A brief introduction to Dedekind cuts

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October 21, 2023.

How to construct real numbers: Dedekind cuts

We assume that rational numbers $\mathbb{Q}$ have been constructed.

Lemma

$\sqrt{2} \notin \mathbb{Q}$, *i.e. there is no* $q \in \mathbb{Q}$ *with* $q^2 = 2$.

Proof.

By contradiction, if $q^2 = 2$ for $q = a/b$, where a, b are coprime integers, then $a^2 = 2b^2$, so $a = 2\alpha$ is even, so $4\alpha^2 = 2b^2$, so $2\alpha = b$, so b is even. This contradicts that a, b are coprime. □

Strategy: think of "$\{\text{rational numbers below } \sqrt{2}\}$" as representing $\sqrt{2}$.

Definition

A **Dedekind cut** $L \subseteq \mathbb{Q}$ is a subset of $\mathbb{Q}$ such that

1. L is non-empty and proper: $L \neq \emptyset$ and $L \neq \mathbb{Q}$,
2. L is "left closed": $\ell \in L \Rightarrow$ all rationals $q \leq \ell$ are in L,
3. L has no maximum: $\ell \in L \Rightarrow \exists \ell' \in L$ with $\ell < \ell'$.

Some side-remarks about the definition

Secretly: we are defining a real number r using only rational numbers. So $\mathbb{Q} = L \sqcup R$ is a disjoint union of a "left interval" $L = (-\infty, r) \cap \mathbb{Q}$, and a "right interval" $R = [r, \infty) \cap \mathbb{Q}$, where $R := \mathbb{Q} \setminus L$. Here $(-\infty, r)$ means the reals $< r$, and $[r, \infty)$ means the reals $\geq r$.

Why not allow maxima in L? Then $\{q \leq \frac{a}{b}\}$ would contain $\frac{a}{b}$? Adding cuts as sets in the obvious way may fail to contain the max:

$$\sqrt{2} + (-\sqrt{2}) = ((-\infty, \sqrt{2}] \cap \mathbb{Q}) + ((-\infty, -\sqrt{2}] \cap \mathbb{Q}) = ((-\infty, 0) \cap \mathbb{Q}).$$

Also, unions of cuts can lead to a cut that may fail to contain the max:

$$\cup_{n \neq 0 \in \mathbb{N}} \{q \in \mathbb{Q} : q \leq -\tfrac{1}{n}\} = \{q \in \mathbb{Q} : q < 0\}.$$

Both issues can be artificially fixed by declaring that rational maxima have to be inserted back in after performing operations, but it is not an elegant approach.

Examples

If $\frac{a}{b} \in \mathbb{Q}$, then we view $\frac{a}{b}$ as the Dedekind cut $L_{\frac{a}{b}} = \{q \in \mathbb{Q} : q < \frac{a}{b}\}$.

Lemma

$L_{\sqrt{2}} := \{q \in \mathbb{Q} : q < 0 \text{ or } q^2 < 2\}$ *is a Dedekind cut.*

Proof.

1. $0 \in L_{\sqrt{2}}$, $2 \notin L_{\sqrt{2}}$, so $L_{\sqrt{2}} \neq \emptyset, \mathbb{Q}$.
2. If $q^2 < 2$ and $\ell < q$, then $\ell^2 < q^2 < 2$, so $\ell \in L_{\sqrt{2}}$.
3. Want: $L_{\sqrt{2}}$ has no max. Suppose $q \in L_{\sqrt{2}}$. For small $0 < h < 1$ in $\mathbb{Q}$, we claim $q + h \in L_{\sqrt{2}}$. If $q < 0$, it is easy to see that such an h exists (easy exercise for you to check, as we understand $\mathbb{Q}$). If $q^2 < 2$, then $q < 2$ (otherwise $q \geq 2$ so $q^2 \geq 4$), and using $h^2 \leq h$ (as $0 < h < 1$):

$$(q+h)^2 = q^2 + 2hq + h^2 < q^2 + 2h \cdot 2 + h \overset{\text{want}}{<} 2.$$

This holds by picking $0 < h < 1$ in $\mathbb{Q}$ with $h < (2 - q^2)/5$. □

Operations and order for Dedekind cuts

We define operations on Dedekind cuts:

$$L + L' = \{\ell + \ell' : \ell \in L, \ell' \in L'\}.$$

Exercise: Define $L \cdot L'$, $-L$, $\frac{1}{L}$ carefully.
We define an order on Dedekind cuts:

$$L \leq L' \text{ means } L \subseteq L'.$$

Exercise: show $\{\text{Dedekind cuts}\}$ is an ordered field (check the axioms).

Theorem

$\mathbb{R} := \{\textit{Dedekind cuts}\}$ *satisfies the least upper bound property.*

Proof.

Given $\emptyset \neq S \subseteq \mathbb{R}$, consider the union of the Dedekind cuts $s \in S$:

$$L = \bigcup_{s \in S} s.$$

Exercise: check L is a Dedekind cut and the least upper bound of S. □

Example

Let $S := \{-\frac{1}{n} \in \mathbb{Q} : n \neq 0 \in \mathbb{N}\}$.

Claim. $\sup S = L_0$.

Proof.

By the Theorem, $\sup S = \cup_{n \neq 0 \in \mathbb{N}} \{q \in \mathbb{Q} : q < -\frac{1}{n}\} = \{q \in \mathbb{Q} < 0\}$.
The last equality is an easy check, because we understand $\mathbb{Q}$ well. □

Let $S := \{q \in \mathbb{Q} : q^2 < 2\}$. (Secretly the set $(-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$).

Claim. $\sup S = L_{\sqrt{2}}$.

Proof.

Observe that $L_{\sqrt{2}} = L_0 \cup S$.
By the Theorem, $L = \cup\{s \in S\}$ is the least upper bound.
As L is an upper bound of S, we have $S \subseteq L$.
Observe that $L_0 \subseteq L$ (since $0 \in S \subseteq L$). So $L_{\sqrt{2}} = L_0 \cup S \subseteq L$.
But $L_{\sqrt{2}}$ is an upper bound for S, as $S \subseteq L_0 \cup S = L_{\sqrt{2}}$.
As L is the *least* upper bound, $L_{\sqrt{2}} \subseteq L$ implies $L_{\sqrt{2}} = L$. □

$\exists$ only "one" ordered field with the lub property

Definition

An **isomorphism** $f : F_1 \to F_2$ of ordered fields is a bijection, preserving operations and order: $f(a+b) = f(a) + f(b)$, $f(a \cdot b) = f(a) \cdot f(b)$, $a \leq b \Rightarrow f(a) \leq f(b)$. Exercise: deduce $f(0) = 0$, $f(1) = 1$.

Theorem

For any ordered field F satisfying the least upper bound property, there is an isomorphism $f : \mathbb{R} \to F$ of ordered fields.

Proof.

Any ordered field contains a copy of $\mathbb{Q}$. Indeed, for $n \in \mathbb{N}$ define $n := 1 + \cdots + 1 \in \mathbb{F}$ (the cancellation property for $+$ and the ordered property ensure that these n are all distinct). Then $-n \in \mathbb{F}$, so $\mathbb{F}$ contains a copy of $\mathbb{Z}$. Finally $\mathbb{Q} \ni \frac{a}{b} = ab^{-1} \in \mathbb{F}$.
More precisely, we built an injection $\phi : \mathbb{Q} \to F$ by $\phi(\frac{a}{b}) = ab^{-1}$. $\square$

End of the proof

Proof continued.

For a general Dedekind cut $L \in \mathbb{R}$, we define $f : \mathbb{R} \to \mathbb{F}$ by:

$$f(L) := \sup \phi(L),$$

where $\phi(L) := \{\phi(q) : q \in L\}$.
Exercise: check $\sup \phi(L)$ is defined, then check f is an isomorphism.
Hint: $\phi(L) \neq \emptyset$ as $L \neq \emptyset$. Also $\phi(L)$ is bounded above by $\phi(r)$ for $r \in R := \mathbb{Q} \setminus L$ since $q < r$ for all $q \in L$ so $\phi(q) < \phi(r)$. □

Remark. There are other ways of constructing $\mathbb{R}$:
Decimal numbers. The disadvantage is that defining operations is messy, and there is the frustration that decimal expansions are not unique (e.g. $0.9999\ldots = 1.0000\ldots$).
Cauchy sequences. One thinks of $r \in \mathbb{R}$ as a limit of $q_n \in \mathbb{Q}$. One demands $|q_n - q_m|$ is arbitrarily small when $n, m \in \mathbb{N}$ are sufficiently large (this secretly ensures convergence to a limit in other constructions of $\mathbb{R}$). One uses equivalence classes: $(q_n) \sim (q'_n)$ if $|q_n - q'_n|$ is small for large n (this ensures they have the same limit). It is messy, but generalisable.