Infinite Groups

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Presentations of groups

Given $G = \langle S \rangle$, by the Universal property, $\exists$ an onto homomorphism $\pi_S : F(S) \to G$, whence G isomorphic to $F(S)/\ker(\pi_S)$.

The elements of $\ker(\pi_S)$ are called relators or relations for $G = \langle S \rangle$.
We are interested in minimal subsets R of $\ker(\pi_S)$ such that $\ker(\pi_S)$ is normally generated by R, that is $\ker(\pi_S) = \langle\langle R \rangle\rangle$.
We say that the elements $r \in R$ are defining relators.
The pair (S, R) defines a presentation of G.
We write $G = \langle S \mid r = 1, \forall r \in R \rangle$ or simply $G = \langle S \mid R \rangle$.
It means G is isomorphic to $F(S)/\langle\langle R \rangle\rangle$.
Equivalently:

- $\forall g \in G$, $g = s_1 \cdots s_n$, for some $n \in \mathbb{N}$ and $s \in S \cup S^{-1}$;
- $w \in F(S)$ satisfies $w =_G 1$ if and only if in $F(S)$

$$w = \prod_{i=1}^{m} r_i^{x_i}, \text{ for some } m \in \mathbb{N}, r_i \in R, x_i \in F(S).$$

Examples of group presentations

1. $\langle a_1, \ldots, a_n \mid [a_i, a_j], 1 \leqslant i, j \leqslant n \rangle$ is a finite presentation of $\mathbb{Z}^n$;
2. $\langle x, y \mid x^n, y^2, yxyx \rangle$ is a presentation of the finite dihedral group D_{2n} ;
3. $\langle x, y \mid x^3, y^2, [y, x] \rangle$ is a presentation of the cyclic group $\mathbb{Z}_6$;
4. $\langle x_1, \ldots, x_n \mid x_i^2, (x_i x_j)^{m_{ij}} \rangle$, is a presentation of a Coxeter group;
5. the Integer Heisenberg group:

$$H_{2n+1}(\mathbb{Z}) := \langle x_1, \ldots, x_n, y_1, \ldots, y_n, z\,;$$

$$[x_i, z] = 1, [y_j, z] = 1, [x_i, x_j] = 1, [y_i, y_j] = 1, [x_i, y_j] = z^{\delta_{ij}}, 1 \leqslant i, j \leqslant n \rangle.$$

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$$H_{2n+1}(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & \dots & \dots & x_n & z \\ 0 & 1 & 0 & \dots & \dots & 0 & y_n \\ 0 & 0 & 1 & \dots & \dots & 0 & y_{n-1} \\ \vdots & \vdots & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 1 & 0 & y_2 \\ 0 & 0 & \dots & \dots & 0 & 1 & y_1 \\ 0 & 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix} \; ; \; x_i, y_j, z \in \mathbb{Z} \right\}$$

Finitely presented groups

A group G is called finitely presented if it admits a finite presentation.

NB While there are continuously many finitely generated groups, there are only countably many finitely presented groups.

Sometimes it is difficult, and even algorithmically impossible, to find a finite presentation of a finitely presented group.

Proposition (Universal Property Generalized)

Let $G = \langle S|R \rangle$. A map $\psi : S \to H$ with target a group H can be extended (uniquely) to a group homomorphism $\Phi : G \to H$ iff for every $r \in R$, $r = s_1 \dots s_n$, $\psi(s_1) \dots \psi(s_n) = 1$ in H.

Generalization of the Universal Property

Proposition

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Proof. The Universal property of free groups $\Rightarrow \psi$ extends to $\tilde{\psi} : F(S) \to H$.

$\langle\langle R\rangle\rangle = \langle R^{F(S)}\rangle$ is generated by elements of the form grg^{-1}, where $g \in F(S), r \in R$.
$\tilde{\psi}(grg^{-1}) = 1 \Rightarrow \langle\langle R\rangle\rangle \leqslant \ker(\tilde{\psi}) \Rightarrow \tilde{\psi}$ defines $\Phi : F(S)/\langle\langle R\rangle\rangle \to H$.

Uniqueness: because every homomorphism is entirely determined by its restriction to a generating set. □

Finite presentability is independent of the generating set

Proposition

Assume $G = \langle S \mid R \rangle$ finite presentation, and $G = \langle X \mid T \rangle$ is such that X is finite. Then $\exists$ finite subset $T_0 \subset T$ such that $G = \langle X \mid T_0 \rangle$.

Proof. $\forall s \in S\ \exists a_s(X)$ word in X s.t. $s = a_s(X)$ in G. (involves a choice)
The map $i_{SX} : S \to F(X)$, $i_{SX}(s) = a_s(X)$ extends to a unique homomorphism $p : F(S) \to F(X)$ (rewriting homomorphism).
We have that $\pi_S = \pi_X \circ p$.
Likewise, $\forall x \in X\ \exists b_x(S)$ in S s.t. $x = b_x(S)$.
The map $i_{XS} : X \to F(S)$, $i_{XS}(x) = b_x(S)$, extends to homomorphism $q : F(X) \to F(S)$ (another rewriting homomorphism).
As previously $\pi_S \circ q = \pi_X$.

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For every $x \in X$,

$$\pi_X(p(q(x))) = \pi_S(q(x)) = \pi_X(x).$$

Whence for every $x \in X$, $x^{-1}p(q(x)) \in \ker(\pi_X)$.
Let N be the normal subgroup of $F(X)$ normally generated by

$$\{p(r) \mid r \in R\} \cup \{x^{-1}p(q(x)) \mid x \in X\}.$$

We have that $N \leqslant \ker(\pi_X)$. Goal: prove equality.
There is a natural projection

$$Q : F(X)/N \to F(X)/\ker(\pi_X).$$

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Let $\bar{p} : F(S) \to F(X)/N$ be the homomorphism induced by p.
$\bar{p}(r) = 1$ for all $r \in R \Rightarrow \bar{p}$ induces a homomorphism

$$Q' : F(S)/\ker(\pi_S) \to F(X)/N.$$

Note that the domain of Q' is isomorphic to G, and Q' is onto:
$F(X)/N$ is generated by $xN = p(q(x))N$, and the latter is the image under Q' of $q(x)\ker(\pi_S)$.
Consider the homomorphism

$$Q \circ Q' : F(S)/\ker(\pi_S) \to F(X)/\ker(\pi_X)$$

.
The isomorphism $G \to F(S)/\ker(\pi_S)$ sends every $x \in X$ to $q(x)\ker(\pi_S)$.
The isomorphism $G \to F(X)/\ker(\pi_X)$ sends every $x \in X$ to $x\ker(\pi_X)$.
Note that $Q \circ Q'\,(q(x)\ker(\pi_S)) = Q(xN) = x\ker(\pi_X)$, whence $Q \circ Q'$ isomorphism $\Rightarrow Q'$ injective $\Rightarrow Q'$ isomorphism $\Rightarrow Q$ isomorphism $\Rightarrow N = \ker(\pi_X)$.

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In particular, $\ker(\pi_X)$ is normally generated by the finite set of relators

$$\mathfrak{R} = \{p(r) \mid r \in R\} \cup \{x^{-1}p(q(x)) \mid x \in X\}.$$

Since $\mathfrak{R} \subset \langle\langle T \rangle\rangle$, every relator $\rho \in \mathfrak{R}$ can be written as a product

$$\prod_{i \in I_\rho} t_i^{v_i}$$

with $v_i \in F(X), t_i \in T$ and I_ρ finite.
Whence $\ker(\pi_X)$ is normally generated by the finite subset

$$T_0 = \bigcup_{\rho \in \mathfrak{R}} \{t_i \mid i \in I_\rho\}$$

of T. □