# Infinite Groups

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# Finite presentability is independent of the generating set

**Proposition**

*Assume $G = \langle S \mid R \rangle$ finite presentation, and $G = \langle X \mid T \rangle$ is such that $X$ is finite. Then $\exists$ finite subset $T_0 \subset T$ such that $G = \langle X \mid T_0 \rangle$.*

Reformulation of the Proposition:

Given a short exact sequence

$$1 \to N \to F(X) \to G \to 1$$

with $X$ finite and $G$ finitely presented, $N$ is normally generated by finitely many elements $n_1, \ldots, n_k$.

This reformulation has a generalization to arbitrary short exact sequences that will appear later on.

# Commutators

### Notation

$[x, y] = xyx^{-1}y^{-1}$= *the commutator of the elements* $x, y$ *in a group* $G$

*For subsets* $A, B$ *in a group* $G$, $[A, B]$= *the subgroup of* $G$ *generated by all the commutators* $[a, b], a \in A, b \in B$.

*For every* $x_1, \dots, x_n$ *in a group* $G$ *we denote by* $[x_1, \dots, x_n]$ *the n-fold left-commutator*

$$[[[x_1, x_2], \dots, x_{n-1}], x_n].$$

*For subsets* $A_1, \dots A_n$ *in a group* $G$, $[A_1, \dots, A_n]$= *the subgroup of* $G$ *generated by all the commutators* $[a_1, \dots, a_n], a_i \in A_i$.

# Nilpotent Groups: first definition

There are two ways of defining nilpotent groups and of measuring "how far they are from being abelian".

First definition: from the group downwards.
The lower central series of a group $G$,

$$C^1G \trianglerighteq C^2G \trianglerighteq \dots \trianglerighteq C^nG \trianglerighteq \dots ,$$

is defined inductively by:

$$C^1G = G\,,\; C^{n+1}G = [C^nG, G]\,.$$

Each $C^kG$ is a characteristic subgroup of $G$ (i.e. for every automorphism $\varphi : G \to G$, $\varphi(C^kG) = C^kG$).
$C^2G = [G, G] = G'$ is the commutator subgroup, or the derived subgroup, of $G$.

# Nilpotent Groups 2

**Definition**

$G$ is $k$-step nilpotent if $C^{k+1}G = \{1\}$. The minimal $k$ for which $G$ is $k$-step nilpotent is called the (nilpotency) class of $G$.

**Examples**

1. Every non-trivial abelian group is nilpotent of class 1.
2. The group $\mathcal{U}_n(\mathbb{K})$ of upper triangular $n \times n$ matrices with 1 on the diagonal and entries in a ring $\mathbb{K}$, is nilpotent of class $n-1$ (see Exercise Sheet 2).
3. The integer Heisenberg group $H_{2n+1}(\mathbb{Z})$ is nilpotent of class 2.

# Basic properties

**Lemma**

*If $G = \langle S \rangle$ ($S$ not necessarily finite, $G$ not necessarily nilpotent), then $\forall k$ the subgroup $C^k G$ is generated by the $k$-fold left commutators in $S$, together with $C^{k+1} G$.*

Proof Induction on $k$ and two formulas:

- $[x, yz] = [x, y]\,[y, [x, z]]\,[x, z]\,;$
- $[xy, z] = [x, [y, z]]\,[y, z]\,[x, z] = [y, z]^x [x, z]\,.$

**Corollary**

*If $G = \langle S \rangle$ is nilpotent, then $C^n G$ is generated by all the $k$-fold left commutators in $S$, where $k \geqslant n$. In particular, if $G$ is finitely generated, each subgroup $C^n G$ is finitely generated.*

# Nilpotent Groups: second definition

Second definition: from $\{1\}$ upwards.
The center $Z(H)$ of a group $H$ is composed of all $z \in H$ s.t. $zh = hz, \forall h \in H$.

Given a group $G$, define inductively an increasing sequence of normal subgroups $Z_i(G) \lhd G$ by:

- $Z_0(G) = \{1\}$.
- If $Z_i(G) \lhd G$ is defined and $\pi_i : G \to G/Z_i(G)$ is the quotient map, then

$$Z_{i+1}(G) = \pi_i^{-1}\left(Z(G/Z_i(G))\right).$$

Note that $Z_{i+1}(G)$ is normal in $G$, as the inverse image of a normal subgroup of a quotient of $G$.
In particular,

$$Z_{i+1}(G)/Z_i(G) \cong Z(G/Z_i(G)).$$

# Nilpotent Groups: second definition 2

**Proposition**

*$G$ is $k$-step nilpotent if and only if $Z_k(G) = G$.*

Proof Assume that $G$ is nilpotent of class $k$.
We prove by induction on $i \geqslant 0$ that $C^{k+1-i}G \leqslant Z_i(G)$.
For $i = 0$ we have equality.
Assume that

$$C^{k+1-i}G \leqslant Z_i(G).$$

For every $g \in C^{k-i}G$ and every $x \in G$, $[g, x] \in C^{k+1-i}G \leqslant Z_i(G)$, whence $gZ_i(G)$ is in the center of $G/Z_i(G)$, i.e. $g \in Z_{i+1}(G)$.

For $i = k$ the inclusion becomes $C^1G = G \leqslant Z_k(G)$, hence $Z_k(G) = G$.

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Conversely, assume $Z_k(G) = G$.

We prove by induction on $j \geqslant 1$ that $C^j G \leqslant Z_{k+1-j}(G)$.

For $j = 1$ the two are equal.

Assume the inclusion true for $j$.

$C^{j+1}G$ generated by $[c, g]$ with $c \in C^j G$ and $g \in G$.

Since $c \in C^j G \leqslant Z_{k+1-j}(G)$, by the definition of $Z_{k+1-j}(G)$, the element $c$ commutes with $g$ modulo $Z_{k-j}(G)$, equivalently $[c, g] \in Z_{k-j}(G)$.

This implies that $[c, g] \in Z_{k-j}(G)$. It follows that $C^{j+1}G \leqslant Z_{k-j}(G)$.

For $j = k + 1$ this gives $C^{k+1}G \leqslant Z_0(G) = \{1\}$, hence $G$ is $k$-step nilpotent. □

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### Definition

The ascending series

$$Z_0(G) = \{1\} \lhd Z_1(G) \lhd \ldots \lhd Z_i(G) \lhd Z_{i+1}(G) \lhd \ldots$$

of normal subgroups of $G$ is called the upper central series of $G$.

A group $G$ is nilpotent if and only if there exists $i$ such that $Z_i(G) = G$, and its nilpotency class is the minimal $k$ such that $Z_k(G) = G$.

The following example shows that the difference between lower and upper central series of groups can be quite substantial:

### Example

We start with the integer Heisenberg group $H$; it is 2-step nilpotent, $C^2H = H' = Z(H) \cong \mathbb{Z}$.
Take $G = H \times \mathbb{Z}$, 2-step nilpotent. $C^2G = C^2H \cong \mathbb{Z}$, while $Z(G) \cong \mathbb{Z}^2$.

# Nilpotent Groups: properties

**Lemma**

1. *Every subgroup of a nilpotent group is nilpotent.*
2. *If $G$ is nilpotent and $N \lhd G$ then $G/N$ is nilpotent.*
3. *The direct product of a family of nilpotent groups is again nilpotent.*

NB (3) not true for semidirect products. Not even for $\mathbb{Z}^n \rtimes \mathbb{Z}$.

# Nilpotent Groups: a key property

**Theorem**

*Every subgroup $H$ of a finitely generated nilpotent group $G$ is finitely generated.*