B1.1 Logic
Lecture 7

Martin Bays

Oxford, MT 2023

The following lemma is a key step in the proof of 6.11; it shows that L_0 implements the rule (PC) of the sequence calculus. It is the only place in the proof of the completeness theorem where (A3) is used.

6.12 Lemma
For any $\alpha, \beta \in \text{Form}(L_0)$,

$$\vdash ((\neg\alpha \to \neg\beta) \to ((\neg\alpha \to \beta) \to \alpha)).$$

Proof: Omitted.

7. Consistency, Completeness and Compactness

7.1 Definition

$\Gamma \subseteq \mathrm{Form}(\mathcal{L}_0)$ is **inconsistent**
if for some formula α,
$\Gamma \vdash \alpha$ and $\Gamma \vdash \neg\alpha$.

Otherwise, Γ is **consistent**.

E.g. $\emptyset$ is consistent by soundness of $\mathcal{L}_0$, since for no α are both α and $\neg\alpha$ tautologies.

7.2. Lemma

If $\Gamma \nvdash \phi$ then $\Gamma \cup \{\neg\phi\}$ is consistent.

Proof: Suppose $\Gamma \cup \{\neg\phi\}$ is inconsistent, say $\Gamma \cup \{\neg\phi\} \vdash \alpha$ and $\Gamma \cup \{\neg\phi\} \vdash \neg\alpha$.

Then by the deduction theorem,
$\Gamma \vdash (\neg\phi \to \alpha)$ and $\Gamma \vdash (\neg\phi \to \neg\alpha)$.

By 6.12 and MP twice, $\Gamma \vdash \phi$.

$\square$

7.3 Lemma
Suppose Γ *is consistent and* $\Gamma \vdash \phi$*.*
Then $\Gamma \cup \{\phi\}$ *is consistent.*

Proof: Suppose not. Then for some α

$$\left.\begin{array}{l}\Gamma \cup \{\phi\} \vdash \alpha \\ \Gamma \cup \{\phi\} \vdash \neg\alpha\end{array}\right\} \Rightarrow_{\mathsf{DT}} \begin{array}{l}\Gamma \vdash (\phi \to \alpha) \\ \Gamma \vdash (\phi \to \neg\alpha)\end{array}$$

$$\overset{\Gamma\vdash\phi}{\Rightarrow_{\mathsf{MP}}} \begin{array}{l}\Gamma \vdash \alpha \\ \Gamma \vdash \neg\alpha\end{array},$$

contradicting consistency of Γ.

$\square$

7.4 Definition
$\Gamma \subseteq \mathrm{Form}(\mathcal{L}_0)$ is **maximal consistent** if
(i) Γ is consistent, and
(ii) for *every* ϕ, either $\Gamma \vdash \phi$ or $\Gamma \vdash \neg\phi$.

7.5 Theorem

Suppose Γ is consistent. Then there is a maximal consistent $\Gamma' \supseteq \Gamma$.

Proof:

Form($\mathcal{L}_0$) is countable, say

$$\text{Form}(\mathcal{L}_0) = \{\phi_1, \phi_2, \phi_3, \ldots\}.$$

Construct consistent sets

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots$$

as follows:

- $\Gamma_0 := \Gamma$.
- Given consistent Γ_n, let

$$\Gamma_{n+1} := \begin{cases} \Gamma_n \cup \{\phi_{n+1}\} & \text{if } \Gamma_n \vdash \phi_{n+1} \\ \Gamma_n \cup \{\neg\phi_{n+1}\} & \text{if } \Gamma_n \nvdash \phi_{n+1} \end{cases}$$

  Then Γ_{n+1} is consistent by 7.3 and 7.2.

Now let $\Gamma' := \bigcup_{n=0}^{\infty} \Gamma_n$.

Then Γ' is consistent:

Any proof of $\Gamma' \vdash \alpha$ and $\Gamma' \vdash \neg\alpha$ would use only finitely many formulas from Γ', so for some n, $\Gamma_n \vdash \alpha$ and $\Gamma_n \vdash \neg\alpha$ – contradicting the consistency of Γ_n.

Finally, Γ' is maximal consistent: for all n,
either $\phi_n \in \Gamma'$ or $\neg\phi_n \in \Gamma'$,
so in particular either $\Gamma' \vdash \phi_n$ or $\Gamma' \vdash \neg\phi_n$.

$\square$

(Note that this proof did not use Zorn's Lemma; countability of the language was crucial for this.)

7.6 Lemma

Suppose Γ *is maximal consistent.*
Then for every $\psi, \chi \in \mathrm{Form}(\mathcal{L}_0)$*:*
(a) $\Gamma \vdash \neg\psi$ *iff* $\Gamma \nvdash \psi$*.*
(b) $\Gamma \vdash (\psi \to \chi)$ *iff either* $\Gamma \nvdash \psi$ *or* $\Gamma \vdash \chi$*.*

Proof:

(a) **'⇒':** by consistency.
'⇐': by maximality.

(b) **'⇒':** Suppose $\Gamma \vdash (\psi \to \chi)$ but $\Gamma \vdash \psi$ and $\Gamma \nvdash \chi$.
By MP, $\Gamma \vdash \chi$, contradicting consistency.

'⇐': Suppose $\Gamma \nvdash \psi$. Then $\Gamma \vdash \neg\psi$ by (a).
$\Gamma \vdash (\neg\psi \to (\psi \to \chi))$ (Problem sheet 2 Q3)
$\Rightarrow_{\mathrm{MP}}$ $\Gamma \vdash (\psi \to \chi)$.

Suppose $\Gamma \vdash \chi$.
$\Gamma \vdash (\chi \to (\psi \to \chi))$ (Axiom A1)
$\Rightarrow_{\mathrm{MP}}$ $\Gamma \vdash (\psi \to \chi)$.

□

7.7 Theorem

Suppose Γ is maximal consistent.
Then Γ is satisfiable.

Proof:

Define a valuation v by

$$v(p_i) = T \text{ iff } \Gamma \vdash p_i.$$

Claim: for all $\phi \in \text{Form}(\mathcal{L}_0)$:

$$\widetilde{v}(\phi) = T \text{ iff } \Gamma \vdash \phi.$$

Proof by induction on the length n of ϕ.

If $n = 1$, then $\phi = p_i$ for some i and we are done by the definition of v.

Suppose $n = \text{length}(\phi) > 1$.
IH: Claim true for all $n' < n$.

Case 1: $\phi = \neg\psi$

$$\begin{array}{llll} \tilde{v}(\phi) = T & \text{iff} & \tilde{v}(\psi) = F & \text{tt } \neg \\ & \text{iff} & \Gamma \nvdash \psi & \text{IH} \\ & \text{iff} & \Gamma \vdash \neg\psi & \text{7.6(a)} \\ & \text{iff} & \Gamma \vdash \phi & \end{array}$$

Case 2: $\phi = (\psi \to \chi)$

$$\begin{array}{llll} \tilde{v}(\phi) = T & \text{iff} & \tilde{v}(\psi) = F \text{ or } \tilde{v}(\chi) = T & \text{tt } \to \\ & \text{iff} & \Gamma \nvdash \psi \text{ or } \Gamma \vdash \chi & \text{IH} \\ & \text{iff} & \Gamma \vdash (\psi \to \chi) & \text{7.6(b)} \\ & \text{iff} & \Gamma \vdash \phi & \end{array}$$

So $\tilde{v}(\phi) = T$ for all $\phi \in \Gamma$, i.e. v satisfies Γ.

$\square$

7.8 Corollary

Let $\Gamma \subset \text{Form}(\mathcal{L}_0)$. Then
Γ is consistent if and only if Γ is satisfiable.

Proof:

$\Rightarrow$: By 7.5 + 7.7:
If Γ is consistent,
then by 7.5 it extends to a maximal consistent set,
which by 7.7 is satisfiable,
hence also Γ is satisfiable.

$\Leftarrow$: By soundness:
Suppose Γ inconsistent,
say $\Gamma \vdash \alpha$ and $\Gamma \vdash \neg\alpha$.
Then $\Gamma \models \alpha$ and $\Gamma \models \neg\alpha$ by soundness,
so Γ is not satisfiable.

$\square$

7.9 The Completeness Theorem
If $\Gamma \models \phi$ then $\Gamma \vdash \phi$.

Proof:

Suppose $\Gamma \nvdash \phi$.

$\Rightarrow$ by 7.2, $\Gamma \cup \{\neg\phi\}$ is consistent
$\Rightarrow$ by 7.8, $\Gamma \cup \{\neg\phi\}$ is satisfiable
$\Rightarrow$ there is some valuation v such that
$\tilde{v}(\psi) = T$ for $\psi \in \Gamma$, but $\tilde{v}(\phi) = F$
$\Rightarrow \Gamma \not\models \phi$. $\square$

7.10 Corollary
(7.9 Completeness + 6.5 Soundness)

$$\Gamma \models \phi \text{ iff } \Gamma \vdash \phi$$

7.11 The Compactness Theorem for $\mathcal{L}_0$

$\Gamma \subseteq \mathrm{Form}(\mathcal{L}_0)$ *is satisfiable iff every finite subset of* Γ *is satisfiable.*

Proof: By 7.8, this is equivalent to:
$\Gamma \subseteq \mathrm{Form}(\mathcal{L}_0)$ *is consistent iff every finite subset of* Γ *is consistent.*

But indeed, by finiteness of proofs,
$\Gamma \vdash \alpha$ and $\Gamma \vdash \neg\alpha$ iff already
$\Gamma_0 \vdash \alpha$ and $\Gamma_0 \vdash \neg\alpha$ for some finite $\Gamma_0 \subseteq \Gamma$.

$\square$