Infinite Groups

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An example of ping-pong

Example

For any real number $r \geqslant 2$ the matrices

$$g_1 = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \text{ and } g_2 = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$$

generate a free subgroup of $SL(2, \mathbb{R})$.

Note that for $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we can write
$g_2 = I \circ g_1^{-1} \circ I^{-1}$.

Why is $\langle g_1, g_2\rangle$ free.

The group $SL(2,\mathbb{R})$ acts on the upper half plane $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ by linear fractional transformations

$$z \mapsto \frac{az+b}{cz+d}\,.$$

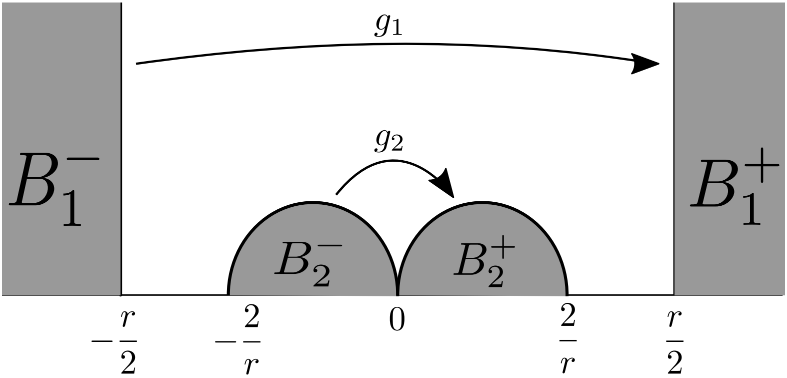
$g_1(z) = z + r,\ g_2(z) = \frac{z}{rz+1}$.
Define quarter-planes

$$B_1^+ = \{z \in \mathbb{H}^2 : \Re(z) \geqslant r/2\}, \quad B_1^- = \{z \in \mathbb{H}^2 : \Re(z) \leqslant -r/2\}$$

and half-disks $B_2^+ := \left\{z \in \mathbb{H}^2 : |z - \frac{1}{r}| \leqslant \frac{1}{r}\right\} = I(B_1^-)$ and

$$B_2^- := \left\{z \in \mathbb{H}^2 : |z + \frac{1}{r}| \leqslant \frac{1}{r}\right\} = I(B_1^+).$$

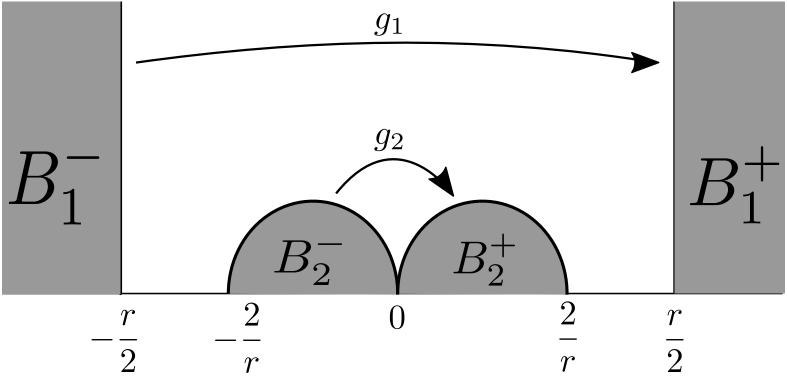
Example of ping-pong.



$g_1(\mathbb{H}^2 \setminus B_1^-) \subset B_1^+$, $g_1^{-1}(\mathbb{H}^2 \setminus B_1^+) \subset B_1^-$.

$g_2 = I \circ g_1^{-1} \circ I^{-1}$. $I(B_1^+) = B_2^-$ and $I(B_1^-) = B_2^+$. Therefore $g_2(\mathbb{H}^2 \setminus B_2^-) \subset B_2^+$, $g_2^{-1}(\mathbb{H}^2 \setminus B_2^+) \subset B_2^-$.

Example of ping-pong.



Take a reduced word in $\{g_1^{\pm 1}, g_2^{\pm 1}\}$, say $g_1 g_2^{-1} g_1 g_2$.

$B_2^+ \sqcup B_1^- \sqcup B_1^+ \xrightarrow{g_2} B_2^+ \xrightarrow{g_1} B_1^+ \xrightarrow{g_2^{-1}} B_2^- \xrightarrow{g_1} B_1^+$

General Ping-pong Lemma

Let $g_1, g_2 \in Bij(X)$ ("ping-pong partners") and $B_i^{\pm} \subset X, \quad i = 1, 2.$

Given $i \in \{1, 2\}$, let j be such that $\{i, j\} = \{1, 2\}$.
Define

$$C_i^+ := B_i^+ \cup B_j^- \cup B_j^+, C_i^- := B_i^- \cup B_j^- \cup B_j^+ .$$

Assume that:
$C_i^{\pm} \not\subset B_j^{\pm}$ and $C_i^{\pm} \not\subset B_j^{\mp}$ for all choices of i, j and $+, -$.

Theorem (Ping–pong, or table–tennis, lemma)

If

$$g_i^{\pm 1}(C_i^{\pm}) \subset B_i^{\pm}, \quad i = 1, 2,$$

then the bijections g_1, g_2 generate a free subgroup of $Bij(X)$.

Proof of General Ping-pong Lemma

Let w be a non-empty reduced word in $\{g_1, g_1^{-1}, g_2, g_2^{-1}\}$, of length at least 2.

w has the form

$$w = g_i^{\pm 1} u g_j^{\pm 1}.$$

We prove by induction on the length of w that

$$w(C_j^{\pm}) \subset B_i^{\pm}, \text{ hence } w \neq \mathrm{id}.$$

Length 2. $w = g_i^{\pm 1} g_j^{\pm 1}$.

$C_j^{\pm} \xrightarrow{g_j^{\pm 1}} B_j^{\pm} \xrightarrow{g_i^{\pm 1}} B_i^{\pm}$

The last transformation is true because the word is reduced, hence $B_j^{\pm} \neq B_i^{\mp}$, hence $B_j^{\pm}$ is contained in $C_i^{\pm}$.

Proof of General Ping-pong Lemma 2

Suppose it is true for all words w' of length n, we prove it for words w of length $n+1$.

Such a w has the form

$$w = g_i^{\pm 1} w', \quad \mathrm{length}(w') = n.$$

In its turn w' can be written as

$$w' = g_j^{\pm 1} u g_k^{\pm 1}, \quad g_j^{\pm 1} \neq g_i^{\mp 1}.$$

By the induction hypothesis

$$w'(C_k^{\pm}) \subset B_j^{\pm}.$$

Since $g_j^{\pm 1} \neq g_i^{\mp 1}$, we have that $B_j^{\pm} \neq B_i^{\mp}$,therefore $B_j^{\pm} \subset C_i^{\pm}$ and

$$w(C_k^{\pm})) = g_i^{\pm 1} w'(C_k^{\pm}) \subset g_i^{\pm 1}(C_i^{\pm}) \subset B_i^{\pm}.$$

Cayley graphs

Goal: to endow a group with a geometry, so first of all a metric.
Let $G = \langle S \rangle$, with $1 \notin S$ and $s^{-1} \in S$ for every $s \in S$.
We write the latter condition as $S^{-1} = S$.
The Cayley graph of G with respect to S is the directed/oriented graph $\mathrm{Cayley}_{\mathrm{dir}}(G, S)$ with

- set of vertices G;
- set of oriented edges (g, gs), with $s \in S$.

We label the oriented edge (g, gs) by s.
The underlying non-oriented graph $\mathrm{Cayley}(G, S)$ of $\mathrm{Cayley}_{\mathrm{dir}}(G, S)$ is the graph with

- set of vertices G;
- set of edges $\{g, h\}$ such that $h = gs$, with $s \in S$.

It is also called the Cayley graph of G with respect to S.
Occasionally, we will use the notation $\overline{gh}$ and $[g, h]$ for the edge $\{g, h\}$.

Cayley graphs 2

- The definition of the graph makes sense for every $S \subset G$.
- $1 \notin S$ prevents edges from composing loops (monogons).
- $S^{-1} = S$ ensures that every edge in $\mathrm{Cayley}(G, S)$ appears in $\mathrm{Cayley}_{\mathrm{dir}}(G, S)$ with both orientations.
- By definition $\mathrm{Cayley}(G, S)$ is a simplicial graph if $1 \notin S$ (i.e. no monogons, no two edges with same endpoints).
- The valency of every vertex g in $\mathrm{Cayley}(G, S)$ (i.e. number of edges having g as an endpoint) is $k = \mathrm{card}(S)$.Thus $\mathrm{Cayley}(G, S)$ is k-regular (all vertices of same valency k).

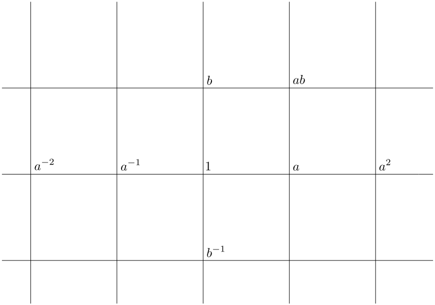
Lemma

$\mathrm{Cayley}(G, S)$ *is connected (i.e. every two vertices can be joined by an edge path) if and only if S generates G.*

Cayley graphs of $\mathbb{Z}^2$

Example

Consider $\mathbb{Z}^2$ *and* $S = \{a = (1,0), b = (0,1), a^{-1}, b^{-1}\}$.

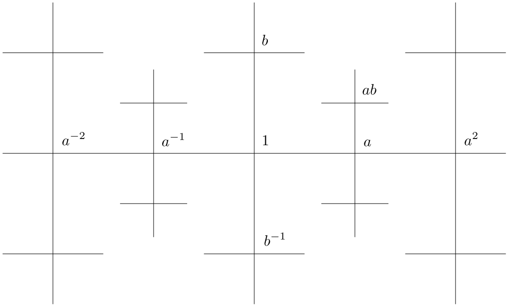


The Cayley graph of $\mathbb{Z}^2$ with respect to $\{\pm(1,0), \pm(1,1)\}$ has the same set of vertices as the above, but the vertical lines are replaced by diagonal lines.

Cayley graph of a free non-abelian group

Example

Let G be the free group on two generators a, b. Take $X = \{a, b\}$, $G = F(X)$ and $S = X \sqcup X^{-1}$. The Cayley graph $\mathrm{Cayley}(G, S)$ is the 4-valent tree.



A tree is a simplicial connected graph with no circuits. A *k*-valent tree is a *k*-regular tree.

Word metric

Convention

When talking about Cayley graphs, the group G is always assumed to be finitely generated, and S is always assumed to be finite.

We endow $\mathrm{Cayley}(G, S)$ with a distance such that edges have length 1.

$$\mathrm{dist}_S(x, y) = \text{length of the shortest path joining } x, y.$$

The restriction of dist_S to $G \times G$ is called word metric.

Exercise

Prove that for every $g, h \in G$, $\mathrm{dist}_S(g, h)$ is the length k of the shortest word $w = s_1 \dots s_k$, where $s_i \in S, \forall i$, such that $g = hw$.

Word metric 2

Notation

- *We denote by* $|g|_S$ *the distance* $\mathrm{dist}_S(1, g)$, *that is the shortest word in* S *representing* g.
- *We denote by* $B_S(x, r)$ *the closed ball centred in* $x \in \mathrm{Cayley}(G, S)$ *and of radius* $r > 0$ *with respect to* dist_S.

Proposition

The action of G *on itself by multiplications to the left is an action by isometries, that is for every* $g \in G$

$$\mathrm{dist}(gx, gy) = \mathrm{dist}_S(x, y), \forall x, y \in G.$$

It extends to an action by isometries on $\mathrm{Cayley}(G, S)$

Word metric 3

Exercise

1. *Prove that if S and $\bar{S}$ are two finite generating sets of G, then the word metrics dist_S and $\mathrm{dist}_{\bar{S}}$ on G are bi-Lipschitz equivalent, i.e. there exists $L > 0$ such that*

$$\frac{1}{L}\mathrm{dist}_S(g,g') \leqslant \mathrm{dist}_{\bar{S}}(g,g') \leqslant L\mathrm{dist}_S(g,g')\,,\forall g,g' \in G\,. \tag{1}$$

2. *Prove that an isomorphism between two finitely generated groups is a bi-Lipschitz map when the two groups are endowed with word metrics.*

Proposition

A finite index subgroup of a finitely generated group is finitely generated.