B1.1 Logic
Lecture 5

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6. A deductive system for propositional calculus

- We introduced '*logical consequence*' – $\Gamma \models \phi$ means: whenever (each formula of) Γ is true, so is ϕ.

- But we don't know yet how to give an actual **proof** of ϕ from the **hypotheses** Γ.

- A **proof** of ϕ should be a finite sequence $\phi_1, \phi_2, \ldots, \phi_n$ of statements such that $\phi_n = \phi$, and for each $i = 1, \ldots, n$:

  - either $\phi_i \in \Gamma$,
  - or ϕ_i is some **axiom** (which should *clearly* be true),
  - or ϕ_i should follow from previous ϕ_j's by some **rule of inference**.

6.1 Definition
Let $\mathcal{L}_0 := \mathcal{L}_{\text{prop}}[\{\neg, \to\}]$ (which is an adequate language). Then the **system** L_0 consists of the following axioms and rules:

Axioms
An **axiom** of L_0 is any formula of the following form ($\alpha, \beta, \gamma \in \text{Form}(\mathcal{L}_0)$):

A1 $(\alpha \to (\beta \to \alpha))$

A2 $((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$

A3 $((\neg\beta \to \neg\alpha) \to (\alpha \to \beta))$

Rules of inference
Just one rule, **modus ponens**:

MP For any $\alpha, \beta \in \text{Form}(\mathcal{L}_0)$:
From α and $(\alpha \to \beta)$, infer β.

6.2 Definition
Let $\Gamma \subseteq \text{Form}(\mathcal{L}_0)$.

- A finite sequence $\alpha_1, \ldots, \alpha_m \in \text{Form}(\mathcal{L}_0)$ is a **proof** (or **deduction/derivation**) in L_0 of α_m from the **hypotheses** Γ if for each $i = 1, \ldots, m$, at least one of the following holds:

  (a) α_i is an axiom of L_0.
  (b) $\alpha_i \in \Gamma$.
  (c) α_i follows by MP from earlier formulas, i.e. there are $j, k < i$ such that $\alpha_j = (\alpha_k \to \alpha_i)$.

- $\alpha \in \text{Form}(\mathcal{L}_0)$ is **provable** from Γ if there is a proof $\alpha_1, \ldots, \alpha_m = \alpha$ of α from Γ.

  We denote this by:

$$\Gamma \vdash \alpha.$$

In the case $\Gamma = \emptyset$, we just write

$$\vdash \alpha,$$

and we say that α is a **theorem** (of the system L_0).

6.3 Example For any $\phi \in \text{Form}(\mathcal{L}_0)$

$$(\phi \to \phi)$$

is a theorem of L_0.

Proof:

α_1 $(\phi \to (\phi \to \phi))$
[A1 with $\alpha = \beta = \phi$]
α_2 $(\phi \to ((\phi \to \phi) \to \phi))$
[A1 with $\alpha = \phi$, $\beta = (\phi \to \phi)$]
α_3 $((\phi \to ((\phi \to \phi) \to \phi)) \to ((\phi \to (\phi \to \phi)) \to (\phi \to \phi)))$
[A2 with $\alpha = \phi$, $\beta = (\phi \to \phi)$, $\gamma = \phi$]
α_4 $((\phi \to (\phi \to \phi)) \to (\phi \to \phi))$
[MP α_2, α_3]
α_5 $(\phi \to \phi)$
[MP α_1, α_4]

Thus, $\alpha_1, \alpha_2, \ldots, \alpha_5$ is a deduction of $(\phi \to \phi)$ in L_0.

□

6.4 Example

For any $\phi, \psi \in \mathrm{Form}(\mathcal{L}_0)$:

$$\{\phi, \neg\phi\} \vdash \psi$$

Proof:

α_1 $(\neg\phi \to (\neg\psi \to \neg\phi))$
[A1 with $\alpha = \neg\phi, \beta = \neg\psi$]
α_2 $\neg\phi$ [$\in \Gamma$]
α_3 $(\neg\psi \to \neg\phi)$ [MP α_1, α_2]
α_4 $((\neg\psi \to \neg\phi) \to (\phi \to \psi))$
[A3 with $\alpha = \phi, \beta = \psi$]
α_5 $(\phi \to \psi)$ [MP α_3, α_4]
α_6 ϕ [$\in \Gamma$]
α_7 ψ [MP α_5, α_6]

$\square$

6.5 The Soundness Theorem for L_0

L_0 is **sound**, *i.e. for any $\Gamma \subseteq \mathrm{Form}(\mathcal{L}_0)$ and for any $\alpha \in \mathrm{Form}(\mathcal{L}_0)$:*

$$\textit{If } \Gamma \vdash \alpha \textit{ then } \Gamma \models \alpha.$$

In particular, any theorem of L_0 is a tautology.

Proof:
We show by induction on m:

($\star$) If α has a proof of length m from Γ in L_0, then $\Gamma \models \alpha$.

For $m = 0$, there is nothing to prove (no proof has length 0).

So suppose $m \geq 1$ and ($\star$) holds for all $m' < m$, and suppose $\alpha_1, \ldots, \alpha_m$ is a proof in L_0. We have to show that $\Gamma \models \alpha_m$.

Case 1: α_m is an axiom.
One verifies by truth tables (exercise) that our axioms are tautologies, so $\Gamma \models \alpha_m$.

Case 2: $\alpha_m \in \Gamma$.
Then $\Gamma \models \alpha_m$.

Case 3: α_m is obtained by MP.
So say $i, j < m$ and $\alpha_j = (\alpha_i \to \alpha_m)$.

By the inductive hypothesis,
since $\alpha_1, \ldots, \alpha_i$ is a proof of length $i < m$,
we have $\Gamma \models \alpha_i$.
Similarly $\Gamma \models \alpha_j$, i.e. $\Gamma \models (\alpha_i \to \alpha_m)$.

But $\{\alpha_i, (\alpha_i \to \alpha_m)\} \models \alpha_m$ by Lemma 3.4, and it follows (from the definition of $\models$) that $\Gamma \models \alpha_m$.

$\square$

For the proof of the converse

Completeness Theorem

If $\Gamma \models \alpha$ *then* $\Gamma \vdash \alpha$.

we first prove

6.6 The Deduction Theorem for L_0

For any $\Gamma \subseteq \text{Form}(\mathcal{L}_0)$ *and*
for any $\alpha, \beta \in \text{Form}(\mathcal{L}_0)$*:*

If $\Gamma \cup \{\alpha\} \vdash \beta$ *then* $\Gamma \vdash (\alpha \to \beta)$.