B1.1 Logic
Lecture 6

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6.6 The Deduction Theorem for L_0

For any $\Gamma \subseteq \text{Form}(\mathcal{L}_0)$ *and*
for any $\alpha, \beta \in \text{Form}(\mathcal{L}_0)$*:*

$$\textit{If } \Gamma \cup \{\alpha\} \vdash \beta \textit{ then } \Gamma \vdash (\alpha \to \beta).$$

Proof:
We prove by induction on m:

If $\alpha_1, \ldots, \alpha_m$ *is a proof in* L_0 *from* $\Gamma \cup \{\alpha\}$
then $\Gamma \vdash (\alpha \to \alpha_i)$ *for all* $i \leq m$*.*

For $m = 0$, this holds trivially. So suppose $m > 0$.

IH: Holds for $m - 1$.
Then $\Gamma \vdash (\alpha \to \alpha_i)$ for $i < m$,
and we must show $\Gamma \vdash (\alpha \to \alpha_m)$.

Case 1: α_m is an Axiom
Then $\vdash (\alpha \to \alpha_m)$, indeed:

1	α_m	[Axiom]
2	$(\alpha_m \to (\alpha \to \alpha_m))$	[Instance of A1]
3	$(\alpha \to \alpha_m)$	[MP 1,2]

is a proof of $(\alpha \to \alpha_m)$ from hypotheses $\emptyset$.

Note generally that if $\Delta \vdash \psi$ and $\Delta' \supseteq \Delta$, then also $\Delta' \vdash \psi$.

Thus $\Gamma \vdash (\alpha \to \alpha_m)$.

Case 2: $\alpha_m \in \Gamma \cup \{\alpha\}$
If $\alpha_m \in \Gamma$ then same proof as above works (with justification on line 1 changed to '$\in \Gamma$').

If $\alpha_m = \alpha$, then, by Example 6.3, $\vdash (\alpha \to \alpha_m)$, hence $\Gamma \vdash (\alpha \to \alpha_m)$.

Case 3: α_m is obtained by MP from some earlier α_j, α_k, i.e. there are $j, k < m$ such that $\alpha_j = (\alpha_k \to \alpha_m)$.

By IH, we have

$$
\begin{array}{ll}
 & \Gamma \vdash (\alpha \to \alpha_k) \\
\text{and} & \Gamma \vdash (\alpha \to \alpha_j), \\
\text{i.e.} & \Gamma \vdash (\alpha \to (\alpha_k \to \alpha_m))
\end{array}
$$

So say

$$\beta_1, \ldots, \beta_{r-1}, (\alpha \to \alpha_k)$$

and

$$\gamma_1, \ldots, \gamma_{s-1}, (\alpha \to (\alpha_k \to \alpha_m))$$

are proofs in L_0 from Γ.

Then

1	β_1	
⋮	⋮	
r-1	β_{r-1}	
r	$(\alpha \to \alpha_k)$	
r+1	γ_1	
⋮	⋮	
r+s-1	γ_{s-1}	
r+s	$(\alpha \to (\alpha_k \to \alpha_m))$	
r+s+1	$((\alpha \to (\alpha_k \to \alpha_m)) \to$ $((\alpha \to \alpha_k) \to (\alpha \to \alpha_m)))$	[A2]
r+s+2	$((\alpha \to \alpha_k) \to (\alpha \to \alpha_m))$	[MP r+s, r+s+1]
r+s+3	$(\alpha \to \alpha_m)$	[MP r, r+s+2]

is a proof of $(\alpha \to \alpha_m)$ in L_0 from Γ. □

6.7 Remarks

- Only needed instances of A1, A2 and the rule MP.
  So any system that includes A1, A2 and MP satisfies the Deduction Theorem.

- Proof gives a precise **algorithm** for converting any proof showing $\Gamma \cup \{\alpha\} \vdash \beta$ into one showing $\Gamma \vdash (\alpha \to \beta)$.

- Converse is easy:

  If $\Gamma \vdash (\alpha \to \beta)$ then $\Gamma \cup \{\alpha\} \vdash \beta$.

  Proof:

$\vdots$	$\vdots$	proof from Γ
r	$\alpha \to \beta$	
r+1	α	$[\in \Gamma \cup \{\alpha\}]$
r+2	β	[MP r, r+1]

$\square$

6.8 Example of use of DT

If $\Gamma \vdash (\alpha \to \beta)$ and $\Gamma \vdash (\beta \to \gamma)$
then $\Gamma \vdash (\alpha \to \gamma)$.

Proof:

By the deduction theorem ('DT'), it suffices to show that $\Gamma \cup \{\alpha\} \vdash \gamma$.

⋮	⋮	proof from Γ
r	$(\alpha \to \beta)$	
r+1	⋮	
⋮	⋮	proof from Γ
r+s	$(\beta \to \gamma)$	
r+s+1	α	$[\in \Gamma \cup \{\alpha\}]$
r+s+2	β	[MP r, r+s+1]
r+s+3	γ	[MP r+s, r+s+2]

□

From now on we may treat DT as an additional inference rule in L_0.

6.9 Definition

The **sequent calculus** SQ is the system where a **proof** (or **derivation**) of $\phi \in \text{Form}(\mathcal{L}_0)$ from $\Gamma \subseteq \text{Form}(\mathcal{L}_0)$ is a finite sequence of **sequents**,
i.e. expressions of the form

$$\Delta \vdash_{SQ} \psi$$

with $\Delta \subseteq \text{Form}(\mathcal{L}_0)$,
such that $\Gamma \vdash_{SQ} \phi$ is the last sequent,
and each sequent is obtained from previous sequents according to the following rules:

Ass: If $\psi \in \Delta$ then infer $\Delta \vdash_{SQ} \psi$.

MP: From $\Delta \vdash_{SQ} \psi$ and $\Delta' \vdash_{SQ} (\psi \to \chi)$
infer $\Delta \cup \Delta' \vdash_{SQ} \chi$.

DT: From $\Delta \cup \{\psi\} \vdash_{SQ} \chi$
infer $\Delta \vdash_{SQ} (\psi \to \chi)$.

PC: From $\Delta \cup \{\neg\psi\} \vdash_{SQ} \chi$
and $\Delta' \cup \{\neg\psi\} \vdash_{SQ} \neg\chi$,
infer $\Delta \cup \Delta' \vdash_{SQ} \psi$.
('PC' stands for *proof by contradiction*.)

Note: no axioms.

6.10 Example of a proof in SQ

1	$\neg\beta \vdash_{SQ} \neg\beta$	[Ass]
2	$(\neg\beta \to \neg\alpha) \vdash_{SQ} (\neg\beta \to \neg\alpha)$	[Ass]
3	$(\neg\beta \to \neg\alpha), \neg\beta \vdash_{SQ} \neg\alpha$	[MP 1,2]
4	$\alpha, \neg\beta \vdash_{SQ} \alpha$	[Ass]
5	$(\neg\beta \to \neg\alpha), \alpha \vdash_{SQ} \beta$	[PC 3,4]
6	$(\neg\beta \to \neg\alpha) \vdash_{SQ} (\alpha \to \beta)$	[DT 5]
7	$\vdash_{SQ} ((\neg\beta \to \neg\alpha) \to (\alpha \to \beta))$	[DT 6]

So $\vdash_{SQ}$ A3.

Notation: To avoid confusion, we sometimes write '$\Gamma \vdash_{L_0} \phi$' for '$\Gamma \vdash \phi$ in L_0'

6.11 Theorem

L_0 and SQ are equivalent, i.e. for all Γ, ϕ:

$$\Gamma \vdash_{L_0} \phi \text{ iff } \Gamma \vdash_{SQ} \phi.$$

Proof: Exercise