B1.1 Logic
Lecture 9

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9. Interpretations and Assignments

9.1 Definition

Let $\mathcal{L}$ be a language. An **interpretation** of $\mathcal{L}$ is an $\mathcal{L}$-**structure**

$$\mathcal{A} := \langle A; (f^{\mathcal{A}})_{f \in \mathsf{Fct}(\mathcal{L})}, (P^{\mathcal{A}})_{P \in \mathsf{Pred}(\mathcal{L})}, (c^{\mathcal{A}})_{c \in \mathsf{Const}(\mathcal{L})} \rangle,$$

where:

- A is a non-empty set, the **domain** of $\mathcal{A}$;
- For $f \in \mathcal{L}$ a k-ary function symbol, $f^{\mathcal{A}} : A^k \to A$ is a k-ary function;
- For $P \in \mathcal{L}$ a k-ary predicate symbol, $P^{\mathcal{A}}$ is a k-ary relation on A, i.e. $P^{\mathcal{A}} \subseteq A^k$;
- For $c \in \mathcal{L}$ a constant symbol, $c^{\mathcal{A}} \in A$.

9.2 Definition

Let $\mathcal{L}$ be a language and let $\mathcal{A} = \langle A; \ldots \rangle$ be an $\mathcal{L}$-structure.

(1) An **assignment** in $\mathcal{A}$ is a function

$$v : \{x_0, x_1, \ldots\} \to A$$

(2) v determines an assignment

$$\widetilde{v} = \widetilde{v}^{\mathcal{A}} : \text{Terms}(\mathcal{L}) \to A$$

defined recursively as follows:

(i) $\widetilde{v}(x_i) := v(x_i)$ for all $i = 0, 1, \ldots$;

(ii) $\widetilde{v}(c) := c^{\mathcal{A}}$ for each constant symbol $c \in \mathcal{L}$;

(iii) $\widetilde{v}(f(t_1, \ldots, t_k)) := f^{\mathcal{A}}(\widetilde{v}(t_1), \ldots, \widetilde{v}(t_k))$ for each k-ary function symbol $f \in \mathcal{L}$.

(3) v determines a **valuation**

$$\widetilde{v} = \widetilde{v}^{\mathcal{A}} : \text{Form}(\mathcal{L}) \to \{T, F\}$$

as follows:

Define $\widetilde{v}$ on formulas recursively:

- On atomic formulas:
  - For each k-ary predicate symbol $P \in \mathcal{L}$ and for all $t_i \in \mathrm{Term}(\mathcal{L})$:

$$\widetilde{v}(P(t_1, \ldots, t_k)) = \begin{cases} T & \text{if } (\widetilde{v}(t_1), \ldots, \widetilde{v}(t_k)) \in P^{\mathcal{A}} \\ F & \text{otherwise.} \end{cases}$$

  - For all $t_1, t_2 \in \mathrm{Term}(\mathcal{L})$:

$$\widetilde{v}(t_1 \doteq t_2) = \begin{cases} T & \text{if } \widetilde{v}(t_1) = \widetilde{v}(t_2) \\ F & \text{otherwise.} \end{cases}$$

- $\widetilde{v}(\neg\psi) = T$ iff $\widetilde{v}(\psi) = F$
- $\widetilde{v}(\psi \to \chi) = T$ iff $\widetilde{v}(\psi) = F$ or $\widetilde{v}(\chi) = T$
- $\widetilde{v}(\forall x_i \psi) = T$ iff $\widetilde{v}^\star(\psi) = T$ for all assignments $v^\star$ agreeing with v except possibly at x_i.

Notation: Write $\mathcal{A} \models \phi[v]$ for $\widetilde{v}^{\mathcal{A}}(\phi) = T$, read '$\phi$ *is true in* $\mathcal{A}$ *under the assignment* v'.

9.3 Example

Consider $\mathcal{A} = \langle \mathbb{Z}; \cdot \rangle$ as an $\{f\}$-structure (f a binary function symbol). Let v be the assignment $v(x_i) = i (\in \mathbb{Z})$ for $i = 0, 1, \ldots$, and let

$$\phi = \forall x_0 \forall x_1 (f(x_0, x_2) \doteq f(x_1, x_2) \rightarrow x_0 \doteq x_1)$$

Then $\mathcal{A} \models \phi[v]$; indeed:

	$\mathcal{A} \models \phi[v]$
iff	for all $v^\star$ with $v^\star(x_i) = i$ for $i \neq 0$ $\mathcal{A} \models \forall x_1 (f(x_0, x_2) \doteq f(x_1, x_2) \rightarrow x_0 \doteq x_1)[v^\star]$
iff	for all $v^{\star\star}$ with $v^{\star\star}(x_i) = i$ for $i \neq 0, 1$ $\mathcal{A} \models (f(x_0, x_2) \doteq f(x_1, x_2) \rightarrow x_0 \doteq x_1)[v^{\star\star}]$
iff	for all $v^{\star\star}$ with $v^{\star\star}(x_i) = i$ for $i \neq 0, 1$ $v^{\star\star}(x_0) \cdot v^{\star\star}(x_2) = v^{\star\star}(x_1) \cdot v^{\star\star}(x_2)$ implies $v^{\star\star}(x_0) = v^{\star\star}(x_1)$
iff	for all $a, b \in \mathbb{Z}$, $a \cdot 2 = b \cdot 2$ implies $a = b$, which is true.

However, with $v'(x_i) = 0$ for all i, we would have finished with
... iff for all $a, b \in \mathbb{Z}$, $a \cdot 0 = b \cdot 0$ implies $a = b$, which is false. So $\mathcal{A} \not\models \phi[v']$.

9.4 Example

Let P be a unary predicate symbol, $\mathcal{L} = \{P\}$, $\mathcal{A}$ an $\mathcal{L}$-structure,

$$\phi = (\forall x_0 P(x_0) \to P(x_1)),$$

and v any assignment in $\mathcal{A}$. Then $\mathcal{A} \models \phi[v]$.

Proof:
$\mathcal{A} \models \phi[v]$ iff
$\mathcal{A} \models \forall x_0 P(x_0)[v]$ implies $\mathcal{A} \models P(x_1)[v]$.

Now suppose $\mathcal{A} \models \forall x_0 P(x_0)[v]$. Then for all $v^\star$ which agree with v except possibly at x_0, $\mathcal{A} \models P(x_0)[v^\star]$.

In particular, for $v^\star(x_i) = \begin{cases} v(x_i) & \text{if } i \neq 0 \\ v(x_1) & \text{if } i = 0 \end{cases}$
we have $P^{\mathcal{A}}(v^\star(x_0))$, and hence $v(x_1) \in P^{\mathcal{A}}$, i.e. $\mathcal{A} \models P(x_1)[v]$. $\square$

9.5 Definition
Let $\mathcal{L}$ be a language.

- An $\mathcal{L}$-formula ϕ is **logically valid** ('$\models \phi$') if $\mathcal{A} \models \phi[v]$ for *all* $\mathcal{L}$-structures $\mathcal{A}$ and for *all* assignments v in $\mathcal{A}$.

- $\phi \in$ Form($\mathcal{L}$) is **satisfiable** if $\mathcal{A} \models \phi[v]$ for *some* $\mathcal{L}$-structure $\mathcal{A}$ and for *some* assignment v in $\mathcal{A}$.

- For $\Gamma \subseteq$ Form($\mathcal{L}$) and $\phi \in$ Form($\mathcal{L}$), ϕ is a **logical consequence** of Γ, written $\Gamma \models \phi$, if for *all* $\mathcal{L}$-structures $\mathcal{A}$ and for *all* assignments v in $\mathcal{A}$ with $\mathcal{A} \models \psi[v]$ for all $\psi \in \Gamma$, also $\mathcal{A} \models \phi[v]$.

- $\phi, \psi \in$ Form($\mathcal{L}$) are **logically equivalent** if $\{\phi\} \models \psi$ and $\{\psi\} \models \phi$.

Example: $\models \phi$ for ϕ from Example 9.4.

Note:
The symbol '$\models$' is now used in two ways:

- $\Gamma \models \phi$ means: ϕ is a logical consequence of Γ.

- $\mathcal{A} \models \phi[v]$ means: ϕ is satisfied in the $\mathcal{L}$-structure $\mathcal{A}$ under the assignment v.

This shouldn't give rise to confusion, since it will always be clear from the context whether there is a set Γ of $\mathcal{L}$-formulas or an $\mathcal{L}$-structure $\mathcal{A}$ in front of '$\models$'.

9.6 Some abbreviations

We use ...	as abbreviation for ...
$(\alpha \vee \beta)$	$((\alpha \to \beta) \to \beta)$
$(\alpha \wedge \beta)$	$\neg(\neg\alpha \vee \neg\beta)$
$(\alpha \leftrightarrow \beta)$	$((\alpha \to \beta) \wedge (\beta \to \alpha))$
$\exists x_i \phi$	$\neg\forall x_i \neg\phi$

9.7 Lemma

For any $\mathcal{L}$-structure $\mathcal{A}$ and any assignment v in $\mathcal{A}$ one has

$$\begin{array}{rcl} \mathcal{A} \models (\alpha \vee \beta)[v] & \text{iff} & \mathcal{A} \models \alpha[v] \text{ or } \mathcal{A} \models \beta[v] \\ \mathcal{A} \models (\alpha \wedge \beta)[v] & \text{iff} & \mathcal{A} \models \alpha[v] \text{ and } \mathcal{A} \models \beta[v] \\ \mathcal{A} \models (\alpha \leftrightarrow \beta)[v] & \text{iff} & \widetilde{v}(\alpha) = \widetilde{v}(\beta) \\ \mathcal{A} \models \exists x_i \phi[v] & \text{iff} & \text{for some assignment} \\ & & v^\star \text{ agreeing with } v \\ & & \text{except possibly at } x_i \\ & & \mathcal{A} \models \phi[v^\star] \end{array}$$

Proof: Easy exercise.