Infinite Groups

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William Thurston: "Mathematics is not about numbers, equations, computations, or algorithms: it is about understanding."

At the entrace to Plato's Academy, an inscription over the door said:
Let no one destitute of geometry enter here.
The same is written at the entrance to the Mathematical Institute.

Material available on the webpage

- Lecture Notes
- Revision Notes: material seen in courses from previous years, to be used as a reference only. Not examinable.Occasional reminders in lectures.
- Hand out Notes: expand on some notions introduced in the course, as further reading for students who wish to have a better understanding of the latter. Not examinable.
- Mini-projects Broadening. Please send me an email asap, and by week 4 at the latest, specifying your supervisor, the approximate topic of your DPhil and what project on the list you would like. Feel free to suggest other projects as well, as long as they are related to this course.

Themes of the course

We study countable infinite groups.
Methods of study:

- By endowing these groups with a metric, a geometry.
  - Easiest way to do it: using Cayley graphs.
  - Effective for groups with a finite generating set.
  - A recurrent theme: search for a connection between algebraic features and geometric features of Cayley graphs.
- By approximating these groups by finite groups.
  - For instance larger and larger finite quotients.
  - This can be done for residually finite groups.
- By designing algorithms/constructing Turing machines that can find solutions to algebraic questions.
  This can be done for (some) finitely presented groups (groups that can be fully described to a computer via finite data).

Themes of the course 2

- By representing infinite groups as groups of matrices.

  The groups that can be thus represented are called linear groups.

Classes of infinite groups that we study:

- "Small": Abelian finitely generated $\subset$ Nilpotent finitely generated $\subset$ Polycyclic $\subset$ Solvable finitely generated.
- "Large": Free groups $\subset$ Hyperbolic groups.
  Free groups $\subset$ Amalgamated products (in the sense of J.P. Serre).

The families of "small" groups are the object of study of the "Infinite Groups" course.
The families of "large" groups are the object of study of the "Geometric Group Theory" course in Hilary Term.

Constructions of groups, old and new

Direct product

The standard approach: take H_1, H_2, define operation on $H_1 \times H_2$.

Another approach: given G and H_1, H_2 subgroups of G, how to decide if G is isomorphic to $H_1 \times H_2$?
Three conditions:

- H_1, H_2 both normal subgroups.
- $H_1 \cap H_2 = \{1\}$.
- $H_1 H_2 = G$.

Generalization: direct sum

Let $X \neq \emptyset$, $\mathcal{G} = \{G_x \mid x \in X\}$ a collection of groups.
Consider

$$Map_f(X, \mathcal{G}) := \{f : X \to \bigsqcup_{x \in X} G_x \mid f(x) \in G_x\,,\ f(x) \neq 1_{G_x}$$

$$\text{for only finitely many } x \in X\}\,.$$

The direct sum $\bigoplus_{x \in X} G_x$ is $Map_f(X, \mathcal{G})$, endowed with the pointwise multiplication:

$$(f \cdot g)(x) = f(x) \cdot g(x)\,,\ \forall x \in X.$$

When $G_x = G$ for all $x \in X$, the direct sum is denoted by $\bigoplus_{x \in X} G$.

Semidirect product

Given two groups N and H and a group homomorphism $\varphi : H \to \mathrm{Aut}(N)$, one can define a new group $G = N \rtimes_\varphi H$ called semidirect product of N and H with respect to φ:

- As a set, $N \rtimes_\varphi H$ is defined as the cartesian product $N \times H$.
- Binary operation $*$ on G defined by

$$(n_1, h_1) * (n_2, h_2) = (n_1 \varphi(h_1)(n_2), h_1 h_2)\,,\ \forall n_1, n_2 \in N \text{ and } h_1, h_2 \in H\,.$$

- If φ is trivial (i.e. has as image $\{\mathrm{id}_N\}$) then $N \rtimes_\varphi H$ is the direct product $N \times H$.

Semi-direct product 2

Given a group G and two subgroups H, N how to know if G isomorphic to $N \rtimes_\varphi H$ for some φ?

Again three conditions:

- N normal subgroup.
- $N \cap H = \{1\}$.
- $NH = G$.

If the above are satisfied, G isomorphic to $N \rtimes_\varphi H$, where $\varphi(h) =$ conjugation by h of every element in N.

A more general notion

An exact sequence is a sequence of groups and group homomorphisms

$$\dots G_{n-1} \xrightarrow{\varphi_{n-1}} G_n \xrightarrow{\varphi_n} G_{n+1} \dots$$

such that $\mathrm{Im}\, \varphi_{n-1} = \ker \varphi_n$ for every n.

A short exact sequence is an exact sequence of the form:

$$\{1\} \longrightarrow N \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow \{1\}\,. \quad (1)$$

In other words, φ is an isomorphism from N to a normal subgroup $N' \lhd G$ and ψ defines an isomorphism $G/N' \simeq H$.

If G isomorphic to $N \rtimes_\varphi H$ then we have a short exact sequence as above. The converse is in general not true.

Semidirect product and short exact sequence

A short exact sequence splits if there exists a homomorphism $\sigma : H \to G$ (called a section) such that

$$\psi \circ \sigma = \mathrm{id}.$$

A split exact sequence determines a decomposition of G as a semidirect product $\varphi(N) \rtimes \sigma(H)$.

Examples

1. The dihedral group D_{2n} is isomorphic to $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$, where $\varphi(1)(k) = n - k$.
2. The infinite dihedral group D_∞ is isomorphic to $\mathbb{Z} \rtimes_\varphi \mathbb{Z}_2$, where $\varphi(1)(k) = -k$.
3. The permutation group S_n is the semidirect product of A_n and $\mathbb{Z}_2 = \{\mathrm{id}, (12)\}$.

Wreath product

Consider a direct sum $\bigoplus_{x \in H} G$ with index set a group H.

There is a natural action of H on the direct sum:

$$\varphi : H \to \mathrm{Aut}\left(\bigoplus_{h \in H} G\right), \varphi(h)f(x) = f(h^{-1}x), \forall x \in H.$$

Thus, we define the semidirect product

$$\left(\bigoplus_{h \in H} G\right) \rtimes_{\varphi} H. \qquad (2)$$

The semidirect product (2) is called the wreath product of G with H, and it is denoted by $G \wr H$.

The wreath product $G = \mathbb{Z}_2 \wr \mathbb{Z}$ is called the lamplighter group. Its name comes from the way in which $\varphi(1)$ acts.

The wreath product construction is a source of interesting examples of groups, in particular of solvable groups.

Finitely generated groups

Given $S \subset G$ and $H \leq G$, TFAE

- H is the smallest subgroup of G containing S;
- $H = \bigcap_{S \subset K \leq G} K$;
- $H = \{s_1^{\pm 1} s_2^{\pm 1} \dots s_n^{\pm 1} \mid n \in \mathbb{N}, s_i \in S\} \cup \{\mathrm{id}\}$.

H is called the subgroup generated by S. We write $H = \langle S \rangle$.

- If $H = G$ then S is called generating set.
- If S finite then G is called finitely generated.
- If $S = \{x\}$ then $\langle x \rangle$ cyclic subgroup generated by x.
- Rank of G= minimal number of generators.

Finitely generated groups 2

- If G is finitely generated then G is countable.
- There are uncountably many non-isomorphic finitely generated groups.

- G finitely generated $\Rightarrow G/N$ finitely generated, for any N normal subgroup.
- Not inherited by subgroups (not even normal).
- G, H finitely generated $\Rightarrow G \wr H$ finitely generated (Ex. Sheet 1).

  But $\bigoplus_{x \in H} G$ not finitely generated if H infinite.
- If N, H finitely generated and

$$\{1\} \longrightarrow N \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow \{1\}, \tag{3}$$

then G finitely generated (Ex. Sheet 1).

Free groups

What is "the largest infinite group" generated by n elements ?

Finite sets: A larger than $B \Leftrightarrow \mathrm{card}(A) \geq \mathrm{card}(B) \Leftrightarrow$ there exists $f : A \to B$ onto.

Infinite groups: We look for a group $G = \langle X \rangle$, $\mathrm{card}(X) = n$, such that for every group $H = \langle Y \rangle$, $\mathrm{card}(Y) = n$, a bijection $X \to Y$ extends to an onto group homomorphism.

Clearly cannot be done for any group G, e.g. if G is abelian then H would have to be abelian.

So G must be a group with no prescribed relation ("free").