $(x_0^2 + x_1)_{[2]} = x_0^2$, $(x_0^2 + x_1)_{[1]} = x_1$, $(x_0^2 + x_1)_{[0]} = 0$.

Note the following elementary fact. If $P(x_0, \dots, x_n) \in k[x_0, \dots, x_n]$ is homogenous then

$$P(s \cdot x_0, \dots, s \cdot x_n) = s^{\deg(P)} P(x_0, \dots, x_n)$$

for all $s \in k$.

We thus see that if $P(x_0, \dots, x_n) \in k[x_0, \dots, x_n]$ is a homogenous polynomial and $\bar{v} \in k^{n+1}$ is non zero, we have $P(\bar{v}) = 0$ iff $P(s \cdot \bar{v}) = 0$ for all $s \in k^*$.

This gives rise to the following definition.

Let $S \subseteq k[x_0, \dots, x_n]$ be a set of homogenous polynomials. We define

$$\mathrm{Z}(S) := \{[\bar{v}] \in \mathbb{P}^n(k) \,|\, \bar{v} \in k^{n+1} \backslash \{0\}, \forall P \in S : P(\bar{v}) = 0\}.$$

A *projective algebraic set* in $\mathbb{P}^n(k)$ is a subset of the form $\mathrm{Z}(S)$, where $S \subseteq k[x_0, \dots, x_n]$ is a set of homogenous polynomials.

For convenience, we shall extend the operator $\mathrm{Z}(\cdot)$ to non homogenous polynomials.

For any set $S \subseteq k[x_0, \ldots, x_n]$ (not necessarily consisting of homogenous polynomials), we set

$$\mathrm{Z}(S) := \{[\bar{v}] \mid \bar{v} \in k^{n+1} \backslash \{0\}, P_{[i]}(\bar{v}) = 0 \, \forall i \geqslant 0\}.$$

Just as in the affine case, we have $\mathrm{Z}(S) = \mathrm{Z}(S \cdot k[x_0, \ldots, x_n])$.

Hence the projective algebraic sets in $\mathbb{P}^n(k)$ are the sets of the type $\mathrm{Z}(I)$, where $I \subseteq k[x_0, \ldots, x_n]$ is an ideal generated by homogenous elements.

We shall say that an ideal of $k[x_0, \ldots x_n]$ is *homogenous* if it is generated by homogenous elements.

Lemma 1.31

Let $I \subseteq k[x_0, \ldots x_n]$ be an ideal.

Then I is homogenous iff for all $P \in I$ and all $i \geqslant 0$, we have $P_{[i]} \in I$.

If I is homogenous then its radical $\mathfrak{r}(I)$ is also homogenous.

In other words, a homogenous ideal is a graded ideal in $k[x_0, \ldots, x_n]$ (ie a graded $k[x_0, \ldots, x_n]$-submodule of $k[x_0, \ldots, x_n]$).

Proof. See exercises. □

Proposition 1.32

Projective algebraic sets are closed in $\mathbb{P}^n(k)$*.*

Furthermore, if $C \subseteq \mathbb{P}^n(k)$ *is a closed subset and* J *is the ideal generated by the homogenous polynomials which vanish on* C*, then* $\mathrm{Z}(J) = C$*.*

In particular, the closed subsets of $\mathbb{P}^n(k)$ *are precisely the projective algebraic sets.*

Proof. Let $S := \{P_l\}$ be a set of homogenous polynomials in $k[x_0, \ldots, x_n]$.

By construction, we have

$$u_i^{-1}(\mathrm{Z}(S)) = \mathrm{Z}(\{P_l(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)\})$$

so that $u_i^{-1}(\mathrm{Z}(S))$ is closed in k^n.

By Proposition 1.30, the set $\mathrm{Z}(S) \cap U_i$ is thus closed in U_i (for the induced topology).

Since the U_i cover $\mathbb{P}^n(k)$, we thus see that $\mathrm{Z}(S)$ is closed in $\mathbb{P}^n(k)$.

As to the second assertion, we clearly have $\mathrm{Z}(J) \supseteq C$.

So we need to prove that $\mathrm{Z}(J) \subseteq C$.

In other words, we have to prove that if $[\bar{v}] \notin C$, then there is a homogenous polynomial $H \in J$, such that $H([\bar{v}]) \neq 0$.

Now let $j \in \{0, \ldots, n\}$ and suppose that $[\bar{v}] \in U_j$.

We then have $[\bar{v}] \notin C \cap U_j$.

Since $u_j^{-1}(C)$ is the zero set of an ideal in $k[x_0, \ldots \check{x}_j, \ldots, x_n]$, there is a polynomial

$$P(x_0, \ldots, \check{x}_j, \ldots x_n) \in k[x_0, \ldots \check{x}_j, \ldots, x_n]$$

such that $P(u_j^{-1}([\bar{v}])) \neq 0$ and such that $P \in \mathcal{I}(u_j^{-1}(C))$.

Let

$$\beta_j(P) := x_j^{\deg(P_j)} P(\frac{x_0}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j} \ldots, \frac{x_n}{x_j}).$$

This is a homogenous polynomial (the "homogenisation" of P with respect of the variable x_j) such that

$$(\beta_j(P))(x_0, \ldots, x_{j-1}, 1, x_j, \ldots, x_n) = P_j.$$

In particular we have $\mathrm{Z}(\beta_j(P)) \supseteq C \cap U_j$ and

$$(\beta_j(P))([\bar{v}]) = P(u_j^{-1}([\bar{v}])) \neq 0.$$

Now let $Q_j = x^j \beta_j(P)$. Then Q_j is still homogenous and we have $Q_j([\bar{v}]) \neq 0$ and $\mathrm{Z}(Q_j) \supseteq C$ (because x_j vanishes on $\mathbb{P}^n(k) \backslash U_j$).

Hence we may set $H = Q_j$.

This completes the proof. □

If $A \subseteq \mathbb{P}^n(k)$ is a subset, we shall write

$$\mathcal{I}(A) \subseteq k[x_0, \ldots, x_n]$$

for the ideal generated by the homogenous polynomials vanishing on A.

This notation clashes with the notation in the affine case but the context should make it clear which definition of $\mathcal{I}(\cdot)$ we use.

Now we have the analogue of Proposition 1.3:

Proposition 1.33

Let $C \subseteq \mathbb{P}^n(k)$ be a closed subset and let $J \subseteq k[x_0, \ldots, x_n]$ be a homogenous radical ideal.

Suppose that $\mathrm{Z}(J) \neq \emptyset$.

Then $\mathcal{I}(C)$ is a (by definition homogenous) radical ideal and we have

$$\mathrm{Z}(\mathcal{I}(C)) = C$$

and

$$\mathcal{I}(\mathrm{Z}(J)) = J.$$

Proof. We first show that $\mathcal{I}(C)$ is a radical ideal.

To see this, let $H \subseteq \mathfrak{r}(\mathcal{I}(C))$ be the subset of $\mathfrak{r}(\mathcal{I}(C))$ consisting of the homogenous elements of $\mathfrak{r}(\mathcal{I}(C))$.

By the definition of the nilradical of an ideal, all the elements of H vanish on C.

On the other hand, $\mathfrak{r}(\mathcal{I}(C))$ is a homogenous ideal by Lemma 1.31 and so H generates $\mathfrak{r}(\mathcal{I}(C))$.

Hence $\mathfrak{r}(\mathcal{I}(C)) \subseteq \mathcal{I}(C)$ and thus $\mathfrak{r}(\mathcal{I}(C)) = \mathcal{I}(C)$.

The equality $\mathrm{Z}(\mathcal{I}(C)) = C$ is contained in Proposition 1.32.

For the second equality, note first that the inclusion $J \subseteq \mathcal{I}(\mathrm{Z}(J))$ follows from the definitions.

We thus only have to prove that $J \supseteq \mathcal{I}(\mathrm{Z}(J))$.

So let Q be a non zero homogenous polynomial vanishing on $\mathrm{Z}(J)$.

We need to show that $Q \in J$.

Note that $\deg(Q) > 0$. Indeed, if $\deg(Q) = 0$ then Q is a non zero constant polynomial and then $\mathrm{Z}(Q) = \emptyset$, which implies that $\mathrm{Z}(J) = \emptyset$. More generally, J does not contain any constant polynomial.

Now consider the map

$$q : k^{n+1} \backslash \{0\} \to \mathbb{P}^n(k)$$

given by the formula $q(\bar{v}) := [\bar{v}]$.

Note that $q^{-1}(\mathrm{Z}(J))$ is by construction the set of zeroes of J in $k^{n+1} \backslash \{0\}$.

Hence the set of zeroes of J in k^n is the set $q^{-1}(\mathrm{Z}(J)) \cup \{0\}$.

Now Q also vanishes on $q^{-1}(\mathrm{Z}(J)) \cup \{0\}$ and so by the strong Nullstellensatz we have $Q \in \mathfrak{r}(J) = J$. $\square$

Lemma 1.34

Let $J \subseteq k[x_0, \ldots, x_n]$ be a homogenous radical ideal.

Then the subset $\mathrm{Z}(J)$ of $\mathbb{P}^n(k)$ is empty iff

$$J = k[x_0, \ldots, x_n]$$

or

$$J = k[x_0, \ldots, x_n]_+.$$

Here $k[x_0, \ldots, x_n]_+$ is the homogenous ideal of $k[x_0, \ldots, x_n]$ generated by all the non constant homogenous polynomials.

Proof. We first prove the $\Leftarrow$ direction of the equivalence.

So let $\bar{v} = \langle v_1, \ldots, v_n \rangle \in k^{n+1} \backslash \{0\}$.

Suppose that $v_{i_0} \neq 0$ for some $i_0 \in \{0, \ldots, n\}$.

The homogenous polynomial $x_{i_0} \in k[x_0, \ldots, x_n]_+$ does not vanish at $[\bar{v}]$.

Since $\bar{v} \in k^{n+1} \backslash \{0\}$ was arbitrary, we see that $\mathrm{Z}(J)$ is empty if $J = k[x_0, \ldots, x_n]_+$ or $J = k[x_0, \ldots, x_n]$.

We now prove the $\Rightarrow$ direction.

So suppose that $\mathrm{Z}(J) = \emptyset$.

To avoid notational confusion, write $Z_{\mathrm{aff}}(I)$ for the set of common zeroes in k^{n+1} of the elements of a (not necessarily homogenous) ideal $I \subseteq k[x_0, \ldots, x_n]$.

By using the map $q : k^{n+1} \backslash \{0\} \to \mathbb{P}^n(k)$ described in the proof of Proposition 1.33, we see that

$$Z_{\mathrm{aff}}(J) \cap (k^{n+1} \backslash \{0\}) = \emptyset.$$

Now suppose first that J does not contain any non zero constant polynomials.

Then $0 \in Z_{\mathrm{aff}}(J)$ (because J is generated by non constant homogenous polynomials) so that $Z_{\mathrm{aff}}(J) = \{0\}$.

Using the correspondence described after Proposition 1.3, we conclude that J is the radical ideal of $k[x_0, \dots, x_n]$ associated with the point 0, which is $k[x_0, \dots, x_n]_+$.

If J contains a non zero constant polynomial then $J = k[x_0, \dots, x_n]$ (because J contains a unit).

So we conclude that if $\mathrm{Z}(J) = \emptyset$ then either $J = k[x_0, \dots, x_n]_+$ or $J = k[x_0, \dots, x_n]$. $\square$

We shall call the ideal $k[x_0, \dots, x_n]_+$ the *irrelevant* ideal of $k[x_0, \dots, x_n]$.

We conclude from Lemma 1.34 and Proposition 1.33 that there is a correspondence

$$\{\text{closed sets in } \mathbb{P}^n(k)\} \underset{Z}{\overset{\mathcal{I}}{\rightleftarrows}} \{\text{non irrelevant homogenous radical ideals in } R_n\}$$

where the maps $\mathrm{Z}(\cdot)$ and $\mathcal{I}(\cdot)$ are inverse to each other.

A *projective variety* is a variety isomorphic (as a variety) to a closed subvariety of $\mathbb{P}^n(k)$ (for some $n \geqslant 0$).

A *quasi-projective variety* is a variety isomorphic to an open subvariety of a projective variety.

Dimension

Let T be a topological space.

The space T is said to be *noetherian* if for any descending sequence

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$$

of closed subsets of T, there is an $i_0 \geqslant 0$ such that $C_{i_0} = C_{i_0+1} = \dots$.

In this situation, we say that the sequence stabilises at i_0.

Note that any subset of a noetherian topological space is also noetherian (in the induced topology).

Finally, note that a noetherian topological space is quasi-compact (ie any covering of the space has a finite subcovering). See exercises.

The topological space T is said to be *irreducible* if T is not empty and any open subset of T is dense in T.

Example.

The Zariski topology on k^n is noetherian.

Indeed any descending sequence

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$$

of closed subsets of k^n corresponds uniquely to a sequence

$$\mathcal{I}(C_1) \subseteq \mathcal{I}(C_2) \subseteq \mathcal{I}(C_3) \subseteq \dots$$

(see the first section) and such a sequence stabilises for some index because $k[x_1, \dots, x_n]$ is a noetherian ring (by Hilbert's basis theorem).

Consequently, the topology of any algebraic set is noetherian.

A closed subspace Z of k^n is irreducible iff Z is irreducible as an algebraic set.

Lemma 1.35

Let T be a non empty noetherian topological space.

Then there is a unique finite collection $\{T_i\}$ of irreducible closed subsets of T such that

(1) $T = \cup_i T_i$

(2) $T_i \not\subseteq \cup_{j \neq i} T_j$ *for all* i.

Note that a consequence of the lemma is that the T_i are the irreducible closed subsets of T which are maximal for the relation of inclusion among all the irreducible closed subsets contained in T.

Proof. See exercises. ☐

The closed subsets T_i described in Lemma 1.35 are called the *irreducible components* of T.

If T is an algebraic set, the decomposition of T into irreducible components coincides with the decomposition given by Lemma 1.6.

Lemma 1.36

A variety is noetherian.

Proof. Let V be a variety. Let

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$$

be a descending sequence of closed subsets of V.

Let $\{U_i\}$ be a finite covering of V by open affine subvarieties.

Since the U_i are noetherian (as topological spaces) by the remark above and since there is only finitely many U_i, there is an integer $l \geqslant 1$ such that $C_l \cap U_i = C_{l+1} \cap U_i = \dots$ for all i.

Since the U_i cover V, this implies that $C_l = C_{l+1} = \dots$ □

Now consider again a non empty topological space T.

The *dimension* $\dim(T)$ of T is

$$\dim(T) := \sup\{t \mid \text{there are irreducible closed subsets}$$

$$C_0, \dots, C_t \subseteq T \text{ such that } C_0 \subsetneq C_1 \subsetneq \cdots \subsetneq C_t\}.$$

Note that $\dim(T)$ might be infinite.

Dimension is not defined for the empty topological space (note that some authors define the dimension of the empty topological space to be -1).

Lemma 1.37

Let $V \subseteq k^n$ be an algebraic set.

Then $\dim(V) = \dim(\mathcal{C}(V))$.

Here $\dim(\mathcal{C}(V))$ is the dimension of $\mathcal{C}(V)$ as a ring (see Def. 11.1 in CA).

Recall that by definition we have

$$\dim(R) := \sup\{n \mid \exists\, \mathfrak{p}_0, \ldots, \mathfrak{p}_n \in \mathrm{Spec}(R) : \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n\}$$

for any ring R.

Proof. We have already seen that irreducible closed subsets of V correspond to prime ideals of $\mathcal{C}(V)$ (see Lemma 1.5).

Hence the definition of $\dim(\mathcal{C}(V))$ corresponds with the definition of $\dim(V)$ under the correspondence between radical ideals of $\mathcal{C}(V)$ and closed subsets of V described at the beginning of section one. $\square$

Theorem 1.38

(1) *The dimension of k^n is n.*

(2) *The dimension of $\mathbb{P}^n(k)$ is n.*

Proof. (1) We saw in CA that $\dim(k[x_1, \ldots, x_n]) = n$ (see Cor. 11.27 in CA). Hence $\dim(k^n) = n$ by Lemma 1.37.

(2) Apply Q2.7 in exercise sheet 2 to the open covering of $\mathbb{P}^n(k)$ by its standard coordinate charts and use (1). ☐

Definition 1.39

Let T be a topological space.

Let $C \subseteq T$ be a closed irreducible subspace.

The codimension, or height of C is

$$\mathrm{cod}(C, T) = \mathrm{ht}(C, T) := \sup\{t \mid \text{there are irreducible closed subsets}$$

$$C_1, \ldots, C_t \subseteq T \text{ such that } C \subsetneq C_1 \subsetneq \cdots \subsetneq C_t\}$$

We shall sometimes write $\mathrm{cod}(C)$ and $\mathrm{ht}(C)$ instead of $\mathrm{cod}(C, T)$ and $\mathrm{ht}(C, T)$, respectively, when the ambient topological space T is clear from the context.

Note that from the definitions, we have

$$\dim(T) = \sup_{C \text{ closed irreducible subset of } T} \mathrm{ht}(C, T).$$

Suppose that $C, V \subseteq k^n$ are algebraic sets in k^n and that $C \subseteq V$.

Suppose that C is irreducible. Then the height of C in V is the height of the prime ideal $\mathcal{I}(C) \pmod{\mathcal{I}(V)}$ of $\mathcal{C}(V)$ (in the sense of section 11 of CA).

Proposition 1.40

Let V be a variety.

Let $C \subseteq V$ be an irreducible closed subset.

Then $\dim(V)$ and $\mathrm{cod}(C, V)$ are finite.

Proof. See Q6 (4) in Sheet 2. □

Finally, we also have the following difficult result of commutative algebra, which justifies the use of the word "codimension".

Theorem 1.41

Let R be a finitely generated k-algebra.

Suppose that R is an integral domain.

Let $\mathfrak{p} \subseteq R$ be a prime ideal.

Then we have

$$\mathrm{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R)$$

The proof of this theorem is given in the Appendix to the notes.

Corollary 1.42

Let V be an irreducible variety. Let $C \subseteq V$ be an irreducible closed subset. Then

$$\mathrm{cod}(C, V) + \dim(C) = \dim(V)$$

Proof. See notes. ☐

The next result is another fundamental result from the CA course, which is relevant to the theory of dimension.

Theorem 1.43

Let $n \geqslant 0$ *and let* $V, W \subseteq k^n$ *be algebraic sets.*

Suppose that $V \subseteq W$. *Suppose that* $I \subseteq k[x_1, \ldots, x_n]$ *is such that* $\mathrm{Z}(I) = V$.

Let $l \geqslant 1$ *and suppose that the ideal* $I \pmod{\mathcal{I}(W)} \subseteq \mathcal{C}(W)$ *is generated by* l *elements.*

Then every irreducible component of V *has codimension* $\leqslant l$ *in* W.

Furthermore, if C *is an irreducible component of* V *then there is an ideal* $J \subseteq \mathcal{I}(C) \subseteq \mathcal{C}(W)$ *which is generated by* $\mathrm{cod}(C, W)$ *elements and such that* C *is an irreducible component of* $\mathrm{Z}(J) \subseteq W$.

See Cor. 11.15 and Cor. 11.17 in CA for the proof. This is a consequence of Krull's principal ideal theorem.

Rational maps

Let V, W be varieties.

Consider the set $H = H_{V,W}$ whose elements are morphisms $f : U \to W$, where U is a non empty open subvariety of V.

Let $\sim \; = \; \sim_{V,W}$ be the relation on H, such that $f : U \to W$ and $g : O \to W$ are related by $\sim$ iff there is a open subvariety UO of $U \cap O$, which is dense in V and which is such that $f|_{UO} = g|_{UO}$.

The relation $\sim$ is easily seen to be an equivalence relation.

We shall write $\mathrm{Rat}(V, W)$ for the set of equivalences classes of H under the relation $\sim$.

We call elements of $\mathrm{Rat}(V, W)$ *rational maps* from V to W.

Beware that rational maps are not actual maps but equivalence classes of maps.

Suppose now until further notice that V is irreducible.

Note the following.

Let $f : U \to W$ be a representative of a rational map from V to W.

If f is dominant, then any other representative of the same rational map is dominant as well.

Indeed, let $g : O \to W$ be another representative of the rational map defined by f. Then

$$f|_{UO} = g|_{UO}.$$

Suppose for contradiction that g is not dominant. Then $W \backslash g(O)$ contains a non empty open subset W_1.

Since $f : U \to W$ is dominant, we know that $f^{-1}(W_1) \neq \emptyset$.

Thus, since V is irreducible, we have

$$f^{-1}(W_1) \cap UO = g^{-1}(W_1) \cap UO \neq \emptyset.$$

In particular $g^{-1}(W \backslash g(O)) \neq \emptyset$, which is a contradiction. So g is also dominant.

It thus makes sense to speak of a *dominant rational map* from V to W.

We shall write $\mathrm{Rat}_{\mathrm{dom}}(V, W)$ for the set of dominant rational maps from V to W.

We shall write $\kappa(V)$ as a shorthand for $\mathrm{Rat}(V, k)$.

If $f : U \to k$ and $g : O \to k$ are two elements of $H_{V,k}$, one may define a new element $f + g : U \cap O \to k$ of $H_{V,k}$ by declaring that

$$(f + g)(u) = f(u) + g(u)$$

for all $u \in U \cap O$.

Similarly, one may define an element $fg = f \cdot g : U \cap O \to k$ by declaring that

$$(f \cdot g)(u) = f(u) \cdot g(u)$$

for all $u \in U \cap O$.

Finally, if $f : U \to k$ does not vanish on all of U, then we may define $f^{-1} : U \backslash \mathrm{Z}(f) \to k$ by the formula $f^{-1}(u) = 1/f(u)$.

It is easily verified that these operations are compatible with $\sim_{V,k}$ and we thus obtain a structure of field on $\kappa(V)$.

This field is called the *function field* of V.

There is an obvious injection $k \hookrightarrow \kappa(V)$ which makes $\kappa(V)$ into a k-algebra.

Note finally that for any $v \in V$, there is a natural injection

$$\mathcal{O}_{V,v} \hookrightarrow \kappa(V),$$

which sends any representative of an equivalence class in $\mathcal{O}_{V,v}$ to its equivalence class in $\kappa(V)$.

So $\kappa(V)$ naturally contains the local rings at all the points of V.

Now suppose that we are given a dominant morphism of irreducible varieties $a : V \to W$.

Then we may define a map $H_{W,k} \to H_{V,k}$ by the recipe

$$(f : O \to k) \mapsto (f \circ a|_{f^{-1}(O)} : f^{-1}(O) \to k)$$

where O is a non empty open subvariety of W and $f : O \to k$ is an element of $H_{W,k}$.

This definition makes sense because $f^{-1}(O) \neq \emptyset$ as f is dominant.

One checks that this map is compatible with the relations $\sim_{W,k}$ and $\sim_{V,k}$ and also with the operations $+$, $(\cdot)^{-1}$ and $\cdot$.

One thus obtains a map of rings

$$a^{*,\mathrm{rat}} : \kappa(W) \to \kappa(V).$$

Note that since $\kappa(W)$ is a field, the map $a^{*,\mathrm{rat}}$ is injective.

Also, if $a : V \to W$ is the inclusion of an open subvariety of V into W, the map $a^{*,\mathrm{rat}}$ is a bijection.

The construction of $a^{*,\mathrm{rat}}$ is compatible with compositions of dominant morphisms.

We conclude from all this that the homomorphism $a^{*,\mathrm{rat}}$ only depends on the element of $\mathrm{Rat}(V, W)$ defined by a.

In turn, any dominant representative $g : O \to W$ of an element of $\mathrm{Rat}(V, W)$ defines a map of k-algebras

$$g^{*,\mathrm{rat}} : \kappa(W) \to \kappa(V) \simeq \kappa(O)$$

and again this map only depends on the class of g in $\mathrm{Rat}(V, W)$.

So any dominant rational map $\rho \in \mathrm{Rat}_{\mathrm{dom}}(V, W)$ gives rise to an injection of fields

$$\rho^{*,\mathrm{rat}} : \kappa(W) \to \kappa(V).$$

Lemma 1.44

Let X be an irreducible affine variety.

Let $V \subseteq k^n$ be an algebraic set giving rise to X.

Then there is a canonical isomorphism of k-algebras $\kappa(X) \to \mathrm{Frac}(\mathcal{C}(V))$.

This isomorphism is compatible with dominant regular maps between irreducible algebraic sets and the corresponding morphisms of varieties.

Note that by Sheet 2, the fact that V irreducible implies that the ring $\mathcal{C}(V)$ is an integral domain. So it makes sense to talk about the fraction field $\mathrm{Frac}(\mathcal{C}(V))$ of $\mathcal{C}(V)$.

Proof. The proof is similar to the proof of Lemma 1.24 and will be omitted. $\square$

Proposition 1.45

Let V be an irreducible variety.

Then $\kappa(V)$ is finitely generated over k as a field and the dimension of V is equal to the transcendence degree of $\kappa(V)$ over k.

Recall that the transcendence degree of $\kappa(V)$ over k is the largest integer $n \geqslant 0$ such that there exists an injection of k-algebras

$$k[x_1, \ldots, x_n] \hookrightarrow \kappa(V)$$

See section 11.1 of CA for details.

Proof. Let $\{V_i\}$ be a finite open covering of V and suppose that each V_i is an affine variety.

The function field of V_i is isomorphic to the function field of V as a k-algebra.

On the other hand, we have $\dim(V) = \sup_i \dim(V_i))$ by sheet 2.

Hence it is sufficient to show that the transcendence degree of $\kappa(V_i)$ over k is equal to $\dim(V_i)$ for all i.

So we may suppose without restriction of generality that V is affine. In that case, the statement is a consequence of Lemma 1.37, Lemma 1.44 and Cor. 11.28 in CA. □

Proposition 1.46

Let $a : V \to W$ *be a dominant morphism of irreducible subvarieties. Then* $a^{*,\mathrm{rat}} : \kappa(W) \to \kappa(V)$ *is an isomorphism iff there exist open subvarieties*

$$V_0 \subseteq V$$

and

$$W_0 \subseteq W$$

such that $a(V_0) \subseteq W_0$ *and such that the induced morphism*

$$a|_{V_0} : V_0 \to W_0$$

is an isomorphism.

Proof. The $\Leftarrow$ direction of the equivalence is clear so we only have to establish the $\Rightarrow$ direction.

Let $W_{00} \subseteq W$ be an open affine subvariety and let V_{00} be an open affine subvariety of $a^{-1}(W_0)$.

We claim that the map $V_{00} \to W_{00}$ induced by a is also dominant.

To prove this claim, suppose for contradiction that the map $V_{00} \to W_{00}$ is not dominant.

Then there is a non empty subset O of W_{00} such that $O \subseteq W_{00} \backslash a(V_{00})$. Hence $a^{-1}(O) \cap V_{00} = \emptyset$.

Now $a^{-1}(O) \neq \emptyset$ since a is dominant, so this contradicts the irreducibility of V.

We have thus established the claim.

Since the inclusions $V_{00} \to V$ and $W_{00} \to W$ induce isomorphisms of function fields, we may thus assume without restriction of generality that V and W are affine to begin with.

In view of Lemma 1.44 and sheet 2, it is thus sufficient to prove the following statement of commutative algebra.

Let $\phi : A \to B$ be a homomorphism of finitely generated integral k-algebras.

Suppose that $\mathrm{Spm}(\phi)(\mathrm{Spm}(B))$ is dense in $\mathrm{Spm}(A)$ and suppose that the induced map

$$\mathrm{Frac}(\phi) : \mathrm{Frac}(A) \to \mathrm{Frac}(B)$$

is an isomorphism.

Then there is an element $f \in A$ such that the induced map

$$A[f^{-1}] \to B[\phi(f)^{-1}]$$

is an isomorphism.

To prove this assertion, note that by Sheet 1 we already know that under the given assumptions, ϕ must be injective.

Note also that since we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Frac}(A) \arrow[r, "\mathrm{Frac}(\phi)"] & \mathrm{Frac}(B) \\
A \arrow[u] \arrow[r, "\phi"] & B \arrow[u]
\end{tikzcd}$$

all whose maps are injective, the induced map $A[f^{-1}] \to B[\phi(f)^{-1}]$ is injective for any choice of $f \in A \backslash \{0\}$.

Thus we only have to show that there is $f \in A \backslash \{0\}$ such that the induced map $A[f^{-1}] \to B[\phi(f)^{-1}]$ is surjective.

Now let $b_1, \ldots, b_l$ be generators of B as a k-algebra.

Let $\frac{a_1}{c_1}, \ldots, \frac{a_l}{c_l} \in \mathrm{Frac}(A)$ such that

$$\frac{b_i}{1} = \frac{\phi(a_i)}{\phi(c_i)} =: \mathrm{Frac}(\phi)(\frac{a_i}{c_i})$$

for all $i \in \{1, \ldots, l\}$.

Let $f := \prod_i c_i$.

Then $\frac{b_i}{1} = \mathrm{Frac}(\phi)(a_i \frac{\prod_{j \neq i} c_j)}{f})$.

Hence the image of

$$A[f^{-1}] \to B[\phi(f)^{-1}]$$

contains $\frac{b_i}{1}$ for all $i \in \{1, \ldots, l\}$ and also contains

$$\frac{1}{\phi(f)} = \mathrm{Frac}(\phi)(\frac{1}{f}).$$

Since $B[\phi(f)^{-1}]$ is generated as a k-algebra by $\frac{1}{\phi(f)}$ and by the elements $\frac{b_i}{1}$ (use Lemma 5.3 in CA), we see that $A[f^{-1}] \to B[\phi(f)^{-1}]$ is surjective. $\square$

If V and W are irreducible varieties, and $V_0 \subseteq V$ and $W_0 \subseteq W$ are open subvarieties such that $V_0 \simeq W_0$, we shall say that V and W are *birational*, or *birationally isomorphic*.

A *birational map* from V to W is a rational map from V to W which has a representative $f : O \to W$, such that $f(O)$ is open and such that the induced map $O \to f(O)$ is an isomorphism.

A *birational morphism* from V to W is a morphism $V \to W$ which induces a birational map.

Proposition 1.46 implies that a dominant rational map $\rho \in \mathrm{Rat}_{\mathrm{dom}}(V, W)$ is birational iff $a^{*,\mathrm{rat}} : \kappa(W) \to \kappa(V)$ is bijective.

Proposition 1.47

Let V, W be irreducible varieties.

Let $\kappa(W) \hookrightarrow \kappa(V)$ be a field extension compatible with the k-algebra structures.

Then there is an open subvariety V_0 of V and a dominant morphism

$$a : V_0 \to W$$

such that the extension

$$a^{*,\mathrm{rat}} : \kappa(W) \to \kappa(V_0)$$

is isomorphic to

$$\kappa(W) \hookrightarrow \kappa(V)$$

as a $\kappa(W)$-extension.

Proof. We may suppose without restriction of generality that V and W are affine varieties.

Let B (resp. A) be the coordinate ring of V (resp. W).

Let $\iota : \mathrm{Frac}(A) \simeq \kappa(W) \hookrightarrow \kappa(V) \simeq \mathrm{Frac}(B)$ be the given field extension.

We <u>claim</u> that there is an $g \in B \backslash \{0\}$ such that

$$\iota(A) \subseteq B[g^{-1}] \subseteq \mathrm{Frac}(B).$$

To prove this, let $a_1, \ldots, a_l$ be generators of A as a k-algebra.

For all $i \in \{1, \ldots, l\}$ let $b_i, c_i \in B$ be such that $b_i / c_i = \iota(a_i/1)$.

Let $g := \prod_i c_i$.

We then have $\iota(a_i/1) \in B[g^{-1}]$ and thus $\iota(A) \subseteq B[g^{-1}]$, proving the claim.

Now let V_0 be the open affine subvariety associated with $B[g^{-1}]$.

Let

$$\iota_0 : A \to B[g^{-1}]$$

be the map induced by ι and the natural map from A to $\mathrm{Frac}(A)$.

Since the map ι_0 is injective, it induces a dominant map $V_0 \to W$ by Sheet 1.

Hence V_0 and the map $V_0 \to W$ satisfy the requirements of the proposition. ☐

Finally, note the following.

Let V and W be irreducible varieties.

Consider the map

$$\mathrm{Rat}_{\mathrm{dom}}(V, W) \to \text{homomorphisms of } k\text{-algebras } \kappa(W) \to \kappa(V) \quad (*)$$

which sends $a \in \mathrm{Rat}_{\mathrm{dom}}(V, W)$ to $a^{*,\mathrm{rat}} : \kappa(W) \to \kappa(V)$.

Proposition 1.47 implies that this map is surjective.

On the other hand we have

Lemma 1.48

The map $(*)$ is injective.

Proof. Let $a_1, a_2 \in \mathrm{Rat}_{\mathrm{dom}}(V, W)$ and suppose that $a_1^{*,\mathrm{rat}} = a_2^{*,\mathrm{rat}}$. We have to show that $a_1 = a_2$.

We may assume that both V and W are affine and that a_1 (resp. a_2) is represented by a morphism. Let $\alpha_1 : V \to W$ (resp. $\alpha_2 : V \to W$) a morphism representing a_1 (resp. a_2).

Now let B (resp. A) be the coordinate ring of V (resp. W).

Let

$$\iota : \mathrm{Frac}(A) \simeq \kappa(W) \hookrightarrow \kappa(V) \simeq \mathrm{Frac}(B)$$

be the field extension given by $a_1^{*,\mathrm{rat}} = a_2^{*,\mathrm{rat}}$. We have by construction a commutative diagram

$$\begin{tikzcd}
\mathrm{Frac}(A) \arrow[r, "\iota"] & \mathrm{Frac}(B) \\
A \arrow[u] \arrow[r, "\alpha_i^*"] & B \arrow[u]
\end{tikzcd}$$

for $i \in \{1, 2\}$. Since the vertical maps are injective and $a_1^{*,\mathrm{rat}} = a_2^{*,\mathrm{rat}}$, we thus have $\alpha_1^* = \alpha_2^*$. $\square$

In view of the last lemma and the comment preceding it, we thus see that

there is a one-to-one correspondence between dominant rational maps from V to W and $\kappa(W)$-algebra structures on the field $\kappa(V)$.

We shall from now on often write a^* for $a^{*,\mathrm{rat}}$ when V and W are irreducible varieties and $a \in \mathrm{Rat}(V, W)$.

This is justified by the proof of Lemma 1.48.

Products

We wish to endow the cartesian product of two varieties with the structure of a variety.

We shall do this for quasi-projective varieties.

Let V and W be varieties.

A *product* of V and W is a triple $(V \prod W, \pi_V, \pi_W)$, where $V \prod W$ is a variety and

$$\pi_V : V \prod W \to V$$

and

$$\pi_W : V \prod W \to W$$

are morphisms of varieties.

This triple is required to have the following property $\mathrm{(PROD)}$.

$\mathrm{(PROD)}$ If X is a variety and $a : X \to V$ and $b : X \to W$ are morphisms of varieties, then there is a unique morphism of varieties

$$a \prod b : X \to V \prod W$$

Note that property (PROD) characterises the triple $(V \prod W, \pi_V, \pi_W)$ uniquely up to unique isomorphism of triples.

This is an example of categorical product.

Note that if V and W are varieties, it is not clear *a priori* that they have a product.

However, if the product of V and W exists, it is uniquely defined.

Abusing language, we shall often say that $V \prod W$ is the product of V and W without writing the associated morphisms π_V and π_W.

Theorem 1.49

Let $m, n \geqslant 0$. The product $\mathbb{P}^m(k) \prod \mathbb{P}^n(k)$ exists.

Before starting with the proof, we make a construction.

We shall consider the projective space $\mathbb{P}^{mn+m+n}$. This is by definition the set of lines generated by non zero vectors in

$$k^{(mn+m+n)+1=(m+1)(n+1)}.$$

We choose a basis b_{ij} for $k^{(m+1)(n+1)}$ where

$$i \in \{0, \ldots, m\}$$

and

$$j \in \{0, \ldots, n\}.$$

Let $\sigma : \mathbb{P}^m(k) \times \mathbb{P}^n(k) \to \mathbb{P}^{mn+m+n}$ be the map given by the formula

$$\sigma(([X_0, \ldots, X_m], [Y_0, \ldots Y_n])) = [(X_i Y_j)_{ij}]$$

where $(\cdot)_{ij}$ means that we put $(\cdot)$ in the coordinate ij corresponding to b_{ij}.

We will write Z_{ij} for a variable quantity in the coordinate ij. We will write z_{ij} for the homogenous variables of $\mathbb{P}^{mn+m+n}$.

Lemma 1.50

The map σ is injective and $\sigma(\mathbb{P}^m(k))$ is the closed subvariety of $\mathbb{P}^{mn+m+n}$ given by the quadratic equations $z_{ij}z_{rs} = z_{is}z_{rj}$.

Proof. See the notes. ☐

The map σ is called the *Segre embedding*.

Its image is called the *Segre variety*.

Proof. (of Theorem 1.49). Endow $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$ with the variety structure inherited from the Segre variety via the Segre embedding.

We will show that the variety $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$, together with the natural projections to the two factors, is a product.

We first show that the projections

$$\pi_1 : \mathbb{P}^m(k) \times \mathbb{P}^n(k) \to \mathbb{P}^m(k)$$

and

$$\pi_2 : \mathbb{P}^m(k) \times \mathbb{P}^n(k) \to \mathbb{P}^n(k)$$

are morphisms of varieties.

For any $i_0 \in \{0, \dots, m\}$ and any $j_0 \in \{0, \dots, n\}$, let $U_{i_0 j_0} \subseteq \mathbb{P}^{mn+m+n}$ be the open subset of the elements $[Z_{ij}]$ such that $Z_{i_0 j_0} \neq 0$.

Let $\pi_{i_0 j_0, 1} : U_{i_0 j_0} \to \mathbb{P}^m(k)$ be given by the formula

$$\pi_{i_0 j_0, 1}([Z_{ij}]) := [Z_{0 j_0}, Z_{1 j_0}, \dots, Z_{m j_0}]$$

By Sheet 2, this defines a morphism from $U_{i_0 j_0}$ to $\mathbb{P}^m(k)$.

Now suppose that

$$\sigma(([X_0,\ldots,X_m],[Y_0,\ldots Y_n])) = [(X_iY_j)_{ij}] \in U_{i_0j_0}$$

In other words, $X_{i_0}, Y_{j_0} \neq 0$.

Then

$$\begin{aligned} & \pi_{i_0j_0,1}(\sigma(([X_0,\ldots,X_m],[Y_0,\ldots Y_n]))) \\ = \ & \pi_{i_0j_0,1}([(X_iY_j)_{ij}]) = [X_0Y_{j_0}, X_1Y_{j_0},\ldots,X_mY_{j_0}] \\ = \ & [X_0,X_1,\ldots,X_m] = \pi_1(([X_0,\ldots,X_m],[Y_0,\ldots Y_n])) \end{aligned}$$

Hence π_1 is a morphism on the open subset $\sigma^{-1}(U_{i_0j_0})$ of $\mathbb{P}^m(k)\times\mathbb{P}^n(k)$.

Now if we vary the indices i_0 and j_0, the open subsets $\sigma^{-1}(U_{i_0j_0})$ cover all of $\mathbb{P}^m(k)\times\mathbb{P}^n(k)$ and hence π_1 is a morphism.

Similarly π_2 is a morphism.

Choosing $\pi_{\mathbb{P}^m(k)} := \pi_1$ and $\pi_{\mathbb{P}^n(k)} := \pi_2$, we shall now verify (PROD).

So let X be a variety and $a : X \to \mathbb{P}^m(k)$ and $b : X \to \mathbb{P}^n(k)$ be morphisms of varieties.

We have to show that there is a unique morphism of varieties

$$c : X \to \mathbb{P}^m(k) \times \mathbb{P}^n(k)$$

such that $\pi_1 \circ c = a$ and $\pi_2 \circ c = b$.

Now note that the set $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$ is the cartesian product of the sets $\mathbb{P}^m(k)$ and $\mathbb{P}^n(k)$.

Hence, if the morphism c exists, it must be given by the formula

$$c(x) = (a(x), b(x))$$

for all $x \in X$.

Hence we only have to verify that c is a morphism of varieties.

Since by the definition of a Topskf, a morphism is a morphism iff it is every locally a morphism, we may assume that X is affine and that

$$a(X) \subseteq U_{\mathbb{P}^m(k),i_0}$$

and

$$b(X) \subseteq U_{\mathbb{P}^n(k),j_0}$$

for some indices i_0 and j_0.

So let us suppose that X is associated with an algebraic set $V \subseteq k^t$.

The map a is then the restriction to V of a map $k^t \to U_{\mathbb{P}^m(k),i_0}$ of the form

$$\bar{v} \in k^t \mapsto [P_0(\bar{v}), \ldots, P_{i_0-1}(\bar{v}), 1, P_{i_0+1}(\bar{v}), \ldots P_m(\bar{v})]$$

where the P_h are polynomials in the entries $v_1, \ldots, v_t$ of the vector $\bar{v}$.

Similarly, the map b is the restriction to V of a map $k^t \to U_{\mathbb{P}^n(k),j_0}$ of the form

$$\bar{v} \in k^t \mapsto [Q_0(\bar{v}), \ldots, Q_{j_0-1}(\bar{v}), 1, Q_{j_0+1}(\bar{v}), \ldots Q_n(\bar{v})]$$

where the P_l are polynomials in the entries $v_1, \ldots, v_t$ of the vector $\bar{v}$.

We now compute

$$\sigma(c(\bar{v})) = [(P_i(\bar{v})Q_j(\bar{v}))_{ij}]$$

and since $P_{i_0}(\bar{v})Q_{j_0}(\bar{v})) = 1$, we see that $\sigma \circ c$ factors through a morphism $V \to U_{i_0 j_0}$ and in particular is a morphism from V to $\mathbb{P}^{mn+m+n}$.

Applying Lemma 1.28, we conclude that the morphism c is a morphism of varieties. $\square$

In the proof above, we have shown that $\mathbb{P}^m(k) \prod \mathbb{P}^n(k)$ can be realised as the Cartesian product $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$ endowed with a certain variety structure.

Furthermore, the projections $\pi_{\mathbb{P}^m(k)}$ and $\pi_{\mathbb{P}^m(k)}$ are then simply the ordinary projections on the two factors.

We shall thus often write $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$ instead of $\mathbb{P}^m(k) \prod \mathbb{P}^n(k)$.

Lemma 1.51

Let $C_1 \subseteq \mathbb{P}^m(k)$ (resp. $V_1 \subseteq \mathbb{P}^m(k)$) and $C_2 \subseteq \mathbb{P}^n(k)$ (resp. $V_2 \subseteq \mathbb{P}^n(k)$) be closed (resp. open) subsets.

Then the Cartesian product $C_1 \times C_2$ is closed in $\mathbb{P}^m(k) \prod \mathbb{P}^n(k)$ and the Cartesian product $V_1 \times V_2$ is open in $\mathbb{P}^m(k) \prod \mathbb{P}^n(k)$.

Proof. Note that the second statement is a consequence of the first, because the complement of $V_1 \times V_2$ is

$$(\mathbb{P}^m(k) \backslash V_1) \times \mathbb{P}^n(k) \cup \mathbb{P}^m(k) \times (\mathbb{P}^n(k) \backslash V_2),$$

which is closed according to the first statement.

For the proof of the second statement, suppose that C_1 (resp. C_2) is defined by homogenous polynomials $P_1(x_0, \ldots, x_m), \ldots, P_a(x_0, \ldots, x_m)$ (resp. $Q_1(y_0, \ldots, y_n), \ldots, Q_b(y_0, \ldots, y_n)$). Then we have

$$\sigma(C_1 \times C_2) = \bigcap_{i=0,\ldots,m \; j=0,\ldots,n} \mathrm{Z}\Big(P_1(z_{0j}, \ldots, z_{mj}), \ldots, P_a(z_{0j}, \ldots, z_{mj}),$$

$$Q_1(z_{i0}, \ldots, z_{in}), \ldots, Q_a(z_{i0}, \ldots, z_{in})\Big) \bigcap \sigma(\mathbb{P}^m(k) \times \mathbb{P}^n(k))$$

and thus $C_1 \times C_2$ is closed in $\mathbb{P}^m(k) \prod \mathbb{P}^n(k)$. $\square$

Corollary 1.52

Let V and W be two quasi-projective varieties.

Then the product $V \prod W$ exists.

Proof. By assumption, there are integers $m, n \geqslant 0$ and open subvarieties $O_1 \subseteq \mathbb{P}^m(k)$ and $O_2 \subseteq \mathbb{P}^m(k)$ such that V is isomorphic to a closed subvariety of O_1 and W is isomorphic to a closed subvariety of O_2.

Let $C_1 \subseteq \mathbb{P}^m(k)$ and $C_2 \subseteq \mathbb{P}^n(k)$ be closed subsets such that $C_1 \cap O_1 = V$ and $C_2 \cap O_2 = W$.

We then have

$$V \times W = (C_1 \times C_2) \cap (O_1 \times O_2)$$

and hence $V \times W$ is closed in the open set $O_1 \times O_2$ by Lemma 1.51.

We endow the set $V \times W$ with the structure of variety which comes from its inclusion into $O_1 \times O_2$ as a closed subset.

We now claim that $V \times W$, together with the projections on the two factors, is a product of V and W.

To see this, let X be a variety and let $a : X \to V$, $b : X \to W$ be two morphisms of varieties.

Since the set $V \times W$ is the Cartesian product of V and W, we see as before that if the morphism $a \prod b$ exists, it must be given by the unique map

$$a \times b : X \to V \times W$$

sending $x \in X$ to $(a(x), b(x))$.

So we only have to verify that this map is a morphism. But this follows from Theorem 1.49. $\square$

An outcome of the proof of Corollary 1.52 is the following.

Let $m, n \geqslant 0$ and let $O_1 \subseteq \mathbb{P}^m(k)$ and $O_2 \subseteq \mathbb{P}^n(k)$ be open subvarieties.

Suppose that V is a closed subvariety of O_1 and that W is a closed subvariety of O_2.

Then $O_1 \times O_2$ is open in $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$, the Cartesian product $V \times W$ is closed in $O_1 \times O_2$ and the product of V and W is the set $V \times W$ endowed with the variety structure it inherits from $O_1 \times O_2$ as a closed subvariety.

The projections π_V and π_W are then the ordinary projections on the two factors.

Again, this justifies simply writing $V \times W$ instead of $V \prod W$.

Corollary 1.53

Let V_1, V_2 be quasi-projective varieties.

Let $C_1 \subseteq V_1$ and $C_2 \subseteq V_2$ be closed subsets.

Let $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ be open subsets.

Then the set theoretic product $C_1 \times C_2$ (resp. the set theoretic product $U_1 \times U_2$) is closed (resp. open) in $V \times W = V \prod W$.

If $C_1 \times C_2$ (resp. $U_1 \times U_2$) is endowed with its structure of closed (resp. open) subvariety of $V_1 \prod V_2$ and with the natural projection maps on the two factors, then $C_1 \times C_2$ (resp. $U_1 \times U_2$) is a product of C_1 and C_2 (resp. U_1 and U_2).

The next lemma is needed for the following proposition.

Lemma 1.54

Let $I \subseteq k[x_1, \ldots, x_n]$ (resp. $J \subseteq k[y_1, \ldots, y_t]$) be an ideal.
Let $\bar{I}$ (resp. $\bar{J}$) be the ideal generated by I (resp. J) in $k[x_1, \ldots, x_n, y_1, \ldots, y_t]$.
If I and J are radical (resp. prime) then $\bar{I} + \bar{J}$ is radical (resp. prime).

Proof. See the notes. This is an exercise in Commutative Algebra. ☐

Proposition 1.55

Let V and W be irreducible quasi-projective varieties.
Then $V \times W = V \prod W$ is also irreducible.

Proof. We first prove the result in the situation where V and W are affine. So suppose that $V \subseteq k^n$ and $W \subseteq k^t$ are algebraic sets in k^n and k^t, respectively.

By Sheet 3, we know that the subset $V \times W$ of $k^n \times k^t = k^{n+t}$ is an algebraic subset in k^{n+t} and is a product of V and W.

So we have to show that $V \times W$ is irreducible, when endowed with the topology induced from k^{n+t}.

Write $k[x_1, \ldots, x_n]$ for the coordinate ring of k^n and $k[y_1, \ldots, y_t]$ for the coordinate ring of k^t.

Let

$$\bar{\mathcal{I}}(V) = \mathcal{I}(V) \cdot k[x_1, \ldots, x_n, y_1, \ldots, y_t]$$

and

$$\bar{\mathcal{I}}(W) = \mathcal{I}(W) \cdot k[x_1, \ldots, x_n, y_1, \ldots, y_t].$$

By construction we have $\mathrm{Z}(\bar{\mathcal{I}}(V) + \bar{\mathcal{I}}(W)) = V \times W$.

Furthermore, by Lemma 1.54 the ideal $\bar{\mathcal{I}}(V) + \bar{\mathcal{I}}(W)$ is prime.

Hence $\mathcal{I}(V \times W) = \bar{\mathcal{I}}(V) + \bar{\mathcal{I}}(W)$ and thus $V \times W$ is irreducible.

Now suppose that V and W are quasi-projective.

Suppose for contradiction that $V \times W$ is not irreducible.

Let $T_1, \dots, T_l$ be the irreducible components of $V \times W$.

By assumption, we have $l \geqslant 2$. Let $(v_1, w_1) \in T_1$ and $(v_2, w_2) \in T_2$.

Let U_{v_1} be an open affine neighbourhood of v_1 in V and let U_{w_1} be an open affine neighbourhood of w_1 in W. Define U_{v_2} and U_{w_2} similarly.

Then we have $(v_1, w_1) \in U_{v_1} \times U_{w_1}$ and $(v_2, w_2) \in U_{v_2} \times U_{w_2}$.

Now from the first part and Lemma 1.53, we know that $U_{v_1} \times U_{w_1}$ and $U_{v_2} \times U_{w_2}$ are open irreducible subsets of $V \times W$.

Hence $U_{v_1} \times U_{w_1} \subseteq T_1$ and $U_{v_2} \times U_{w_2} \subseteq T_2$.

Also, we have $U_{v_1} \times U_{w_1} \cap U_{v_2} \times U_{w_2} = \emptyset$, for otherwise $T_1 \backslash (T_1 \cap T_2)$ is not dense in T_1.

However, since V and W are irreducible there is a point $z_v \in U_{v_1} \cap U_{v_2}$ and a point $z_w \in U_{w_1} \cap U_{w_2}$.

We have $(z_v, z_w) \in U_{v_1} \times U_{w_1} \cap U_{v_2} \times U_{w_2}$, which is a contradiction. So $V \times W$ is irreducible. □

Proposition 1.56

Let V and W be irreducible quasi-projective varieties. Then

$$\dim(V \times W) = \dim(V) + \dim(W).$$

Proof. Skipped. See the notes. This uses Noether's normalisation lemma to reduce the statement to the case $V = K^n$ and $W = k^t$. □

We end with the following important remark.

One can show that for any varieties V, W the product $V \prod W$ exists.

The proof uses different methods. It proceeds roughly as follows.

One covers V and W with open affine varieties V_i and W_j, respectively. It can be shown using commutative algebra that the products $V_i \prod W_j$ exist (see Sheet 3).

One then constructs the product $V \prod W$ by glueing the varieties $V_i \prod W_j$.

The above construction of the product of quasi-projective varieties bypasses the need for such a cumbersome glueing procedure.

Intersections in affine and projective space

The following proposition is the key to the proof of the *projective dimension theorem*, which follows it.

Proposition 1.57 (affine dimension theorem)

Let $n \geqslant 0$ and let $V, W \subseteq k^n$ be irreducible algebraic sets.

Then every irreducible component of $V \cap W$ has dimension $\geqslant \dim(V) + \dim(W) - n$.

Proof. Note that the Cartesian product $V \times W \subseteq k^{2n}$ is closed and is a product of V and W (see Sheet 3). Let

$$\Delta := \{(a_1, \ldots, a_n, a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in k\}$$

be the diagonal of k^{2n}. Note that we have

$$\Delta = \mathrm{Z}(x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$$

where we write $\mathcal{C}(k^{2n}) = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$. We have a k-algebra map

$$\phi : k[x_1, \ldots, x_n, y_1, \ldots, y_n]/(x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n) \to k[z_1, \ldots, z_n]$$

such that $\phi(x_i) = \phi(y_i) = z_i$ for all $i \in \{1, \ldots, n\}$. The map ϕ has an inverse given by the map

$$z_i \mapsto x_i \pmod{(x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)}.$$

In particular $\mathrm{Spm}(\phi) : k^n \to \Delta$ is an isomorphism of algebraic sets.

By construction, we have

$$\mathrm{Spm}(\phi)^{-1}(V \times W \cap \Delta) = V \cap W.$$

Thus we only have to prove that every irreducible component of $V \times W \cap \Delta$ has dimension $\geqslant \dim(V) + \dim(W) - n$.

Now by construction we have

$$V \times W \cap \Delta = \mathrm{Z}(x_1 - y_1) \cap \mathrm{Z}(x_2 - y_2) \cap \cdots \cap \mathrm{Z}(x_n - y_n) \cap V \times W.$$

Applying Theorem 1.43, we see that for any irreducible component C of $V \times W \cap \Delta$ we have

$$\mathrm{cod}(C, V \times W) \leqslant n$$

and by Corollary 1.42, Proposition 1.55 and Proposition 1.56, this translates as

$$\dim(V \times W) - \dim(C) = \dim(V) + \dim(W) - \dim(C) \leqslant n$$

which is equivalent to the conclusion of the proposition. $\square$

Proposition 1.58 (projective dimension theorem)

Let $n \geqslant 0$ *and let* $V, W \subseteq \mathbb{P}^n(k)$ *be closed irreducible subvarieties.*

Then every irreducible component of $V \cap W$ *has dimension* $\geqslant \dim(V) + \dim(W) - n$.

Furthermore, we have $V \cap W \neq \emptyset$ *if* $\dim(V) + \dim(W) - n \geqslant 0$.

Proof. We first prove the first assertion. Let C be an irreducible component of $V \cap W$.

Let U_i be a standard coordinate chart of $\mathbb{P}^n(k)$ such that $C \cap U_i \neq \emptyset$.

We claim that $C \cap U_i$ is an irreducible component of $(V \cap W) \cap U_i$.

To see this, note that since $C \cap U_i$ is irreducible, there is an irreducible component T of $(V \cap W) \cap U_i$, which contains $C \cap U_i$.

Write $\bar{T}$ for the closure of T in $V \cap W$.

Then $\bar{T}$ is also irreducible by Sheet 2 and hence $\bar{T} \subseteq C$.

On the other hand, by construction, we also have $\bar{T} \supseteq C$ so that $C = \bar{T}$.

Hence $T = \bar{T} \cap U_i = C \cap U_i$ so that $C \cap U_i$ is an irreducible component of $V \cap W$.

Now by Proposition 1.57, we have

$$\dim(C \cap U_i) \geqslant \dim(V \cap U_i) + \dim(W \cap U_i) - n$$

and by Proposition 1.45, we have $\dim(V \cap U_i) = \dim(V)$, $\dim(W \cap U_i) = \dim(W)$ and $\dim(C \cap U_i) = \dim(C)$.

This proves the first assertion.

For the second assertion, consider again the map $q : k^{n+1} \backslash \{0\} \to \mathbb{P}^n(k)$ such that $q(\bar{v}) = [\bar{v}]$ for all $\bar{v} \in k^{n+1} \backslash \{0\}$. Let

$$V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{\dim(V)} = V$$

be an ascending sequence of irreducible closed subsets of V, which is of maximal length.

The closed subvarieties $q^{-1}(V_i)$ of $k^{n+1}\backslash\{0\}$ are all irreducible by Sheet 3. Write $\overline{q^{-1}(V_i)}$ for the closure of $q^{-1}(V_i)$ in k^{n+1}.

The closed subsets $\overline{q^{-1}(V_i)}$ of k^{n+1} are then all irreducible by Sheet 3 and Sheet 2. We thus get an ascending sequence

$$\overline{q^{-1}(V_0)} \subsetneq \overline{q^{-1}(V_1)} \subsetneq \cdots \subsetneq \overline{q^{-1}(V_{\dim(V)})} = \overline{q^{-1}(V)}$$

of closed irreducible subsets of k^{n+1}.

Now note that by maximality the variety V_0 is a point.

We thus have

$$q^{-1}(V_0) = \{\lambda \bar{v}_0 \mid \lambda \in k\} \cap (k^{n+1}\backslash\{0\})$$

for some $\bar{v}_0 \in k^{n+1}\backslash\{0\}$.

We claim that the closure of $L^*_{\bar{v}_0} = \{\lambda \bar{v}_0 \,|\, \lambda \in k\} \cap (k^{n+1} \backslash \{0\})$ in k^{n+1} is $L_{\bar{v}_0} = \{\lambda \bar{v}_0 \,|\, \lambda \in k\}$.

To see this, note that $L_{\bar{v}_0}$ is closed in k^{n+1} and that there is an isomorphism$L_{\bar{v}_0} \simeq k$ sending $0 \in k^{n+1}$ to $0 \in k$. Since the closure of $k \backslash \{0\}$ in k is k, we see that the closure of $L^*_{\bar{v}_0}$ in k^{n+1} is $L_{\bar{v}_0}$.

We thus obtain an ascending sequence of irreducible closet subsets

$$\{0\} \subsetneqq \{\lambda \bar{v}_0 \,|\, \lambda \in k\} = \overline{q^{-1}(V_0)} \subsetneqq \overline{q^{-1}(V_1)} \subsetneqq \cdots \subsetneqq \overline{q^{-1}(V_{\dim(V)})} = \overline{q^{-1}(V)}$$

and we thus see that $\overline{q^{-1}(V)}$ has dimension $\geqslant \dim(V) + 1$.

Similarly, $\overline{q^{-1}(W)}$ is irreducible in k^{n+1} and has dimension $\geqslant \dim(W) + 1$.

We conclude from Proposition 1.57 that every irreducible component of $\overline{q^{-1}(V)} \cap \overline{q^{-1}(W)}$ has dimension larger or equal to

$$\begin{aligned} & \dim(\overline{q^{-1}(V)}) + \dim(\overline{q^{-1}(W)}) - (n+1) \\ \geqslant \; & \dim(V) + \dim(W) + 2 - (n+1) \\ = \; & \dim(V) + \dim(W) - n + 1. \end{aligned}$$

Hence, if $\dim(V) + \dim(W) - n \geqslant 0$ then every irreducible component of

$$\overline{q^{-1}(V)} \cap \overline{q^{-1}(W)}$$

has dimension $\geqslant 1$.

On the other hand, both $\overline{q^{-1}(V)}$ and $\overline{q^{-1}(W)}$ contain the point 0, so $\overline{q^{-1}(V)} \cap \overline{q^{-1}(W)}$ is not empty.

We conclude that $\overline{q^{-1}(V)} \cap \overline{q^{-1}(W)}$ contains points other than 0, or in other words that

$$q^{-1}(V) \cap q^{-1}(W) \neq \emptyset.$$

This implies that $V \cap W \neq \emptyset$. $\square$

Corollary 1.59

Let $n \geqslant 0$ *and let* $V \subseteq \mathbb{P}^n(k)$ *be a closed irreducible subset.*
Let H *be a closed irreducible subset such that* $\mathrm{cod}(H, \mathbb{P}^n(k)) = 1$*.*
If $\dim(V) \geqslant 1$ *then* $H \cap C \neq \emptyset$*.*

Proof. Clear. □

Separatedness and completeness

Separatedness is an algebraic analogue of the Hausdorff property in topology. Completeness is an algebraic analogue of the notion of compactness in topology.

If X is a quasi-projective variety. Write $\delta_X : X \to X \prod X$ for the map $\mathrm{Id}_X \prod \mathrm{Id}_X$.

We shall write $\Delta_X \subseteq X \prod X$ for the image of δ_X.

We call it the *diagonal* in $X \prod X$.

Definition 1.60

Let X be a quasi-projective variety. We say that X is separated if the diagonal in $X \prod X$ is closed.

Note that if Δ_X is closed in $X \prod X$ then δ_X induces an isomorphism between X and Δ_X, where Δ_X is seen as a closed subvariety of $X \prod X$.

Indeed, the map δ_X induces a morphism $X \to \Delta_X$ by Lemma 1.28 and this map has an inverse, given by the projection on the first factor.

To understand this definition, note that if T is a topological space and $T \times T$ is endowed with the product topology, then T is Hausdorff iff the diagonal $\Delta_T \subseteq T \times T$ is closed.

Indeed, let $a, b \in T$ and $a \neq b$. Then $(a, b) \notin \Delta_T$.

If Δ_T is closed then there are open subsets $U, V \subseteq T$ such that $U \times V \cap \Delta_T = \emptyset$ and such that $(a, b) \in U \times V$.

In particular, $a \in U$, $b \in V$ and $U \cap V = \emptyset$.

So a and b have disjoint neighbourhoods.

On the other hand, if a and b have disjoint neighbourhoods U and V, respectively, then $U \times V \cap \Delta_T = \emptyset$ and $(a, b) \in U \times V$.

So $(T \times T) \backslash \Delta_T$ is open, ie Δ_T is closed.

Lemma 1.61

Let X be a separated quasi-projective variety.

Let V be a closed (resp. open) subvariety of X.

Then V is separated.

Proof. Suppose that V is a closed subvariety of X.

The Cartesian product $V \times V \subseteq X \times X$ is closed and represents the product of V with itself as a closed subvariety of $X \times X$ (by Corollary 1.53).

On the other hand, we have $\Delta_V = \Delta_X \cap V \times V$ so Δ_V is closed in $V \times V$ since Δ_X is closed.

In other words, V is separated.

The proof in the situation where V is an open subvariety of X is similar. □

Lemma 1.62

Affine varieties are separated.

Proof. We first prove that the varieties k^t are separated for $t \geqslant 0$.

Recall that by Sheet 3, $k^t \prod k^t \simeq k^{2t}$.

Write $\mathcal{C}(k^{2t}) = k[x_1, \ldots, x_t, y_1, \ldots, y_t]$. Now note that

$$\Delta_{k^t} = \mathrm{Z}(x_1 - y_1, x_2 - y_2, \ldots, x_t - y_t).$$

Hence Δ_{k^t} is closed.

The general case now follows from Lemma 1.61. $\square$

Lemma 1.63

Let X be a quasi-projective variety.

Suppose that for any two points $a, b \in X$ there exists an open affine subvariety $U \subseteq X$ such that $a, b \in U$.

Then X is separated.

Proof. Let $(a, b) \in X \times X \backslash \Delta_X$ (ie $a, b \in X$ and $a \neq b$).

Let $U_{a,b}$ be an open affine subvariety of X such that $a, b \in U_{a,b}$. Then $(a, b) \in U_{a,b} \times U_{a,b}$.

Furthermore,

$$\Delta_{U_{a,b}} = \Delta_X \cap (U_{a,b} \times U_{a,b})$$

and the Cartesian product $U_{a,b} \times U_{a,b}$ is a product of $U_{a,b}$ with itself as an open subvariety of $X \times X$.

Hence $\Delta_{U_{a,b}}$ is closed as a subset of $U_{a,b} \times U_{a,b}$ by Lemma 1.62.

In particular, (a, b) is contained in an open subset of $X \times X$, which is disjoint from (a, b). Since $(a, b) \in X \times X \backslash \Delta_X$ was arbitrary, we conclude that $X \times X \backslash \Delta_X$ is open, ie Δ_X is closed. ☐

Proposition 1.64

Any quasi-projective variety is separated.

Proof. Suppose first that $X = \mathbb{P}^n(k)$ for some $n \geqslant 0$.

Then X is separated by Lemma 1.63 and Sheet 2.

The general case follows from this and Lemma 1.61. ☐

Proposition-Definition 1.65 (The graph of a morphism)

Let X and Y be quasi-projective varieties.

Let $\gamma : X \to Y$ be a morphism. Let

$$\Gamma_\gamma := \{(x, \gamma(x)) \,|\, x \in X\} \subseteq X \times Y$$

be the graph of γ.

Then Γ_γ is closed in $X \times Y$.

Proof. Let $\widetilde{\gamma} : X \times Y \to Y \times Y$ be the morphism such that

$$\widetilde{\gamma}(x, y) := (\gamma(x), y)$$

for all $(x, y) \in X \times Y$. We have

$$\Gamma_\gamma = \widetilde{\gamma}^{-1}(\Delta_Y)$$

and so Γ_γ is closed since Δ_Y is closed by Proposition 1.64. □

Definition 1.66

Let X be a quasi-projective variety.

We say that X is complete if for any quasi-projective variety B and any closed subset $C \subseteq X \times B$, the set $\pi_B(C)$ is closed.

Here $\pi_B : X \times B \to B$ is the projection on the second factor.

Lemma 1.67

Let X be a complete quasi-projective variety.

Then any closed subvariety of X is also complete.

Proof. Unroll the definitions and use Corollary 1.53. □

Theorem 1.68

Projective varieties are complete.

Proof. By Lemma 1.67, we only need to prove this for $X = \mathbb{P}^n(k)$.

So let B be a quasi-projective variety and let $\{B_i\}$ be an open affine covering of B.

Let $C \subseteq \mathbb{P}^b(k) \times B$ be a closed subset.

By Corollary 1.53, the Cartesian product $\mathbb{P}^b(k) \times B_i$ is open in $\mathbb{P}^b(k) \times B$ and if $\mathbb{P}^b(k) \times B_i$ is viewed as an open subvariety of $\mathbb{P}^b(k) \times B$ it is a product of $\mathbb{P}^n(k)$ and B_i.

Now $\pi_B(C)$ is closed iff $\pi_B(C) \cap B_i$ is closed in B_i for all i and we have $\pi_B(C) \cap B_i = \pi_{B_i}(C \cap (\mathbb{P}^n(k) \times B_i))$.

Hence we may suppose from the start that B is affine.

In that case B is a closed subvariety of k^t for some $t \geqslant 0$.

By Corollary 1.53 again, the subset $\mathbb{P}^n(k) \times B \subseteq \mathbb{P}^n(k) \times k^t$ is closed and is a product of $\mathbb{P}^n(k)$ and B if $\mathbb{P}^n(k) \times B$ is viewed as a closed subvariety of $\mathbb{P}^n(k) \times k^t$.

Furthermore, $\pi_B(C)$ is closed in B iff it is closed in k^t.

Some we might suppose that $B = k^t$.

Now let $i \in \{0, \dots, n\}$ and let $U_i \subseteq \mathbb{P}^n(k)$ be the well-known coordinate chart.

Recall that there is an isomorphism $u_i : k^n \to U_i$ given by the formula

$$u_i(\langle X_0, \dots, \check{X}_i, \dots, X_n \rangle) = [X_0, \dots, X_{i-1}, 1, X_{i+1}, \dots, X_n] \in \mathbb{P}^n(k).$$

By Sheet 3, the variety $U_i \times k^t$ is affine and we have

$$\mathcal{C}(k^n \times k^t) = k[x_0, \dots, \check{x}_i, \dots, x_n, y_1, \dots, y_t]$$

where the x_j are the coordinates of k^n and the y_j are the coordinates of k^t.

Write

$$\phi_i : k[x_0, \dots x_n, y_1, \dots, y_t] \to k[x_0, \dots, \check{x}_i, \dots, x_n, y_1, \dots, y_t]$$

for the map of k-algebras such that $\phi(x_j) = x_j$ for all $j \neq i$, $\phi(x_i) = 1$ and $\phi(y_j) = y_j$ for all j.

Let $I_i := \mathcal{I}((u_i \times \mathrm{Id}_{k^t})^{-1}(C)) \subseteq k[x_0, \dots, \check{x}_i, \dots, x_n, y_1, \dots, y_t]$.

Note the following. Suppose that $H \in k[x_0, \dots x_n, y_1, \dots, y_t]$ and that H is homogenous in the x-variables. Then $H \in \phi_i^{-1}(I_i)$ iff

$$H(X_0, \dots, X_n, Y_1, \dots, Y_t) = 0$$

for all

$$[X_0, \dots, X_n] \times \langle Y_1, \dots, Y_t \rangle \in C \cap (U_i \times k^t).$$

This follows directly from the definitions.

In particular a polynomial $H \in k[x_0, \dots x_n, y_1, \dots, y_t]$ which is homogenous in the x-variables lies in $\cap_i \phi_i^{-1}(I_i)$ iff $H(X_0, \dots, X_n, Y_1, \dots, Y_t) = 0$ for all

$$[X_0, \dots, X_n] \times \langle Y_1, \dots, Y_t \rangle \in C.$$

For any $N \geqslant 0$, write $S_N \subseteq k[x_0, \dots x_n, y_1, \dots, y_t]$ for the polynomials, which are homogenous in the x-variable and which are of degree N in the x-variable.

This gives $k[x_0, \dots x_n, y_1, \dots, y_t]$ the structure of a graded ring with $S_0 = k[y_1, \dots, y_t]$.

In particular S_N is a $S_0 = k[y_1, \dots, y_t]$-submodule of $k[x_0, \dots x_n, y_1, \dots, y_t]$.

We also write $A_N = S_N \cap (\cap_i \phi_i^{-1}(I_i))$.

It follows from the definitions that $\oplus_{l \geq 0} A_l$ is then a graded ideal in (= graded sub-$k[x_0, \dots x_n, y_1, \dots, y_t]$-module of) $k[x_0, \dots x_n, y_1, \dots, y_t]$.

In particular, A_N is a $S_0 = k[y_1, \dots, y_t]$-submodule of S_N.

Now let $\bar{w} = \langle W_1, \dots W_t \rangle \in k^t$ and suppose that $\bar{w} \notin \pi_B(C)$.

Let $\bar{m} = (y_1 - W_1, \dots, y_t - W_t) \subseteq k[y_1, \dots, y_t]$ be the maximal ideal associated with $\bar{w}$. Let $i \in \{0, \dots, n\}$.

By assumption, we have

$$I_i + \mathfrak{m} \cdot k[x_0, \dots, \check{x}_i, \dots, x_n, y_1, \dots, y_t] = k[x_0, \dots, \check{x}_i, \dots, x_n, y_1, \dots, y_t]$$

(since the zero set of $\mathfrak{m} \cdot k[x_0, \dots, \check{x}_i, \dots, x_n, y_1, \dots, y_t]$ is $k^n \times \{w\}$ and by assumption $u_i^{-1}(C) = \mathrm{Z}(I_i)$, which does not meet $k^n \times \{w\}$).

In particular, there is a polynomial $P_i \in I_i$ and polynomials $M_{il} \in \mathfrak{m}$ and $G_{il} \in k[x_0, \dots, \check{x}_i, \dots, x_n, y_1, \dots, y_t]$ such that

$$1 = P_i + \sum_l M_{il} \cdot G_{il}$$

Hence, for any $N \geqslant 0$ we have

$$\begin{aligned} x_i^N &= x_i^{N-\deg_x(P_i)}\left(x_i^{\deg_x(P_i)} P_i(x_0/x_i, \ldots, \check{x}_i, \ldots, x_n/x_i, y_1, \ldots, y_t)\right) \\ + & \sum_l M_{il}(y_1, \ldots, y_t) \\ & \left[x_i^{N-\deg_x(G_{il})}\left(x_i^{\deg_x(G_{il})} G_{il}(x_0/x_i, \ldots, \check{x}_i, \ldots, x_n/x_i, y_1, \ldots, y_t)\right)\right] \end{aligned}$$

Now note that the polynomial

$$x_i^{\deg_x(P_i)} P_i(x_0/x_i, \ldots, \check{x}_i, \ldots, x_n/x_i, y_1, \ldots, y_t)$$

is by construction homogenous in the x-variable and of x-degree $\deg_x(P_i)$. The same polynomial also lies in $\phi_i^{-1}(I_i)$ since

$$\phi_i(x_i^{\deg_x(P_i)} P_i(x_0/x_i, \ldots, \check{x}_i, \ldots, x_n/x_i, y_1, \ldots, y_t)) = P_i.$$

Furthermore, by definition, the polynomial

$$x_i^{\deg_x(P_i)+1} P_i(x_0/x_i, \ldots, \check{x}_i, \ldots, x_n/x_i, y_1, \ldots, y_t)$$

vanishes when evaluated on $\langle X_0, \ldots, X_n, Y_1, \ldots, Y_t \rangle$ whenever

$$[X_0, \ldots, X_n] \times \langle Y_1, \ldots, Y_t \rangle \in C$$

(remember that x_i vanishes on $(\mathbb{P}^n(k) \backslash U_i) \times k^t$).

Hence

$$x_i^{\deg_x(P_i)+1} P_i(x_0/x_i, \ldots, \check{x}_i, \ldots, x_n/x_i, y_1, \ldots, y_t) \in A_{\deg_x(P_i)+1}$$

by the above discussion.

Similarly, the polynomial $x_i^{\deg_x(G_{il})} G_{il}(x_0/x_i, \ldots, \check{x}_i, \ldots, x_n/x_i, y_1, \ldots, y_t)$ is also homogenous in the x-variable and is of x-degree $\deg_x(G_{il})$.

So if N is larger than $\deg_x(P_i) + 1$ and also larger than $\deg_x(G_{il})$ for all l, we have an equality

$$x_i^N = T_i + \sum_l M_{il} H_{il}$$

where $T_i \in A_N$ and $H_{il} \in S_N$.

Since there is only a finite number of indices i, there is thus a natural number N_0 such that

$$x_i^N \in A_N + \mathfrak{m} S_N$$

for all $N \geqslant N_0$ and all $i \in \{0, \ldots, n\}$.

Now note that if N_1 is sufficiently large, any monomial of degree $\geqslant N_1$ in the x_i becomes divisible by $x_j^{N_0}$ for some x_j.

So if N_1 is sufficiently large then for all $N \geqslant N_1$ we have

$$S_N \subseteq (\oplus_{s\geqslant 0} S_s)(A_{N_0} + \mathfrak{m}S_{N_0})$$

Since $\oplus_{s\geqslant 0} A_s$ is a graded ideal, we then have

$$S_N \subseteq S_{N-N_0}(A_{N_0} + \mathfrak{m}S_{N_0}) \subseteq A_N + \mathfrak{m}S_N.$$

In particular, we have $(S_N/A_N) = \mathfrak{m}(S_N/A_N)$ where the quotient S_N/A_N is quotient of $k[y_1, \dots, y_t]$-modules.

We conclude from the generalised form of Nakayama's lemma (see Q4 in Sheet 1 of CA) that there is $Q \in 1 + \mathfrak{m}$ such that $Q \cdot (S_N/A_N) = 0$.

In particular $Q \cdot x_i^N \in A_N$ for all $i \in \{0, \dots, n\}$. In other words, for any i we have

$$X_i^N Q(X_0, \dots, X_n, Y_1, \dots, Y_t) = X_i^N Q(Y_1, \dots, Y_t) = 0$$

for all $[X_0, \dots, X_n] \times \langle Y_1, \dots, Y_t \rangle \in C$ (see the discussion above).

In particular, whenever $Q(Y_1, \dots, Y_t) \neq 0$ we have

$$C \cap (U_i \times \{\langle Y_1, \dots, Y_t \rangle\}) = \emptyset.$$

Since this holds for all $i \in \{0, \dots, n\}$, the set

$$C \cap (\mathbb{P}^n(k) \times \{\langle Y_1, \dots, Y_t \rangle\})$$

is empty whenever $Q(Y_1, \dots, Y_t) \neq 0$.

Said differently, if $\langle Y_1, \dots, Y_t \rangle \in k^t \backslash \mathrm{Z}(Q)$ then $\langle Y_1, \dots, Y_t \rangle \notin \pi_B(C)$.

Finally, we have $Q(\bar{w}) \neq 0$ since $Q \in 1 + \mathfrak{m}$, so $k^t \backslash \mathrm{Z}(Q)$ is a neighbourhood of $\bar{w}$.

Since $\bar{w} \in k^t \backslash \pi_B(C)$ was arbitrary, we conclude that $k^t \backslash \pi_B(C)$ is open, ie $\pi_B(C)$ is closed. $\square$

Remark. Suppose given polynomials $H_1, \dots, H_l \in k[x_0, \dots, x_n, y]$.

Suppose that the H_j are homogenous in the variables x_i. Let

$$C := \{[X_0, \dots, X_n] \times \langle Y \rangle \in \mathbb{P}^n(k) \times k \mid \forall j \in \{1, \dots, l\} : H_j(X_0, \dots, X_n, Y) = 0\}$$

It can easily be shown that C is a closed subset of $\mathbb{P}^n(k) \times k$.

By Theorem 1.68, the set

$$\pi_k(C) := \{Y \in k \mid \exists [X_0, \dots, X_n] \in \mathbb{P}^n(k) : \forall j : H_j(X_0, \dots, X_n, Y) = 0\}$$

is then closed. In other words, there is a unique polynomial $Q(y) \in k[y]$, which is a product of distinct linear factors, and such that $Q(y) = 0$ iff there is $X_0, \dots, X_n \in k^{n+1} \backslash \{0\}$ such that

$$H_1(X_0, \dots, X_n, Y) = H_2(X_0, \dots, X_n, Y) = \dots = H_l(X_0, \dots, X_n, Y) = 0.$$

This result is called the *main theorem of elimination theory.*

The polynomial $Q(y)$ is called the *resultant* of the polynomials $H_1, \dots, H_l$.

Corollary 1.69 (of Theorem 1.68)

Let X, Y be quasi-projective varieties and suppose that X is complete.

Let $\phi : X \to Y$ is a morphism.

Then $\phi(X)$ is closed.

Proof. The image of $\phi(X)$ is the projection of the graph $\Gamma_\phi \subseteq X \times Y$ by the projection to Y. Hence Proposition-Definition 1.65 implies the result. $\square$

Proposition 1.70

A complete quasi-projective variety is projective.

Proof. Let X be a quasi-projective complete variety.

By definition, we may suppose that there is an open subvariety U of $\mathbb{P}^n(k)$ such that X is a closed subvariety of U.

By Corollary 1.69, X is closed in $\mathbb{P}^n(k)$.

Hence, from the definition of subvarieties, X is a closed subvariety of $\mathbb{P}^n(k)$. Hence X is projective. $\square$

Lemma 1.71

Let X be an affine complete variety. Then X consists of a finite number of points.

Proof. By Sheet 3, $\mathcal{C}(X)$ is a finite dimensional k vector space.

In particular, $\mathcal{C}(X)$ is finite over k.

We deduce from Prop. 8.12 in CA that $\mathcal{C}(X)$ has only finitely maximal ideals.

Hence X has only finitely many points by the discussion before Lemma 1.8. □

Smoothness

A variety is smooth if it has "no kinks".

For a curve C in the plane given by one equation $f(x,y)=0$, this can analysed by looking at its gradient $\mathrm{grad}(f)=\langle\frac{\partial}{\partial x}f,\frac{\partial}{\partial y}f\rangle$.

The curve will be smooth if $\mathrm{grad}(f)$ does not vanish for any point of C.

The general definition has a similar flavour.

Definition 1.72

Let $V\subseteq k^n$ be an algebraic set.

Suppose that $\mathcal{I}(V)=(P_1,\ldots,P_t)\subseteq k[x_1,\ldots,x_n]$. Let $\bar{v}\in V$.

We say that V is nonsingular at $\bar{v}$ if the matrix $[(\frac{\partial}{\partial x_j}P_i)(\bar{v})]_{ij}$ has rank $n-\mathrm{cod}(\{v\},V)$.

Note that when C is a curve in the plane, we recover the definition given above.

To make sense of this definition, we need to show that it does not depend on the polynomials P_i.

In fact, we will show that the definition only depends on the coordinate ring $\mathcal{C}(V)$.

On the way to this result, we first make another definition.

Definition 1.73

Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k_0 := R/\mathfrak{m}$.

We say that R is a regular local ring if $\dim(R) = \dim_{k_0} \mathfrak{m}/\mathfrak{m}^2$.

Note that with the notation of the last definition, we have $\dim(R) = \mathrm{ht}(\mathfrak{m})$.

On the other hand, by Nakayama's lemma (see Cor. 3.6 in CA), the ideal $\mathfrak{m}$ can be generated by $\dim_{k_0} \mathfrak{m}/\mathfrak{m}^2$ elements.

Hence by a corollary of Krull's theorem (see CA Cor. 11.15), we have

$$\dim(R) = \mathrm{ht}(\mathfrak{m}) \leqslant \dim_{k_0} \mathfrak{m}/\mathfrak{m}^2.$$

The local ring R is regular iff this last inequality is an equality.

Proposition 1.74

Let $V \subseteq k^n$ be an algebraic set.

Then V is nonsingular at $\bar{v} \in V$ iff the local ring $\mathcal{O}_{V,v} \simeq \mathcal{C}(V)_{\mathcal{I}(\{\bar{v}\})}$ is regular.

For the proof, we shall need the

Lemma 1.75

Let R be a ring and let $\mathfrak{m} \subseteq R$ be a maximal ideal.

Let $\phi : R \to R_{\mathfrak{m}}$ be the natural map of rings. Let $n \geqslant 0$.

Then the unique maximal ideal $\underline{\mathfrak{m}}$ of $R_{\mathfrak{m}}$ is the ideal of $R_{\mathfrak{m}}$ generated by $\phi(\mathfrak{m})$.

Furthermore, we have $\phi^{-1}(\underline{\mathfrak{m}}^n) = \mathfrak{m}^n$ and the map of R-modules induced by ϕ

$$\mathfrak{m}^n/\mathfrak{m}^{n+1} \to \underline{\mathfrak{m}}^n/\underline{\mathfrak{m}}^{n+1}$$

is an isomorphism.

Note that the lemma is obviously false if $\mathfrak{m}$ is not maximal (look eg at the case $n = 0$).

Proof. (of Lemma 1.75) Skipped. See the notes. ☐

Proof. (of Proposition 1.74)

Let $\bar{v} = \langle v_1, \ldots, v_n \rangle \in V \subseteq k^n$.

Suppose that $\mathcal{I}(V) = (P_1, \ldots, P_t)$.

Write

$$\mathfrak{m} := \mathcal{I}(\{\bar{v}\}) = (x_1 - v_1, \ldots, x_n - v_n)$$

be the maximal ideal of $k[x_1, \ldots, x_n]$ associated with $\bar{v}$.

Let $\mathfrak{n} = \mathfrak{m} \,(\mathrm{mod}\, \mathcal{I}(V)) \subseteq \mathcal{C}(V)$ be the maximal ideal of $\mathcal{C}(V)$ associated with $\bar{v}$.

Define a map of k-vector space $\phi : \mathfrak{m} \to k^n$ by the formula

$$\phi(Q) = \langle (\frac{\partial}{\partial x_1} Q)(\bar{v}), \ldots, (\frac{\partial}{\partial x_n} Q)(\bar{v}) \rangle.$$

Since $\mathfrak{m}^2$ is generated by the elements $(x_i - v_i)(x_j - v_j)$, we see that $\phi(\mathfrak{m}^2) = 0$.

We thus obtain a k-linear map $\mathfrak{m}/\mathfrak{m}^2 \to k^n$.

This map is surjective because $\phi(x_i - v_i)$ is the i-the element of the standard basis of k^n.

On the other hand, $\mathfrak{m}/\mathfrak{m}^2$ is generated by n elements as a $R/\mathfrak{m} = k$-vector space and so is of dimension $\leqslant n$.

Hence the map $\mathfrak{m}/\mathfrak{m}^2 \to k^n$ is an isomorphism of k-vector spaces.

Now the image $(\mathcal{I}(V) + \mathfrak{m}^2)/\mathfrak{m}^2$ of $\mathcal{I}(V) \subseteq \mathfrak{m}$ in $\mathfrak{m}/\mathfrak{m}^2$ is generated by $P_1 \pmod{\mathfrak{m}^2}, \ldots, P_t \pmod{\mathfrak{m}^2}$ as a $R/\mathfrak{m} = k$-vector space. Hence

$$\dim_k((\mathcal{I}(V) + \mathfrak{m}^2)/\mathfrak{m}^2) = \dim_k(\phi(\mathcal{I}(V)))$$

$$= \quad \mathrm{rk}\begin{pmatrix} (\frac{\partial}{\partial x_1}P_1)(\bar{v}) & \ldots & (\frac{\partial}{\partial x_n}P_1)(\bar{v}) \\ (\frac{\partial}{\partial x_1}P_2)(\bar{v}) & \ldots & (\frac{\partial}{\partial x_n}P_2)(\bar{v}) \\ \vdots & \vdots & \vdots \\ (\frac{\partial}{\partial x_1}P_t)(\bar{v}) & \ldots & (\frac{\partial}{\partial x_n}P_t)(\bar{v}) \end{pmatrix} =: \mathrm{rk}[(\frac{\partial}{\partial x_j}P_i)(\bar{v})]_{ij}.$$

On the other hand, we have by construction a complex of $R/\mathfrak{m} = k$-vector spaces

$$0 \to (\mathcal{I}(V) + \mathfrak{m}^2)/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0 \quad (*)$$

We claim that $(*)$ is exact.

The second arrow from the left is injective by definition and likewise it follows from the definitions that the third arrow from the left is surjective.

So we only have to show that the complex is exact at $\mathfrak{m}/\mathfrak{m}^2$.

To see this, suppose that $P \in \mathfrak{m}$ and that $P \pmod{\mathcal{I}(V)} \in \mathfrak{n}^2$.

Since $\mathfrak{n}^2 = (\mathfrak{m}^2 + \mathcal{I}(V))/\mathcal{I}(V)$, there is $Q \in \mathfrak{m}^2 + \mathcal{I}(V)$ such that

$$P \pmod{\mathcal{I}(V)} = Q \pmod{\mathcal{I}(V)}.$$

We then have $(P - Q) \pmod{\mathcal{I}(V)} = 0$, or in other words $P - Q \in \mathcal{I}(V)$.

Hence P is the sum of an element of $\mathcal{I}(V)$ and an element of $\mathfrak{m}^2$.

This shows that $(*)$ is exact at $\mathfrak{m}/\mathfrak{m}^2$ and is thus an exact complex.

We conclude that

$$\mathrm{rk}[(\frac{\partial}{\partial x_j}P_i)(\bar{v})]_{ij} + \dim_k(\mathfrak{n}/\mathfrak{n}^2) = n. \tag{2}$$

Now we have $\mathrm{cod}(V, \{\bar{v}\}) = \mathrm{ht}(\mathfrak{n}) = \dim(\mathcal{C}(V)_{\mathcal{I}(\{\bar{v}\})})$ (see Lemma 11.2 in CA).

Using Lemma 1.75, we see that the local ring $\mathcal{C}(V)_{\mathcal{I}(\{\bar{v}\})}$ is regular iff

$$\mathrm{rk}[(\frac{\partial}{\partial x_j}P_i)(\bar{v})]_{ij} = n - \mathrm{cod}(V, \{\bar{v}\}).$$

This proves the first assertion.

For the second assertion, note that if V is irreducible, we have

$$\mathrm{cod}(V, \{\bar{v}\}) = \dim(V)$$

by Theorem 1.41 (note that a point has dimension 0). □

Remark. (1) Keep the notation of the proof of Proposition 1.74.

From the remark preceding the proposition, we have $\dim_k(\mathfrak{n}/\mathfrak{n}^2) \geqslant \mathrm{cod}(V, \{\bar{v}\})$ and so we always have

$$\mathrm{rk}[(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij} = n - \dim_k(\mathfrak{n}/\mathfrak{n}^2) \leqslant n - \mathrm{cod}(V, \{\bar{v}\})$$

even if V is singular at $\bar{v}$.

(2) Note that equation (2) gives an effective way to compute $\dim_k(\mathfrak{n}/\mathfrak{n}^2)$.

We also record the following lemma, which will be useful in calculations.

Lemma 1.76

Keep the assumptions and notation of Proposition 1.74.

Let $Q_1, \ldots Q_s \in \mathcal{I}(V)$.

Suppose that $[(\frac{\partial}{\partial x_j} Q_i)(\bar{v})]_{ij}$ *has rank* $n - \mathrm{cod}(V, \{v\})$.

Then V *is nonsingular at* $\bar{v}$.

This lemma will allow us to check nonsingularity in situations where it is difficult to find generators of $\mathcal{I}(V)$.

Proof. We use the notation of the proof of Proposition 1.74.

Let $J \subseteq \mathcal{I}(V)$ be the ideal generated by $Q_1, \ldots, Q_s$.

It was shown in the proof of Proposition 1.74 that

$$\mathrm{rk}[(\frac{\partial}{\partial x_j} Q_i)(\bar{v})]_{ij} = \dim_k(\phi(J))$$

and in particular that $\mathrm{rk}[(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij} = \dim_k(\phi(\mathcal{I}(V)))$.

On the other hand, we have $\dim_k(\phi(\mathcal{I}(V))) \geqslant \dim_k(\phi(J))$ since $J \subseteq \mathcal{I}(V)$.

Hence by the remark preceding the lemma, we have

$$\mathrm{rk}[(\frac{\partial}{\partial x_j} Q_i)(\bar{v})]_{ij} \leqslant \mathrm{rk}[(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij} \leqslant n - \mathrm{cod}(V, \{\bar{v}\}).$$

The assumptions of the lemma now imply that the two last inequalities are equalities, hence the conclusion. ☐

Let now X be any variety.

We shall write $\mathrm{Sing}(X)$ for the set of points $x \in X$ such that the local ring $\mathcal{O}_{X,x}$ is a regular local ring.

This clearly specialises to Definition 1.72 when X is an affine variety.

A variety X is *nonsingular* or *smooth* if $\mathrm{Sing}(X) = \emptyset$.

Proposition 1.77

Let X be a non empty irreducible variety.

Then the set $\mathrm{Sing}(X)$ is closed and $\mathrm{Sing}(X) \neq X$.

Let R be a UFD with fraction field K.

If

$$Q(x) = x^m + r_{m-1}x^{m-1} + \cdots + r_0 \in R[x],$$

we define the *content* $\mathrm{cont}(Q)$ to be the gcd of the coefficients of Q (the gcd is only well-defined up to multiplication by a unit of R).

If $Q(x) \in K[x]$, we define the content of $Q(x)$ to be $\mathrm{cont}(d \cdot Q)/d$, where $d \in R$ is such that $d \cdot Q(x) \in R[x]$.

One can show that this last definition does not depend on the choice of d.

Moreover, one can show that $\mathrm{cont}(Q_1 \cdot Q_2) = \mathrm{cont}(Q_1) \cdot \mathrm{cont}(Q_2)$ for any two $Q_1, Q_2 \in K[x]$. Note that if $Q(x) \in K[x]$ and $\mathrm{cont}(Q)$ is a unit, then $Q(x) \in R[x]$.

The all-important result concerning the content function is the

Lemma (generalisation of Gauss's lemma). The irreducible elements of $R[x]$ are the irreducible elements of R and the polynomials $P(x) \in R[x]$, whose content is a unit and which are irreducible (and hence non constant) in $K[x]$.

See IV, §2 in S. Lang's book *Algebra* (Springer) for more details.

Proposition 1.78

Let X be a non empty irreducible variety.

Then X is birational to an algebraic set $V \subseteq k^n$ such that $\mathcal{I}(V) \subseteq k[x_1, \dots, x_n]$ is prime and principal.

Proof. (of Proposition 1.78) We shall only prove this in the situation where $\mathrm{char}(k) = 0$. So suppose that $\mathrm{char}(k) = 0$.

Restricting to an open affine subset of X, we may assume wlog that X is an irreducible affine variety. Let $K := \mathrm{Frac}(\mathcal{C}(X))$ be the function field of X.

Since the k-algebra $\mathcal{C}(X)$ is finitely generated over k, the field K is finitely generated as a field over k.

Let $b_1, \dots, b_t \in K$ be a transcendence basis for K over k.

By definition, this means that the b_i are algebraically independent over k and that the field extension $K|k(b_1, \dots, b_t)$ is algebraic.

Since $\mathrm{char}(k) = 0$, the extension $K|k(b_1, \dots, b_t)$ is a separable extension.

$K|k(b_1,\dots,b_t)$ is also a finite extension because K is finitely generated as a field over $k(b_1,\dots,b_t)$.

Hence the extension $K|k(b_1,\dots,b_t)$ is a simple extension by the *primitive element theorem* (see Galois theory) and so there is an element $b\in K$, such that $K=k(b_1,\dots,b_t)(b)$ and an irreducible polynomial $Q(x)\in k(b_1,\dots,b_t)[x]$ such that $Q(b)=0$.

Now note that every element of $k(b_1,\dots,b_t)$ can be written as quotient c/d, where $c,d\in k[b_1,\dots,b_t]$.

Write

$$Q(x)=x^m+\frac{c_{m-1}}{d_{m-1}}x^{m-1}+\cdots+\frac{c_1}{d_1}x+\frac{c_0}{d_0}$$

where $c_i,d_i\in k[b_1,\dots,b_t]$. Let $d=\prod_i d_i$.

Consider the polynomial $dQ\in k[b_1,\dots,b_t][x]$ and let

$$P:=dQ/\mathrm{cont}(dQ)\in k[b_1,\dots,b_t][x],$$

where $\mathrm{cont}(dQ)\in k[b_1,\dots,b_t]$ is an arbitrary representative of the content of dQ.

By construction, the polynomial $P(x)$ is irreducible in $k(b_1, \ldots, b_t)[x]$ and its content is a unit.

By the generalised Gauss lemma, $P(x)$ is thus irreducible in $k[b_1, \ldots, b_t][x]$.

Now let

$$\phi : k[b_1, \ldots, b_t][x] \to K$$

be the homomorphism of k-algebras sending the b_i to themselves and x to b.

The kernel $\ker(\phi)$ is then a prime ideal (since the image of ϕ is a domain) and by construction we have $P(x) \in \ker(\phi)$.

Now the ideal $(P) \subseteq k[b_1, \ldots, b_t][x]$ is also prime, since P is irreducible.

Hence $\mathrm{cod}((P), k[b_1, \ldots, b_t][x]) = 1$ by Krull's principal ideal theorem (see Th. 11.13 in CA).

On the other hand, the fraction field of

$$\mathrm{Im}(\phi) = k[b_1, \ldots, b_t, b] \simeq k[b_1, \ldots, b_t][x]/\ker(\phi)$$

is the field K and K has transcendence degree t by assumption.

Thus

$$\dim(k[b_1,\dots,b_t][x]/\ker(\phi)) = t$$

by Corollary 11.28 in CA.

Using Theorem 1.41, we deduce that

$$\operatorname{cod}(\ker(\phi), k[b_1,\dots,b_t][x]) = \dim(k[b_1,\dots,b_t][x]) - t = t+1-t = 1.$$

Hence we must have $\ker(\phi) = (P)$, for otherwise we would have $\operatorname{cod}(\ker(\phi), k[b_1,\dots,b_t][x]) \geqslant 2$.

So we conclude that $k[b_1,\dots,b_t][x]/(P) \simeq k[b_1,\dots,b_t,b]$.

Now the b_i are algebraically independent and thus the k-algebra $k[b_1,\dots,b_t][x]$ can be viewed as the coordinate ring of k^{t+1}.

The ring $k[b_1,\dots,b_t][x]/(P)$ is thus isomorphic to the coordinate ring of an irreducible algebraic set V in k^{t+1}, whose (prime) radical ideal is generated by a single irreducible polynomial.

Since the function field of V is isomorphic to K as a K-algebra, it satisfies the conclusion of the proposition (by Proposition 1.46). $\square$

Proof. (of Proposition 1.77) We first show that $\mathrm{Sing}(X)$ is closed.

Let $\{U_i\}$ be an open affine covering of X. By Proposition 1.74, a point $x \in U_i$ is nonsingular in X iff it is nonsingular in U_i, ie we have $\mathrm{Sing}(X) \cap U_i = \mathrm{Sing}(U_i)$.

On the other hand, the set $\mathrm{Sing}(X)$ is closed iff $\mathrm{Sing}(X) \cap U_i$ is closed for all i.

Hence we may assume that X is isomorphic to an algebraic set $V \subseteq k^n$ for some n.

Let $P_1, \ldots, P_t$ be generators of $\mathcal{I}(V) \subseteq k[x_1, \ldots, x_n]$.

From the remark following the proof of Proposition 1.74, we have

$$\mathrm{Sing}(V) = \{\bar{v} \in V \mid \mathrm{rk}[(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij} < n - \dim(V)\}.$$

Now recall that

$$\begin{aligned}
& \mathrm{rk}[(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij} \\
= \quad & \max\{h \in \mathbb{N} \mid \text{there exists a } h \times h\text{-submatrix } M \\
& \text{in } [(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij} \text{ such that } \det(M) \neq 0\}
\end{aligned}$$

and hence

$$\begin{aligned}
& \mathrm{Sing}(V) = \{\bar{v} \in V \mid \det(M) = 0 \\
& \text{for all the } (n - \mathrm{dim}(V)) \times (n - \mathrm{dim}(V))\text{-submatrices } M \\
& \text{in } [(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij}\}
\end{aligned}$$

and hence $\mathrm{Sing}(V)$ is the zero set of a set of polynomials and is thus closed.

We now prove that $\mathrm{Sing}(X) \neq X$.

Again, we only show this when $\mathrm{char}(k) = 0$ (but the statement holds without that assumption).

We may replace wlog X by any of its open subsets and so thanks to Proposition 1.78 we may suppose that X is an algebraic set $V \subseteq k^n$ such that $\mathcal{I}(V) = (P)$, where $P \in k[x_1, \ldots, x_n]$ is an irreducible polynomial.

In this situation, we have to show that

$$\begin{aligned} \mathrm{Sing}(V) = \{\bar{v} \in V \mid (\frac{\partial}{\partial x_1} P)(\bar{v}) = \\ = \quad (\frac{\partial}{\partial x_2} P)(\bar{v}) = \cdots = (\frac{\partial}{\partial x_n} P)(\bar{v}) = 0\} \neq V. \end{aligned}$$

Suppose for contradiction that $\mathrm{Sing}(V) = V$.

By construction, we have $\frac{\partial}{\partial x_i} P \in (P)$ for all i, since (P) is a prime ideal.

In other words, $P | \frac{\partial}{\partial x_i} P$ for all i.

Now let i_0 be such that P has a monomial divisible by x_{i_0}.

This exists since P is irreducible and in particular not constant. In that case $\frac{\partial}{\partial x_{i_0}} P \neq 0$ (note that we use the fact that $\mathrm{char}(k) = 0$ here) and

$$\deg_{x_{i_0}}(\frac{\partial}{\partial x_{i_0}} P) < \deg_{x_{i_0}}(P).$$

In particular, $\frac{\partial}{\partial x_{i_0}} P$ is not divisible by P. This is a contradiction, so $\mathrm{Sing}(V) \neq V$. $\square$

Blowing up

The blow-up construction is a geometric construction, which replaces the ambient variety of a closed subvariety by a new variety, which lies over it and such that the inverse image of the closed subvariety is locally defined by one equation.

This new variety often has better properties than the new one - eg the blow-up of a variety at a singular point tends to be "less" singular than the original variety.

This construction is best understood in the language of schemes.

In this section, we explain in the language of varieties how to blow up an affine variety at a point.

We can only establish few properties of such blow-ups in our setting.

Let $n \geqslant 1$. Let $x_1, \ldots, x_n$ be variables for k^n and let $y_1, \ldots, y_n$ be homogenous variables for $\mathbb{P}^{n-1}(k)$.

Note that contrary to what is customary, the index of the homogenous variables runs between 1 and n here (not 0 and $n-1$).

Let Z be the subset of $k^n \times \mathbb{P}^{n-1}(k)$ defined by the equations $\{x_i y_j - x_j y_i = 0\}_{i,j \in \{1,\ldots,n\}}$ (note that this makes sense because the polynomials are homogenous in the y-variables).

The set Z is called the *blow-up* of k^n at the origin of k^n.

Let $\phi : Z \to k^n$ the map obtained by restricting the projection $k^n \times \mathbb{P}^{n-1}(k) \to k^n$ to Z.

Proposition 1.79

(1) *The set Z is a closed subvariety of $k^n \times \mathbb{P}^{n-1}(k)$.*

(2) *The closed subvariety $\phi^{-1}(\{0\})$ of Z is canonically isomorphic to $\mathbb{P}^{n-1}(k)$. The points of $\phi^{-1}(0)$ are in one-to-one correspondence with the lines going through the origin of k^n.*

(3) *The restriction of ϕ to the open subvariety $\phi^{-1}(k^n \backslash \{0\})$ of Z induces an isomorphism $\phi^{-1}(k^n \backslash \{0\}) \simeq k^n \backslash \{0\}$.*

Proof. **(1)** On the open affine subset $k^n \times U^{n-1}_{j_0}$, Z is given by the equations

$$\{x_i y_j - x_j y_i = 0, x_i - x_{j_0} y_i = 0\}_{i \in \{1,\dots,n\}, j \in \{1,\dots,j_0-1,j_0+1,\dots,n\}}.$$

The set $Z \cap k^n \times U^{n-1}_{j_0}$ is thus closed in $k^n \times U^{n-1}_{j_0}$. Since the $k^n \times U^{n-1}_{j}$ cover $k^n \times \mathbb{P}^{n-1}(k)$, we see that Z is closed.

(2) It follows from the definitions that $\phi^{-1}(\{0\}) = \{0\} \times \mathbb{P}^{n-1}(k)$.

(3) Suppose that $\langle X_1, \dots, X_n \rangle \neq 0$. Then there is an i_0 such that $X_{i_0} \neq 0$.

The equations for Z then give $Y_j = X_j(Y_{i_0}/X_{i_0})$ for all j.

Up to multiplication of all the Y_j by a non zero scalar factor, the only solution to this set of equations is $\langle X_1, \dots, X_n \rangle$.

In particular, we have

$$\phi^{-1}(\langle X_1, \dots, X_n \rangle) = \{\langle X_1, \dots, X_n \rangle\} \times \{[X_1, \dots, X_n]\}.$$

This shows that the morphism $\phi^{-1}(k^n \backslash \{0\}) \to k^n \backslash \{0\}$ is a bijection.

To show that it is an isomorphism, we shall provide an inverse morphism.

For this, consider the morphism $q : k^n \backslash \{0\} \to \mathbb{P}^{n-1}(k)$ introduced in sheet 3.

We define a map $k^n \backslash \{0\} \to Z$ by the formula

$$g := \mathrm{Id}_{k^n \backslash \{0\}} \prod q.$$

By construction, this gives an inverse of the morphism

$$\phi^{-1}(k^n \backslash \{0\}) \to k^n \backslash \{0\}.$$

☐

Let now $X \subseteq k^n$ be a closed subvariety (ie an algebraic set).

Let $\bar{v} := \langle v_1, \ldots, v_n \rangle \in X$ and suppose that $\{\bar{v}\}$ is not an irreducible component of X.

Let $\tau_{\bar{v}} : k^n \to k^n$ be the map such that

$$\tau_{\bar{v}}(\langle w_1, \ldots, w_n \rangle) = \langle w_1 + v_1, \ldots, w_n + v_n \rangle$$

for all $\bar{w} = \langle w_1, \ldots, w_n \rangle \in k^n$.

Let $Y := \tau_{-\bar{v}}(X)$.

Note that by construction we have $0 \in Y$.

We define the *blow-up* $\mathrm{Bl}(X, \bar{v})$ of X at $\bar{v}$ to be the closure of $\phi^{-1}(Y \backslash \{0\})$ in Z.

Let $b : \mathrm{Bl}(X, \bar{v}) \to X$ be the morphism $\tau_{\bar{v}} \circ \phi|_{\mathrm{Bl}(X,\bar{v})}$.

Proposition 1.80

(1) *We have* $\phi(\mathrm{Bl}(X,\bar{v})) = Y$*.*

(2) *Suppose that* X *is irreducible.*

Then $\mathrm{Bl}(X,\bar{v})$ *is an irreducible component of* $\phi^{-1}(Y) \subseteq k^n \times \mathbb{P}^{n-1}(k)$*.*

The morphism b *is birational.*

If $X \neq k^n$*, the irreducible components of* $\phi^{-1}(Y)$ *are* $\mathrm{Bl}(X,\bar{v})$ *and* $\{0\} \times \mathbb{P}^{n-1}(k)$*.*

The closed set $b^{-1}(\{v\}) = \mathrm{Bl}(X,\bar{v}) \cap (\{0\} \times \mathbb{P}^{n-1}(k))$ is called the *exceptional divisor* of $\mathrm{Bl}(X,\bar{v})$.

Proof. (1) Note first that $\bar{v}$ lies in the closure of $X\backslash\{\bar{v}\}$.

To see this, let C be the irreducible component of X containing $\bar{v}$.

Then $C\backslash\{\bar{v}\}$ is non-empty (by assumption) and it is open in C (since $\{\bar{v}\}$ is closed).

Furthermore, $C\backslash\{\bar{v}\}$ is not closed in C, for otherwise C would be disconnected and hence reducible.

Thus $\bar{v}$ lies in the closure of $C\backslash\{0\}$ in C (which must be C) and hence $\bar{v}$ lies in the closure of $X\backslash\{\bar{v}\}$ in X.

Now since $\mathbb{P}^{n-1}(k)$ is complete (see Theorem 1.68), we know that $\phi(\mathrm{Bl}(X, \bar{v}))$ is closed.

By (3) of Proposition 1.79, we know that $\phi(\mathrm{Bl}(X, \bar{v}))\backslash\{0\} = Y\backslash\{0\}$ and thus by the reasoning in the last paragraph, we see that $0 \in \phi(\mathrm{Bl}(X, \bar{v}))$.

In particular, $\phi(\mathrm{Bl}(X, \bar{v})) = Y$.

(2) From (3) of Proposition 1.79 we know that the natural morphism

$$\phi^{-1}(Y\backslash\{0\}) \to Y\backslash\{0\}$$

is an isomorphism.

Now if X is irreducible, so is Y and so is $Y\backslash\{0\}$.

Hence $\mathrm{Bl}(X, \bar{v})$ is irreducible by sheet 2.

On the other hand, $\mathrm{Bl}(X, \bar{v}) \subseteq \phi^{-1}(Y)$ since $\phi^{-1}(Y)$ is closed in Z.

Since $\mathrm{Bl}(X, \bar{v})$ contains the non empty open subset set $\phi^{-1}(Y\backslash\{0\})$ of $\phi^{-1}(Y)$, we see that $\mathrm{Bl}(X, \bar{v})$ is an irreducible component of $\phi^{-1}(Y)$.

Since $\phi^{-1}(Y\backslash\{0\}) \to Y\backslash\{0\}$ is an isomorphism, the morphism b is birational.

On the other hand, we have by construction

$$\phi^{-1}(Y) = \mathrm{Bl}(X, \bar{v}) \cup (\{0\} \times \mathbb{P}^{n-1}(k)).$$

Now suppose that $X \neq k^n$.

We then have $\{0\} \times \mathbb{P}^{n-1}(k) \not\subseteq \mathrm{Bl}(X, \bar{v})$ because

$$\dim(\{0\} \times \mathbb{P}^{n-1}(k)) = n - 1 \geqslant \dim(\mathrm{Bl}(X, \bar{v})) = \dim(X) \leqslant n - 1$$

(use Proposition 1.45, sheet 2 and Theorem 1.41).

Since $\{0\} \times \mathbb{P}^{n-1}(k)$ is irreducible (since it is isomorphic to $\mathbb{P}^{n-1}(k)$) we see that the irreducible components of $\phi^{-1}(Y)$ are

$$\mathrm{Bl}(X, \bar{v})$$

and

$$\{0\} \times \mathbb{P}^{n-1}(k).$$

□

Example. Let C be the curve $y^2 = x^3$ in k^2.

Let $b : \mathrm{Bl}(C, 0) \to C$ of C be the blow-up of C at the origin.

(1) We have $\mathrm{Bl}(C, 0) \simeq k$.

(2) The map b is a homeomorphism but is not an isomorphism.

Use the terminology of the last two propositions, letting $n = 2$ and $X = \mathrm{Z}(x_2^2 - x_1^3) = Y$.

We first compute $\phi^{-1}(X)$. Let $\pi : k^n \times \mathbb{P}^1(k) \to k^n$ be the natural projection. By definition

$$\phi^{-1}(X) = \pi^{-1}(X) \cap Z = \mathrm{Z}(x_1 y_2 - x_2 y_1, x_2^2 - x_1^3)$$

Let $U_1 := \{[1, Y_2] \mid Y_2 \in k\} \subset \mathbb{P}^1(k)$.

In $k^2 \times U_1$, we have

$$\begin{aligned} & \phi^{-1}(X) \cap (k^2 \times U_1) = \mathrm{Z}(x_1 y_2 - x_2, x_2^2 - x_1^3) \\ = \; & \mathrm{Z}(x_1 y_2 - x_2, x_1^2 y_2^2 - x_1^3) = \mathrm{Z}(x_1 y_2 - x_2, x_1) \cup \mathrm{Z}(x_1 y_2 - x_2, y_2^2 - x_1) \\ = \; & (\{0\} \times U_1) \cup \mathrm{Z}(x_1 y_2 - x_2, y_2^2 - x_1) \end{aligned}$$

The closed set $\mathrm{Z}(x_1 y_2 - x_2, y_2^2 - x_1)$ does not contain $\{0\} \times U_1$.

Also $\phi^{-1}(X) \cap (k^2 \times U_1)$ has at most two irreducible components by Proposition 1.80 (2) so we conclude that

$$\mathrm{Z}(x_1 y_2 - x_2, y_2^2 - x_1) = \mathrm{Bl}(X, 0) \cap (k^2 \times U_1).$$

On the other hand, $\mathrm{Z}(x_1 y_2 - x_2, y_2^2 - x_1) \cap (\{0\} \times U_1) = \{0\} \times \{[1, 0]\}$.

We now repeat the above reasoning for $U_2 := \{[Y_1, 1] \mid Y_1 \in k\} \subseteq \mathbb{P}^1(k)$ instead of U_1. We have

$$\begin{aligned} & \phi^{-1}(X) \cap (k^2 \times U_2) = \mathrm{Z}(x_1 - x_2 y_1, x_2^2 - x_1^3) \\ = \ & \mathrm{Z}(x_1 - x_2 y_1, x_2^2 - x_2^3 y_1^3) = \mathrm{Z}(x_1 - x_2 y_1, x_2) \cup \mathrm{Z}(x_1 - x_2 y_1, 1 - x_2 y_2^3) \\ = \ & (\{0\} \times U_2) \cup \mathrm{Z}(x_1 - x_2 y_1, 1 - x_2 y_2^3) \end{aligned}$$

As before, we have

$$\mathrm{Z}(x_1 - x_2 y_1, 1 - x_2 y_2^3)) \cap (k^2 \times U_2) = \mathrm{Bl}(X, 0) \cap (k^2 \times U_2).$$

On the other hand, a simple calculation shows that

$$\mathrm{Z}(x_1 - x_2 y_1, 1 - x_2 y_2^3) \cap (\{0\} \times U_2) = \emptyset.$$

So we conclude that the exceptional divisor of $\mathrm{Bl}(X, 0)$ consists of the one point $\{0\} \times \{[1, 0]\}$.

In particular, the map $b : \mathrm{Bl}(X, 0) \to X$ is bijective.

Since $\mathbb{P}^1(k)$ is complete, the morphism b sends closed sets to closed sets (see Theorem 1.68 and Corollary 1.69) and thus (since b is bijective), b sends open sets to open sets.

Hence b is a homeomorphism.

Taking into account (1), which we will establish below, we see that b is not an isomorphism because k is smooth whereas X has a singularity at 0. This establishes (2).

We now turn to (1). We have

$$\begin{aligned} & \phi^{-1}(X) \cap k^2 \times (\mathbb{P}^1 \backslash U_1) = \mathrm{Z}(x_1 y_2 - x_2 y_1, x_2^2 - x_1^3, y_1) \\ = \quad & \mathrm{Z}(x_1, y_1, x_2) = \{0\} \times \{[0,1]\} \end{aligned}$$

and this set is not in $\mathrm{Bl}(X, 0)$ by the above. Hence

$$\mathrm{Bl}(X,0) = \mathrm{Z}(x_1 y_2 - x_2, y_2^2 - x_1) \subseteq \{0\} \times U_1 \subseteq k^3$$

We claim that the map $A(t) = \langle t^2, t^3, t \rangle$ gives an isomorphism between k and $\mathrm{Z}(x_1 y_2 - x_2, y_2^2 - x_1)$.

Indeed this map has an inverse, which is the restriction to $\mathrm{Z}(x_1 y_2 - x_2, y_2^2 - x_1)$ of the map $B : k^3 \to k$ given by the formula $B(X_1, X_2, Y_2) = Y_2$.

To verify this, note first that we clearly have $A(t) \in \mathrm{Z}(x_1 y_2 - x_2, y_2^2 - x_1)$ and $B(A(t)) = t$.

Secondly, for $\langle X_1, X_2, Y_2\rangle \in \mathrm{Z}(x_1 y_2 - x_2, y_2^2 - x_1)$ we have

$$A(B(X_1, X_2, Y_2)) = (Y_2^2, Y_2^3, Y_2)$$

and we have

$$Y_2^2 = X_1,\ Y_2^3 = X_1 Y_2 = X_2.$$

We conclude that $\mathrm{Bl}(X, 0) \simeq k$. This establishes (1).